(c) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{du/u}{\sqrt{u^2-a^2}}$

$\displaystyle = \int \frac{du}{u\sqrt{u^2-a^2}}$

Then,

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1} = \frac{1}{2}\int \frac{du}{u^2+a^2}$$

$$= \frac{1}{2}\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\tan^{-1}(2x+1) + C$$

Example :

$\displaystyle\int \frac{dx}{e^x+e^{-x}}$

$\displaystyle = \int \frac{du}{u^2+1}$ Let $u = e^x$, $u^2 = e^{2x}$, $du = e^x\,dx$.

$= \tan^{-1}u + C$ Integrate with respect to u.

$= \tan^{-1}(e^x) + C$ Replace u by e^x.

Example

(a) $\displaystyle\int \frac{dx}{\sqrt{4x-x^2}}$ **(b)** $\displaystyle\int \frac{dx}{4x^2+4x+2}$

Solution

(a) we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2-4x) = -(x^2-4x+4)+4 = 4-(x-2)^2.$$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

(b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

$$= 4\left(x + \frac{1}{2}\right)^2 + 1 = (2x + 1)^2 + 1.$$

.4 HYPERBOLIC FUNCTIONS

The hyperbolic sine and hyperbolic cosine functions are defined by:

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$= \sinh 2x.$$

Derivatives and Integrals of Hyperbolic Functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \text{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\text{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\text{sech}\, u) = -\text{sech}\, u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\text{csch}\, u) = -\text{csch}\, u \coth u \frac{du}{dx}$$

proof :

1- $$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right)$$

$$= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2}$$