Chapter 3
Fredholm Integral Equations

3.1 Introduction

It was stated in Chapter 1 that Fredholm integral equations arise in many scientific applications. It was also shown that Fredholm integral equations can be derived from boundary value problems. Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in Acta Mathematica played a major role in the establishment of operator theory. As stated before, in Fredholm integral equations, the integral containing the unknown function u(x) is characterized by fixed limits of integration in the form

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \qquad (3.1)$$

where a and b are constants, The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. For the first kind Fredholm integral equations, the unknown function $u(x)$ occurs only under the integral sign in the form

$$f(x) = \int_a^b K(x,t)u(t)dt. \qquad (3.2)$$

However, Fredholm integral equations of the second kind, the unknown function $u(x)$ occurs inside and outside the integral sign. When $f(x) = 0$, the equation (3.1) is said to be homogeneous.

3.2 Fredholm Integral Equations of the Second Kind

We will first study Fredholm integral equations of the second kind given by

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \qquad (3.1)$$

The unknown function $u(x)$, that will be determined, occurs inside and outside the integral sign. The kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. Infinite geometric series

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r}, & if\ |r| < 1 \\ \infty, & if\ \ |r| \geq 1 \end{cases}$$

3.2.1 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian and was used before in Chapter 2. The Adomian method will be briefly outlined. The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.2}$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \tag{3.3}$$

where the components $u_0(x),\ n \geq 0$ will be determined recurrently. The Adomian decomposition method concerns itself with finding the components $u_0, u_1, u_2, \cdots$ individually. As we have seen before, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (3.2) into the Fredholm integral equation (3.1) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t) \left(\sum_{n=0}^{\infty} u_n(t)\right) dt, \tag{3.4}$$

or equivalently

$$u_0(x) + u_1(x) + \cdots = f(x) + \lambda \int_a^b K(x,t)\,(u_0(t) + u_1(t) + u_2(t) + \cdots)dt$$

$$u_0(x) = f(x), \text{or } \ u_0(x) = u(0)$$

$$u_1(x) = \lambda \int_a^b K(x,t)\, u_0(t) dt,$$

$$u_2(x) = \lambda \int_a^b K(x,t)\, u_1(t) dt,$$

$$u_3(x) = \lambda \int_a^b K(x,t)\, u_2(t) dt,$$

and so on for other components. In view of (3.4), the components $u_0(x), u_1(x),\ u_2(x), u_3(x), \ldots$ are completely determined. As a result, the solution $u(x)$ of the Fredholm integral equation (3.1) is readily obtained in a series form by using the series assumption in (3.2). It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown that if an exact solution exists for the problem, then the obtained series converges very rapidly to that exact solution. The convergence concept of the decomposition series was thoroughly investigated by many researchers to confirm the rapid convergence of the resulting series. However, for concrete problems, where a closed form solution is not obtainable, a truncated number of terms is usually used for numerical purposes. The more components we use the higher accuracy we obtain.

Example 3.1 Solve the following Fredholm integral equation

Solve the following Fredholm integral equation

$$u(x) = e^x - x + x\int_0^1 tu(t)dt.$$

The Adomian decomposition method assumes that the solution $u(x)$ has a series form. Substituting the decomposition series into both sides gives

$$\sum_{n=0}^{\infty} u_n(x) = e^x - x + x\int_0^1 t\sum_{n=0}^{\infty} u_n\,(t)dt,$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation

$$u_0(x) = e^x - x,$$

$$u_1(x) = x\int_0^1 tu_0(t)dt = x\int_0^1 t(e^t - t)dt = x - \frac{x}{3} = \frac{2}{3}x,$$

$$u_2(x) = x\int_0^1 tu_1(t)dt = \frac{2}{3}x\int_0^1 t^2dt = \frac{2}{9}x,$$

$$u_3(x) = x\int_0^1 tu_2(t)dt = \frac{2}{9}x\int_0^1 t^2dt = \frac{2}{27}x,$$

and so on. the series solution

$$u(x) = e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right) = e^x - x + \frac{2}{3}x\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n.$$

Notice that the infinite geometric series at the right side $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ has $a = 1$, and the ratio $= \frac{1}{3}$. The sum of the infinite series is therefore given by

$$S = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

$$u(x) = e^x - x + \frac{2}{3}x.\frac{3}{2} = e^x.$$

Example 3.2 Solve the following Fredholm integral equation

$$u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} u(t)dt.$$

Proceeding as before, we substitute the decomposition series into both sides of to find

$$\sum_{n=0}^{\infty} u_n(x) = \sin x - x + x\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty} u_n(t)\,dt$$

Consequently, we obtain

$$u_0(x) = \sin x - x,$$

$$u_1(x) = x\int_0^{\frac{\pi}{2}} u_0(t)dt = x - \frac{\pi^2}{8}x = x(1 - \frac{\pi^2}{8}),$$

$$u_2(x) = x\int_0^{\frac{\pi}{2}} u_1(t)dt = \frac{\pi^2}{8}x - \frac{\pi^4}{64}x = x\frac{\pi^2}{8}(1 - \frac{\pi^2}{8})$$

$$u_3(x) = x\int_0^{\frac{\pi}{2}} u_2(t)dt = \frac{\pi^4}{64}x - \frac{\pi^6}{512}x = x\frac{\pi^4}{8^2}(1 - \frac{\pi^2}{8}).$$

Gives the series solution

We can easily observe the appearance of the noise terms, i.e. the identical terms with opposite signs. Canceling these noise terms of

$u(x) = \sin x - x + x - \frac{\pi^2}{8}x + \frac{\pi^2}{8}x - \frac{\pi^4}{64}x + \frac{\pi^4}{64}x - \frac{\pi^6}{512}x + \frac{\pi^6}{512}x - \frac{\pi^8}{4096}x + \cdots$gives the exact solution $u(x) = \sin x$.

Example 3.3 Solve the following Fredholm integral equation

$$u(x) = x + e^x - \frac{4}{3} + \int_0^1 tu(t)dt.$$

Substituting the decomposition series into both sides gives

$$\sum_{n=0}^{\infty} u_n(x) = x + e^x - \frac{4}{3} + \int_0^1 t \sum_{n=0}^{\infty} u_n(t)\, dt.$$

$$u_0(x) = x + e^x - \frac{4}{3},$$

$$u_1(x) = \int_0^1 tu_0(t)dt = \int_0^1 t(t + e^t - \frac{4}{3})dt = \frac{2}{3},$$

$$u_2(x) = \int_0^1 tu_1(t)dt = \frac{2}{3}\int_0^1 tdt = \frac{1}{3},$$

$$u_3(x) = \int_0^1 tu_2(t)dt = \frac{1}{3}\int_0^1 tdt = \frac{1}{6},$$

$$u_4(x) = \int_0^1 tu_3(t)dt = \frac{1}{6}\int_0^1 tdt = \frac{1}{12},$$

and so on. the series solution

$$u(x) = x + e^x - \frac{4}{3} + \frac{2}{3}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = x + e^x - \frac{4}{3} + \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$$

$$= x + e^x - \frac{4}{3} + \frac{4}{3} = = x + e^x$$

Example 3.4 Solve the following Fredholm integral equation

$$u(x) = 2 + \cos x + \int_0^{\pi} tu(t)dt.$$

We next set the following recurrence relation

$$u_0(x) = 2 + \cos x,$$

$$u_1(x) = \int_0^{\pi} tu_0(t)dt = \pi^2 - 2,$$

$$u_2(x) = \int_0^{\pi} tu_1(t)dt = \frac{\pi^4}{2} - \pi^2,$$

$$u_3(x) = \int_0^{\pi} tu_2(t)dt = \frac{\pi^6}{4} - \frac{\pi^4}{2},$$

$$u_4(x) = \int_0^{\pi} t u_3(t)dt = \frac{\pi^8}{8} - \frac{\pi^6}{4},$$

gives the series solution

$$u(x) = 2 + \cos x + (\pi^2 - 2) + \left(\frac{\pi^4}{2} - \pi^2\right) + (\frac{\pi^6}{4} - \frac{\pi^4}{2}) + (\frac{\pi^8}{8} - \frac{\pi^6}{4}) + \cdots.$$

gives the exact solution $u(x) = \cos x$.

Example 3.5 Solve the following Fredholm integral equation

$$u(x) = 1 + \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u(t)dt.$$

Consequently, we obtain $u_0(x) = 1$,

$$u_1(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_0(t)dt = \frac{\pi}{8}\sec^2 x,$$

$$u_2(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_1(t)dt = \frac{\pi}{16}\sec^2 x,$$

$$u_3(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_2(t)dt = \frac{\pi}{32}\sec^2 x,$$

$$u_4(x) = \frac{1}{2}\sec^2 x \int_0^{\frac{\pi}{4}} u_3(t)dt = \frac{\pi}{64}\sec^2 x,$$

and so on. the series solution

$$u(x) = 1 + \frac{\pi}{8}\sec^2 x\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) = 1 + \frac{\pi}{8}\sec^2 x \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n.$$

The sum of the geometric series at the right side is $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$. The series solution converges to the closed form solution $u(x) = 1 + \frac{\pi}{4}\sec^2 x$.

Example 3.6 Solve the following Fredholm integral equation

$$u(x) = \pi x + \sin 2x + x\int_{-\pi}^{\pi} t u(t)dt.$$

Proceeding as before we find

$$u_0(x) = \pi x + \sin 2x,$$

$$u_1(x) = x\int_{-\pi}^{\pi} tu_0(t)dt = -\pi x + \frac{2\pi^4 x}{3},$$

$$u_2(x) = x\int_{-\pi}^{\pi} tu_1(t)dt = -\frac{2\pi^4 x}{3} + \frac{4\pi^7 x}{9},$$

$$u_3(x) = x\int_{-\pi}^{\pi} tu_2(t)dt = -\frac{4\pi^7 x}{9} + \frac{8\pi^{10} x}{27},$$

$$u_4(x) = x\int_{-\pi}^{\pi} u_3(t)dt = -\frac{8\pi^{10} x}{27} + \frac{16\pi^{13} x}{81},$$

gives the series solution

$$u(x) = \pi x + \sin 2x + \left(-\pi + \frac{2\pi^4}{3}\right)x + \left(-\frac{2\pi^4}{3} + \frac{4\pi^7}{9}\right)x + \left(-\frac{4\pi^7}{9} + \frac{8\pi^{10}}{27}\right)x + \cdots.$$

We can easily observe the appearance of the noise terms, i.e the identical terms with opposite signs. Canceling these noise terms in (4.32) gives the exact solution $u(x) = \sin 2x$.

Exercises 4.2.1 In Exercises 1–20, solve the following Fredholm integral equations by using the Adomian decomposition method

1. $u(x) = e^x + 1 - e + \int_0^1 u(t)dt.$

2. $u(x) = \cos x + 2x + x\int_0^{\pi} tu(t)dt.$

3. $u(x) = e^x + \dfrac{e^{x+1} - 1}{x+1} - \int_0^1 e^{xt}u(t)dt.$

4. $u(x) = x + (1-x)e^x + x^2\int_0^1 e^{t(x-1)}u(t)dt$

5. $u(x) = 1 - \dfrac{19}{15}x^2 + \int_{-1}^1 (xt + x^2t^2)\, u(t)dt$

3.2.2 The Modified Decomposition Method

$$u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt, \qquad (3.1)$$

Set

$$f(x) = f_1(x) + f_2(x).$$

The modified decomposition method admits the use of the modified recurrence relation:

$$u_0(x) = f_1(x),\ u_1(x) = f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt,$$

$$u_{k+1}(x) = \lambda \int_a^b K(x,t)u_k(t)\,dt,\ k \geq 1. \qquad (3.5)$$

A rule that may help for the proper choice of $f_1(x)$ and $f_2(x)$ could not be found yet. Second, if $f(x)$ consists of one term only, the modified decomposition method cannot be used in this case.

Example 3.7 Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4) + \int_0^1 tu(t)dt.$$

We first decompose f(x) given by $f(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4)$, into two parts, namely $f_1(x) = 3x + e^{4x}, f_2(x) = -\frac{1}{16}(17 + 3e^4)$. We next use the modified recurrence formula (4.66) to obtain

$$u_0(x) = f_1(x) = 3x + e^{4x},\ u_1(x) = -\frac{1}{16}(17 + 3e^4) + \int_0^1 tu_0(t)dt = 0,$$

$u_2(x) = 0, u_3(t) = 0, \cdots$. It is obvious that each component of uj , j 1 is zero. This in turn gives the exact solution by $u(x) = 3x + e^{4x}$.

Example 4.8 Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = \frac{1}{1+x^2} - 2\sinh\frac{\pi}{4} + \int_{-1}^1 e^{\tan^{-1} t}\, u(t)dt.$$

Proceeding as before we split f(x) given by $f(x) = \frac{1}{1+x^2} - 2\ \sinh\frac{\pi}{4}$,

into two parts, namely

$f_1(x) = \frac{1}{1+x^2}$, $f_2(x) = -2\sinh\frac{\pi}{4}$. We next use the modified recurrence formula to obtain $u_0(x) = f_1(x) = \frac{1}{1+x^2}$, ,

$$u_1(x) = f_2(x) + \int_{-1}^1 e^{\tan^{-1} t}\, u_0(t)dt = -2\ \sinh\frac{\pi}{4} + \int_{-1}^1 e^{\tan^{-1} t}\,\frac{1}{1+t^2}\,dt,$$

$$= -2\ \sinh\frac{\pi}{4} + \int_{-1}^{1} e^{\tan^{-1} t}\,(\tan^{-1} t)' dt = -2\ \sinh\frac{\pi}{4} + e^{\tan^{-1} 1} - e^{\tan^{-1} -1}$$

$$= -2\ \sinh\frac{\pi}{4} + 2\ \sinh\frac{\pi}{4} = 0$$

$$u_2(x) = 0 = u_3(x) = \cdots,\ \ u(x) = \frac{1}{1+x^2}$$

Example 3.9 Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = x + \sin^{-1}\frac{x+1}{2} + \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2\int_{-1}^{1} u(t)dt.$$

We decompose f(x) into two parts given by

$f_1(x) = x + \sin^{-1}\frac{x+1}{2}$, $f_2(x) = \frac{2-\pi}{2}x^2$.

We next use the modified recurrence formula to obtain

$$u_0(x) = x + \sin^{-1}\frac{x+1}{2},$$

$$u_1(x) = \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2\int_{-1}^{1} u_0(t)dt$$

$$= \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2\int_{-1}^{1}(t + \sin^{-1}\frac{t+1}{2})dt\,, y = \ \sin^{-1}\frac{t+1}{2}, t = 2\sin y - 1$$

$$= \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2(0 + 2\int_{0}^{\frac{\pi}{2}} y\cos y\, dy = \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2(\pi - 2) = 0$$

$$u_2(x) = 0 = u_3(x) = \cdots,\ \ u(x) = x + \sin^{-1}\frac{x+1}{2}$$

Example 3.10 Solve the Fredholm integral equation by using the modified decomposition method

$$u(x) = \sec^2 x + x^2 + x - \int_{0}^{\frac{\pi}{4}}(\frac{4}{\pi}x^2 + x\,u(t))dt$$

$$= \sec^2 x + x^2 + x - x^2 - \int_{0}^{\frac{\pi}{4}} x\,u(t)\,dt$$

$$u(x) = \sec^2 x + x - \int_{0}^{\frac{\pi}{4}} x\,u(t)\,dt$$

$$f_1(x) = \sec^2 x\,, \qquad f_2(x) = x.$$
$$u_0(x) = \sec^2 x\,,$$
$$u_1(x) = x - x\int_0^{\frac{\pi}{4}} u_0(t)dt = x - x\int_0^{\frac{\pi}{4}} \sec^2 t\ dt$$
$$= x - x\int_0^{\frac{\pi}{4}} (1+\tan^2 t)dt = x - \frac{\pi}{4}x - x\int_0^{\frac{\pi}{4}} \tan^2 t\, dt,$$
$$y = \tan t\,, \cos t = \frac{1}{\sqrt{1+y^2}},\quad dy = \sec^2 t\, dt, dt = \cos^2 t\, dy = \frac{dy}{1+y^2}$$
$$u_1(x) = x - \frac{\pi}{4}x - x\int_0^1 \frac{y^2}{1+y^2}dy = x - \frac{\pi}{4}x - x\int_0^1 (1 - \frac{1}{1+y^2})dy$$
$$= x - \frac{\pi}{4}x - x + x\int_0^1 \frac{1}{1+y^2}dy = x - \frac{\pi}{4}x - x + \frac{\pi}{4}x = 0$$
$$u_2(x) = 0 = u_3(x) = \cdots,\ \ u(x) = \sec^2 x.$$

Exercises 3.2.2 Use the modified decomposition method to solve the following Fredholm integral equations:

1. $u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} tu(t)\, dt$
2. $u(x) = e^x + \frac{1}{2}x^2 + (3+e)x - 4 - \int_0^1 (x-t)u(t)dt$
3. $u(x) = (\pi - 2)x + \sin^{-1}\frac{x+1}{2} + \frac{1}{2} - \sin^{-1}\frac{x-1}{2} - \int_0^1 xu(t)dt$
4. $u(x) = x + x^4 + 9e^x + 1 - 23e^x - \int_0^1 e^{x+t}u(t)dt$
5. $u(x) = xe^x + \frac{xe^{x+1}+1}{(x+1)^2} - \int_0^1 e^{xt}u(t)dt$
6. $u(x) = \frac{2}{15} + \frac{7}{12}x + x^2 + x^3 - \int_0^1 (1+x-t)u(t)dt$
7. $u(x) = e^x(e^{\frac{1}{3}} - 1) + e^{2x} - \frac{1}{3}\int_0^1 e^{x-\frac{5}{3}t}u(t)dt$
8. $u(x) = (\pi-2)(x+1)^2 + x\tan^{-1} x - \int_0^1 (1+x-t)u(t)dt$
9. $u(x) = \ln\frac{1+e}{2}x + \frac{e^x}{1+e^x} - \int_0^1 xu(t)dt$

3.2.5 The Direct Computation Method

In this section, the direct computation method will be applied to solve the Fredholm integral equations. It is important to point out that this method will be applied for the degenerate or separable kernels of the form

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t). \qquad (3.6)$$

Examples of separable kernels are $x - t, xt, x^2 - t^2, xt^2 + x^2t$, etc. The direct computation method can be applied as follows:

1. We first substitute (3.6) into the Fredholm integral equation the form

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt. \qquad (3.7)$$

2. This substitution gives

$$u(x) = f(x) + g_1(x)\int_a^b h_1(t)u(t)dt + g_2(x)\int_a^b h_2(t)u(t)dt + \cdots$$

$$\cdot + g_n(x)\int_a^b h_n(t)u(t)dt. \, (3.8)$$

3. Each integral at the right side depends only on the variable t with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (3.8) becomes

$$u(x) = f(x) + \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x), \qquad (3.9)$$

where $\alpha_i = \int_a^b h_i(t)u(t)dt, 1 \le i \le n.$ (3.10)

4. Substituting (3.9) into (3.10) gives a system of n algebraic equations that can be solved to determine the constants $\alpha_i, 1 \le i \le n$. Using the obtained numerical values of αi into (4.132), the solution u(x) of the Fredholm integral equation (3.9) is readily obtained.

Example 3.19 Solve the Fredholm integral equation by using the direct computation method

$$u(x) = 3x + 3x^2 + \frac{1}{2}\int_0^1 x^2tu(t)dt. \qquad (3.11)$$

The kernel $K(x,t) = x^2t$ is separable. Consequently, we rewrite as

$$u(x) = 3x + 3x^2 + \frac{1}{2}x^2\int_0^1 tu(t)dt. \qquad (3.12)$$

The integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation (3.12) can be rewritten as

$$u(x) = 3x + 3x^2 + \frac{1}{2}\alpha x^2, \qquad (3.13)$$

where

$$\alpha = \int_0^1 t u(t) dt. \qquad (3.14)$$

To determine α, we substitute (3.13) into (3.14) to obtain

$$\alpha = \int_0^1 t(3t + 3t^2 + \frac{1}{2}\alpha t^2) dt. \qquad (3.15)$$

Integrating the right side of (3.15) yields

$$\alpha = \frac{7}{4} + \frac{1}{8}\alpha, \qquad (3.16)$$

that gives $\alpha = 2$. Substituting into (3.13) leads to the exact solution

$$u(x) = 3x + 4x^2,$$

Example 3.20 Solve the Fredholm integral equation by using the direct computation method

$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}x\int_0^{\frac{\pi}{3}} u(t) dt.$$

The integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation can be rewritten as

$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}\alpha x, \quad where \quad \alpha = \int_0^{\frac{\pi}{3}} u(t) dt.$$

To determine α, we substitute to obtain

$$\alpha = \frac{1}{3}\int_0^{\frac{\pi}{3}} (\frac{1}{3}t + \sec t \tan t - \frac{1}{3}\alpha t) dt.$$

Integrating the right side yields

$$\alpha = 1 + \frac{1}{54}\pi^2 - \frac{1}{54}\alpha\pi^2,$$

that gives $\alpha = 1$. Substituting into gives the exact solution

$$u(x) = \sec x \tan x.$$

Example 3.21 Solve the Fredholm integral equation by using the direct computation method

$$u(x) = 11x + 10x^2 + x^3 - \int_0^1 (30xt^2 + 20x^2t)u(t)dt.$$

The kernel $K(x,t) = 30xt^2 + 20x^2t$ is separable. Consequently, we rewrite as

$$u(x) = 11x + 10x^2 + x^3 - 30x\int_0^1 t^2u(t)dt - 20x^2\int_0^1 tu(t)dt.$$

Consequently, Equation can be rewritten as

$$u(x) = (11 - 30\alpha)x + (10 - 20\beta)x^2 + x^3,$$

where $\alpha = \int_0^1 t^2u(t)dt$, $\beta = \int_0^1 tu(t)dt$.

To determine the constants α and β, we substitute to obtain

$$\alpha = \int_0^1 t^2\big((11 - 30\alpha)t + (10 - 20\beta)t^2 + t^3\big)dt = \frac{59}{12} - \frac{15}{2}\alpha - 4\beta,$$

$$\beta = \int_0^1 t\big((11 - 30\alpha)t + (10 - 20\beta)t^2 + t^3\big)dt = \frac{191}{30} - 10\alpha - 5\beta.$$

Solving this system of algebraic equations gives $\alpha = \frac{11}{30}, \beta = \frac{9}{20}$.

Substituting into gives the exact solution $u(x) = x^2 + x^3$.

Example 3.22 Solve the Fredholm integral equation by using the direct computation method

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 (1 + 30xt^2 + 12x^2t)u(t)dt.$$

The kernel $K(x,t) = 1 + 30xt^2 + 12x^2t$ is separable. Consequently, we rewrite as

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 u(t)dt - 30x\int_0^1 t^2u(t)dt - 12x^2\int_0^1 tu(t)dt.$$

Each integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, Equation can be rewritten as

$$u(x) = (4 - \alpha) + (45 - 30\beta)x + (26 - 12\gamma)x^2,$$

where