Chapter Five

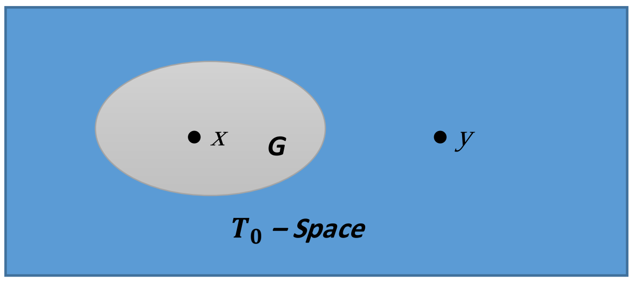
Separation Axioms

5.1 T_0 - Space

5.1.1 Definition:

A topological space (X,τ) is called $\boldsymbol{T_0}$ **– *Space*** iff it satisfies the following axiom of Kolomogorov:

[T_0] If x and y are two distinct points of X, then there exists an open set which contains one of them but not the other, $\forall x , y \in X, x \neq y$, $\exists G \in \tau$, s. t. $x \in G , y \notin G$.



5.1.2 Example:

Let $X = \{a,b\}$, $\tau = \{\{X,\emptyset,\{a\}\}$ then (X,τ) is T_0 – Space ,since a , b ∈ X , a ≠ b , $\exists\{a\} \in \tau$, s. t. $x \in \{a\}$, $y \notin \{a\}$.

5.1.3 Example:

Let $X = \{a,b,c\}$, $\tau = \{\{X,\emptyset,\{a,b\}\}$ then (X,τ) is not T_0 – Space ,since a , b ∈ X , a ≠ b , every open set contain a contain b.

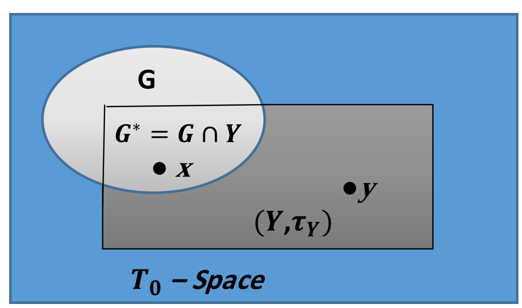
5.1.4 Theorem:

T_0 – Space is a hereditary property.

Proof:

Let (Y,τ_Y) be a subspace of a T_0 – Space(X,τ). We want to prove that (Y,τ_Y) is T_0 – Space.

Let $x, y \in Y, x \neq y$. Since $Y \subset X$ then $x, y \in X$ but X is T_0 – Space then $\exists G \in \tau$, s.t. $x \in G, y \notin G$. Let $G^* = G \cap Y$ then $x \in G^*$(since $x \in G$, $x \in Y$) But $y \notin G^*$(since $y \notin G$, $y \in Y$) , so (Y,τ_Y) is T_0 – Space. □



Exercise:

Prove that T_0 – Space is a topological property.

5.1.5 Theorem:

A topological space (X,τ) is called T_0 – Space iff the closures of distinct points are distinct.

Proof:

$\Longrightarrow$

Suppose that $x \neq y$ implies that $\overline{\{x\}} \neq \overline{\{y\}}$ and that x and y are distinct points of X. Since the sets $\overline{\{x\}}$ and $\overline{\{y\}}$ are not equal, there must exist some point $z \in X$ which is contained in one of them but not the other.

Suppose that $z \in \overline{\{x\}}$ but $z \notin \overline{\{y\}}$. If we had $x \in \overline{\{y\}}$, then we would have $\overline{\{x\}} \subseteq \overline{\overline{\{y\}}} = \overline{\{y\}}$ and so $z \in \overline{\{x\}} \subseteq \overline{\{y\}}$, which is a contradiction. Hence $x \notin \overline{\{y\}}$ and so $\overline{\{y\}}^c$ is an open set containing x but not y.

$\Longleftarrow$

Let us suppose that X is a T_0 – Space, and that x and y are two distinct points of X. By $[T_0]$, there exists an open set G containing one of them but not the other.

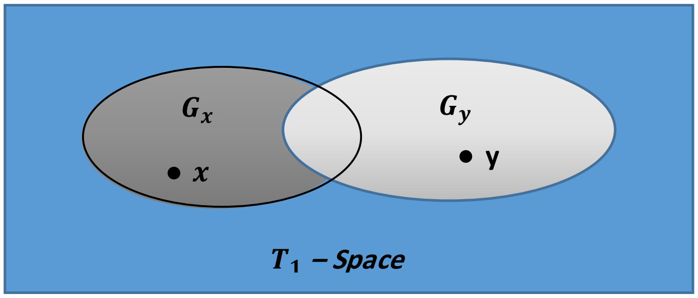
Suppose that $x \in G$ but $y \notin G$. Clearly, G^c is a closed set containing y but not x. From the definition of $\overline{\{y\}}$ as the intersection of all closed sets containing $\{y\}$ we see that $y \in \overline{\{y\}}$, but $x \notin \overline{\{y\}}$ because of G^c. Hence, $\overline{\{x\}} \neq \overline{\{y\}}$. □

5.2 T_1 - Space

5.2.1 Definition:

A topological space (X,τ) is called $\boldsymbol{T_1}$ **– *Space*** iff it satisfies the following axiom of Fréchet:

[T_1] If x and y are two distinct points of X, then there exists two open sets one containing x not y, and the other containing y but not x**,** i.e. $\forall x\ ,\ y \in X,\ x \neq y$, $\exists G_x\ ,\ G_y \in$ τ, s. t. $x \in G_x\ ,\ y \notin G_x$ and y$\in G_y\ ,\ x \notin G_y$.



5.2.2 Example:

Let $X = \{$a,b$\}$, $\tau = \{\{X, \emptyset\ , \{a\}\ , \{b\}\}$ then (X,τ) is T_1 – Space ,since a , b ∈ X , a ≠ b , $\exists\{a\}\ , \{b\} \in$ τ, s. t. a$\in\{a\}$,b $\notin \{a\}$ and b$\in \{b\}$, a $\notin \{b\}$.

5.2.3 Remark:

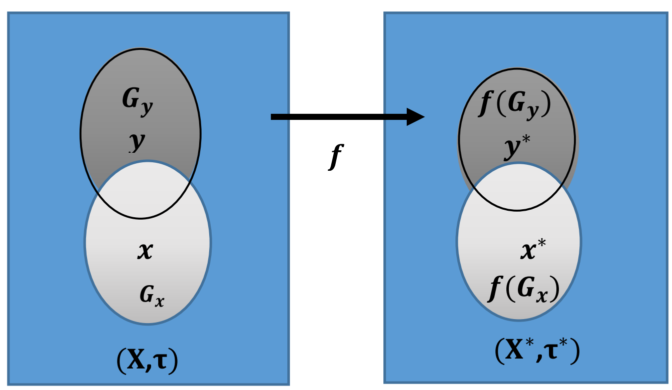
Every T_1 – Space is obviously a T_0 – Space, the converse is not true as the following example:

5.2.4 Example:

Let $X = \{$a,b$\}$, $\tau = \{\{X, \emptyset, \{a\}\}$ then (X,τ) is T_0 – Space not T_1 – Space, since X is the only open set contain a and b.

5.2.5 Theorem:

T_1 – Space is a topological property.



Proof:

Let f: (X,τ) ⟶ (X^*,τ^*) be A homeomorphism from a T_1 – Space (X,τ) to the topological space (X^*,τ^*),we want to show that (X^*,τ^*) is T_1 – Space .

Let x^*,$y^* \in X^*$, $x^* \neq y^*$.Since f is onto then ∃x,y∈X, s. t. $f(x) = x^*$,$f(y) = y^*$. Since f is 1-1 and $x^* \neq y^*$then $x \neq y$. Since (X,τ) is T_1 – Space then $\exists G_x\ ,\ G_y \in$ τ, s. t. $x \in G_x$, y $\notin G_x$ and y$\in G_y\ ,\ x \notin G_y$, so $x^* \in f(G_x)$,y$^* \notin f(G_x)$ and $y^* \in f(G_y)$,$x^* \notin f(G_y)$. Since f is open function then $f(G_x)$,$f(G_y) \in \tau^*$, $x^* \in f(G_x)$,$y^* \in f(G_y)$. So (X^*,τ^*) is T_1 – Space. □

Exercise:

Prove that T_1 – Space is a hereditary property.

5.2.6 Theorem:

A topological space (X,τ) is called T_1 – Space iff every singleton is closed.

Proof:

$\Longrightarrow$

If x and y are distinct points of a space X in which subsets consisting of exactly one point are closed, then $\{x\}^c$ is an open set containing y but not x, while $\{y\}^c$ is an open set containing x but not y. Thus (X,τ) is a T_1 – Space.

$\Longleftarrow$

Suppose that (X,τ) is a T_1 – Space, and that x is a point of X. By **[T_1]** if $y \neq x$, there exists an open set G_y containing y but not x, that is, $y \in G_y \subseteq \{x\}^c$. But then $\{x\}^c = \bigcup\{G_y : y \neq x\}$ and so $\{x\}^c$ is the union of open sets, and hence is itself open. Thus $\{x\}$ is a closed set for every $x \in X$.□

5.2.7 Example:

Let $X = \mathbb{N}$ the set of positive integers, and let τ be the family consisting of $\emptyset$, X and all subsets of the form $\{1 , 2 , ... , n\}$ then $(\mathbb{N},\tau)$ is not a T_1 – Space, since $\forall n \in \mathbb{N}, \{n\}$ is not a closed set (Note that $(\mathbb{N},\tau)$ is a T_0 – Space).

5.2.8 Example:

Let $X = \mathbb{R}$ the set of real numbers, and let τ be the family consisting of $\emptyset$ and all subsets of $\mathbb{R}$ whose complement is finite then $(\mathbb{R},\tau)$ is a T_1 – Space, since $\forall p \in \mathbb{R}, \{p\}$ is a closed set.

5.2.9 Theorem:

In a T_1 – Space (X,τ), a point x is a limit point of a set E iff every open set containing x contains an infinite number of distinct points of E.

Proof:

$\Longrightarrow$

The sufficiency of the condition is obvious, since if G is an open set containing x and $G \cap E$ contains an infinite number of distinct points of E, i.e. $G \cap E/\{x\} \neq \emptyset$. So that $x \in d(E)$.

$\Longleftarrow$

To prove the necessity, suppose there were an open set G containing x for which $G \cap E$ was finite. If we let $G \cap E/\{x\} = \bigcup_{i=1}^{n}\{x_i\}$, then each set $\{x_i\}$ would be closed by the above theorem, and the finite union $\bigcup_{i=1}^{n}\{x_i\}$ would also be a closed set. But then $(\bigcup_{i=1}^{n}\{x_i\})^c \cap G$ would be an open set containing x with $((\bigcup_{i=1}^{n}\{x_i\})^c \cap G) \cap E/\{x\} = ((\bigcup_{i=1}^{n}\{x_i\})^c \cap \bigcup_{i=1}^{n}\{x_i\}) = \emptyset$.Thus x would not be a limit point of E.□

5.2.10 Corollary:

The finite subset of T_1 – Space (X,τ) has no limit point.

Proof:

Suppose A be a finite subset of X. If A has a limit point $x \in X$ (i.e. $x \in d(E)$) then by theorem 5.2.9 every open set G containing x contains infinite number of A but A is finite set and this contradiction, so A has no limit points. □

5.2.11 Remark:

Countably compact spaces are more useful in T_1 – Spaces, since we may then characterize them in a way that is exactly analogous to that for compact spaces. The following theorem, in fact, explains why we chose the name "countably compact."

5.2.12 Theorem:

A T_1 – Space (X,τ) is countably compact iff every countable open covering of X is reducible to a finite subcover.

Proof:

$\Longrightarrow$

Suppose $\{G_n\}_{n\in\mathbb{N}}$ is a countable open covering of the countably compact space X which has no finite subcover. This means that $\bigcup_{i=1}^{n} G_i$ does not contain X for any $n \in \mathbb{N}$. If we let $F_n = (\bigcup_{i=1}^{n} G_i)^c$, then each F_n is a nonempty closed set contained in the preceding one. From each F_n let us choose a point x_n , and let $E=\bigcup_{n\in\mathbb{N}}\{x_n\}$. The set E cannot be finite because there would then be some point in an infinite number, and hence all of the sets F_n , and this would contradict the fact that the family $\{G_n\}_{n\in\mathbb{N}}$ is a covering of X. Since E must be infinite, we may use the countable compactness of X to obtain a limit point x of E.

By theorem 5.2.9, every open set containing x contains an infinite number of points of E. and so x must be a limit point of each of the sets $E_n = \bigcup_{i>n}\{x_i\}$. For each n, however, E_n is contained in the closed set F_n , and so x must belong to F_n for every $n \in \mathbb{N}$. This again contradicts the fact that the family $\{G_n\}_{n\in\mathbb{N}}$ is a covering of $\boldsymbol{X}$. Hence the condition is necessary.

$\Longleftarrow$

Now let us suppose that E is an infinite subset of X and that E has no limit points. Since E is infinite, we may choose an infinite sequence of distinct points x_n from E. The set $A=\bigcup_{n\in\mathbb{N}}\{x_n\}$ has no limit points since it is a subset of E, and so, in particular, each point x_n is not a limit point of A**.** This means that for every $n \in \mathbb{N}$ there exists an open set G_n containing x_n such that $A \cap G_n/\{x_n\} = \emptyset$ **.** From the definition of A we see that $A \cap G_n = \{x_n\}$ for every $n \in \mathbb{N}$**.** Since A has no limit points, it is a closed set, and hence A^c is open. The collection $A^c \cup \{G_n\}_{n\in\mathbb{N}}$ is then a countable open covering of X which has no finite subcover, since the set G_n is needed to cover the point x_n for every $n \in \mathbb{N}$. Thus, the condition is sufficient. □

5.2.13 Corollary:

A T_1 – Space (X,τ) is countably compact iff every countable family of closed sets having the finite intersection property has a nonempty intersection.

5.2.14 Example:

Every finite T_1 – *Space* has the discrete topology.

Solution:

Let (X,τ) be a finite T_1 – Space, so every subset of X is finite, i.e. equal a union of finite numbers of singleton and therefore closed. Hence every subset of X is also open, i.e. X is a discrete topology.

5.2.15 Remark:

Although countable compactness is a topological property, we noted from remark 4.1.32 that it may not be preserved by continuous mappings. With the aid of one-to-oneness, we may show that it is preserved by continuous mappings of T_1 – Spaces .

5.2.16 Theorem:

If *f is a continuous mapping of the T_1 – Space (X,τ) into the topological space (X^*,τ^*), then f maps every countably compact subset of X onto a countably compact subset of X^*.*

Proof:

Suppose E is a countably compact subset of X and $\{G_n^*\}_{n\in\mathbb{N}}$ is a countable open covering of $f(E)$. We need only show that there is a finite subcovering of $f(E)$, since we noted above that the condition of theorem 5.2.12 is always sufficient. Since f is continuous, $\{f^{-1}(G_n^*)\}_{n\in\mathbb{N}}$ is a countable open covering of E**.** In the induced topology, $\{E \cap f^{-1}(G_n^*)\}_{n\in\mathbb{N}}$ is a countable open covering of the countably compact T_1 – Space ***E*****.** By theorem 5.2.12, there exists some finite subcovering

$\left\{E \cap f^{-1}(G_{n_i}^*)\right\}_{i=1}^{k}$, and clearly the family $\left\{G_{n_i}^*\right\}_{i=1}^{k}$ is the desired finite subcovering of $f(E)$. □

5.2.17 Example:

Let (X,τ) be a T_1 – Space and let $\mathcal{B}_p$ be a local base at $p \in X$. Show that if $q \in X$ distinct from p then some member of $\mathcal{B}$ does not contain q.

Solution:

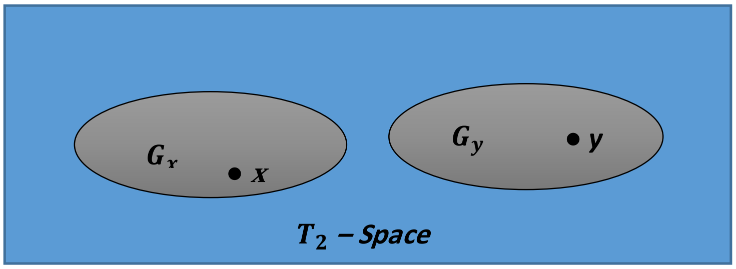
Since $p \neq q$ and X satisfies $[T_1]$,∃ an open set $G \subset X$ consisting p but not q. Now $\mathcal{B}_p$ is a local base at p, so G is contain of some $B \in \mathcal{B}_p$ and B also does not contain q.

5.3 T_2 - Space

5.3.1 Definition:

A topological space (X,τ) is called ***T_2* – *Space* or Hausdorff space** iff it satisfies the following axiom of Hausdorff:

[T_2] If x and y are two distinct points of X, then there exists two disjoint open sets one containing x and the other containing y . $\forall x\ ,y \in X, x \neq y\ ,\ \exists G_x\ ,G_y \in \tau$, s. t. $x \in G_x$ and $y \in G_y\ ,G_x \cap G_y = \emptyset$.



5.3.2 Example:

Let $X = \{a,b\}$, $\tau = \{\{X,\emptyset,\{a\},\{b\}\}$ then (X,τ) is T_2 – Space, a,b∈X,a $\neq b$,∃{a},{b}∈τ and {a}∩{b}=∅, s. t. a∈{a} , b∈{b}.

5.3.3 Remark:

From definition of T_2 – Space we get

$$T_2 - \text{Space} \ \underset{\not\Leftarrow}{\Rightarrow}\ T_1 - \text{Space} \ \underset{\not\Leftarrow}{\Rightarrow}\ T_0 - \text{Space}$$

5.3.4 Example:

Let (X,τ) be the co-finite topology then (X,τ) is T_1 – Space not T_2 – Space.

Solution:

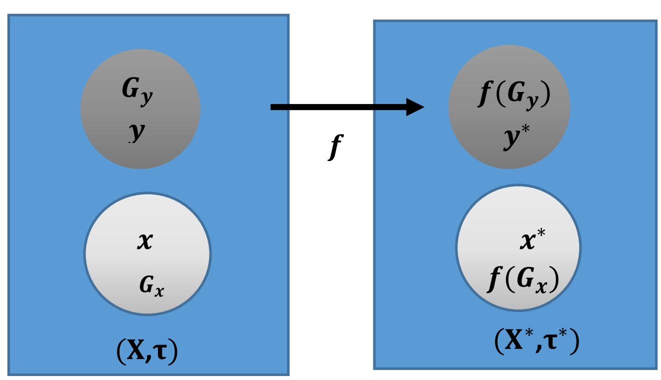
If $G,H \in \tau$ then $G^c,H^c$are finite sets. If $H \cap G = \emptyset$ then $G \subseteq H^c$and this is contradiction ,since $H^c$is finite set and G is infinite set. Then $H \cap G \neq \emptyset$.So (X,τ) is not T_2 – Space.

5.3.5 Theorem:

***T_2* – Space is a topological property.**

Proof:

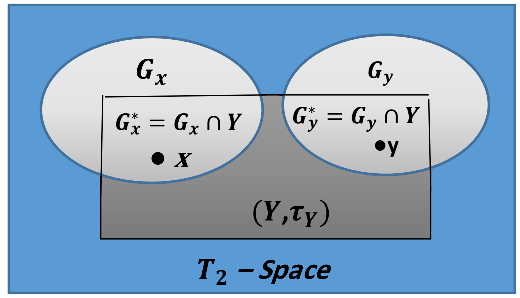
Let f: (X,τ) ⟶ (X*,τ*) be A homeomorphism from a T_2 – Space (X,τ) to the topological space (X*,τ*),we want to show that (X*,τ*) is T_2 – Space .



Let $x^*,y^* \in X^*$, $x^* \neq y^*$.Since f is onto then $\exists x$,y∈X, s.t. $f(x) = x^*,f(y) = y^*$. Since f is 1-1 and $x^* \neq y^*$then $x \neq y$. Since (X,τ) is T_2 – Space then $\exists G_x$, G_y∈ τ,$G_x \cap G_y = \emptyset$, s. t. x∈ G_x , y∈ G_y . Since f is open function then $f(G_x)$,$f(G_y) \in \tau^*$. Since f is 1-1 and $G_x \cap G_y = \emptyset$ then $f(G_x) \cap f(G_y) = \emptyset$.Since x∈ G_x , y∈ G_y then $x^* \in f(G_x)$,$y^* \in f(G_y)$. So (X*,τ*) is T_2 – Space.□

5.3.6 Theorem:

T_2 – Space is a hereditary property.



Proof:

Let (Y,τ_Y) be a subspace of a T_2 – Space(X,τ). We want to prove that (Y,τ_Y) is T_2 – Space.

Let $x, y \in Y$, $x \neq y$.Since $Y \subset X$ then $x, y \in X$ but X is T_2 – Space then $\exists$ G_x , G_y∈ τ, $G_x \cap G_y = \emptyset$, s. t. x∈ G_x , y∈ G_y.By definition of subspace let $G_x^* = G_x \cap Y$, $G_y^* = G_y \cap Y$ are τ_Y – open sets. Furthermore $x \in G_x^*$(since $x \in G_x$, $x \in Y$), $y \notin G_y^*$(since y $\notin G_y$, y $\in Y$) and and $G_x \cap G_y = \emptyset$ then$(G_x \cap Y) \cap (G_y \cap Y) = (G_x \cap G_y) \cap Y = \emptyset \cap Y = \emptyset$.So (Y,τ_Y) is T_2 – Space. □

5.3.7 Remark:

Compact sets are more useful in T_2 – Spaces since we may prove a part of the Heine-Borel Theorem which does not hold in general topological spaces.

5.3.8 Theorem:

Every compact subset E of a Hausdorff space X is closed.

Proof:

Let x be a fixed point in E^c. By $[T_2]$, for each point $y \in E$,there exist two disjoint open sets G_x and G_y such that $x \in G_x$ and y∈ G_y.The family of sets $\{G_y : y \in E\}$ is an open covering of E. Since E is compact, there must be some finite subcovering $\{G_{y_i}\}_{i=1}^{n}$. Let $\{G_{y_i}\}_{i=1}^{n}$be the corresponding open sets containing x, and let $G = \bigcap_{i=1}^{n} G_{x_i}$.Then G is an open set containing x since it is the intersection of a finite number of open sets containing x. Furthermore, we see that $G = \bigcap_{i=1}^{n} G_{x_i} \subseteq \bigcap_{i=1}^{n} {G_{y_i}}^c = \left(\bigcup_{i=1}^{n} G_{y_i}\right)^c \subseteq E^c$. Thus each point in E^c is contained in an open set which is itself contained in E^c.Hence E^c is an open set, and so E must be closed.□

5.3.9 Corollary:

If f is a one-to-one continuous mapping of the compact topological space (X,τ) onto the T_2 – Space (X^*,τ^*), then f is also open, and so f is a homeomorphism.

Proof:

Let G be open in X, so that G^c is closed. By theorem 3.5.6, $G^c$is compact. By theorem 4.1.23 $f(G^c)$ is compact. By theorem 5.3.8, $f(G^c)$ is closed. Thus $\left(f(G^c)\right)^c$is open. Since f is one-to-one and onto, $\left(f(G^c)\right)^c = f(G)$ which is open.□

5.3.10 Theorem:

Every metric space is T_2 – Space (Hausdorff space).

Proof:

Let $a,b \in X$ be distinct points $d(a,b) = \varepsilon > 0$. Consider the open spheres $G = B_{\frac{1}{3}\varepsilon}(a)$ and $H = B_{\frac{1}{3}\varepsilon}(b)$ centered at a and b respectively.

We claim that $G \cap H = \emptyset$ if not then $\exists x \in G \cap H$ s.t. $d(a,x) = \frac{1}{3}\varepsilon$ and $d(x,b) = \frac{1}{3}\varepsilon$ hence by Triangle Inequality, $d(a,b) \le d(a,x) + d(x,b) < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon$ but this is contradicts the fact that $d(a,b) = \varepsilon$. Hence G and H are disjoint, i.e. a and b belong respectively to the disjoint open spheres G and H. So X is Hausdorff space. □

5.3.11 Remark:

The following theorem shows in T_2 – Space we can separate a point from compact set by using open sets.

5.3.12 Theorem:

In T_2 – Space we can separate any point and compact subset not contain the point by disjoint open sets.

Proof:

Let (X,τ) be a T_2 – Space ,F compact subset of X ,$x \in X$and $x \notin F$.Let $y \in F$ then $y \neq x$. Since (X,τ) is T_2 – Space then $\exists G_x$, $H_y \in$ τ, s. t. $x \in G_x$ and y$\in H_y$,$G_x \cap H_y = \emptyset$.

The family $\{H_y : y \in F\}$ is an open cover for F. Since F is compact then there exist $\left\{H_{y_i}\right\}_{i=1}^{n}$finite subcover for F corresponding $\{G_i\}_{i=1}^{n}$family of finite open sets contain x.Let $H = \bigcup_{i=1}^{n} H_{y_i}$,$G = \bigcap_{i=1}^{n} G_i$, i.e. $x \in G,F \subseteq H$ and $G \cap H = \emptyset$.□

5.3.13 Remark:

Since the notion of a convergent sequence of real numbers plays such a basic role in the study of the real number system, we might expect that the equivalent notion for topological spaces would be as primitive a concept as the closure. Although convergence has been used as the primitive notion for abstract spaces, we will see below that some of the natural properties fail to hold in more general spaces than Hausdorff spaces.

5.3.14 Definition:

Let (X,τ) be a topological space and let $\langle x_n \rangle$ be a sequence in X. We say that $\langle x_n \rangle$ converge in X if $\exists x \in X$ (denote by $x_n \to x$) such that

for every open set G contain x , $\exists k \in \mathbb{N}$, s.t. $x_n \in G, \forall n > k$.

5.3.15 Example:

Let $\langle a_1, a_2, \dots \rangle$ be a sequence of points in an indiscrete topological space (X,τ). Since X is only open set containing any point $b \in X$ and X contains every term of the sequence $\langle a_n \rangle$, so the sequence $\langle a_1, a_2, \dots \rangle$ converge to every point of $b \in X$.

5.3.16 Example:

Let $\langle a_1, a_2, \dots \rangle$ be a sequence of points in a discrete topological space (X,τ).Since $\forall b \in X$ the singleton set $\{b\}$ is an open set contain b , so if $a_n \to b$ then the set $\{b\}$ must contain almost all of the terms of the sequence. In other words the sequence $\langle a_n \rangle$ converges to a point $b \in X$ iff the sequence is of the form $\langle a_1, a_2, \dots, a_{n_0}, b, b, b, \dots \rangle$.

5.3.17 Example:

Let τ be the topology on an infinite set X which consists of $\emptyset$ and the complements of countable sets . A sequence $\langle a_1, a_2, \dots \rangle$ in X converges to $b \in X$ iff the sequence is also of the form $\langle a_1, a_2, \dots, a_{n_0}, b, b, b, \dots \rangle$, i.e. the set A consisting of the terms of $\langle a_n \rangle$ different from b is finite .Now A is countable and so A^c is an open set containing b. Hence if $a_n \to b$ then A^c contains all except a finite number of the terms of the sequence ,so A is finite

5.3.18 Remark:

It is the failure of limits of sequences to be unique that makes this concept unsatisfactory in general topological spaces. The following example shows that a T_0 – Space in which limits of sequences need not be unique.

5.3.19 Example:

Let $X = \mathbb{N}$, and let τ be the family consisting of $\emptyset$, X, and all subsets of the form $\{n, n+1, n+2, \dots\}$ then $(\mathbb{N}, \tau)$ is $T_0 - Space$ not $T_2 - Space$,(since if $n_1, n_2 \in \mathbb{N}$. $n_1 \neq n_2$ with $n_2 < n_1$then there exists $\{n_1, n_1+1, \dots\}$ contain $n_1$not n_2 if $n_1 < n_2$ then there exists $\{n_2, n_2+1, \dots\}$ contain $n_2$not n_1) but the sequence $< a_n = n >$ for which converges to every point of that space, i.e. $< n >$ converge to ,2,3,… .

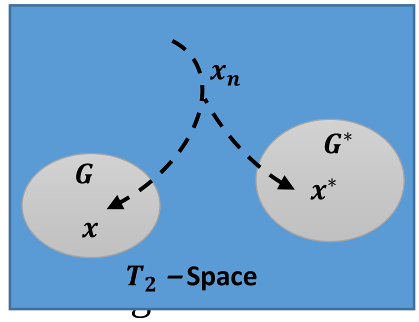
5.3.20 Remark:

The following theorem shows that this anomalous behavior cannot occur in a Hausdorff space.

5.3.21 Theorem:

In a Hausdorff space, a convergent sequence has a unique limit.

Proof:



Suppose a sequence $\langle x_n \rangle$ converged to two distinct points x and x^* in a Hausdorff space X. By $[T_2]$, there exist two disjoint open sets G and G^* such that $x \in G$ and $x^* \in G^*$. Since $x_n \to x$, there exists an integer k such that $x_n \in G$ whenever $n > k$. Since $x_n \to x^*$ there exists $x_n \in G^*$ whenever $n > k^*$. If m is any integer greater than both k and k^* , then x_m must be in both G and G^*,which contradicts the fact that G and $G^*$are disjoint.□

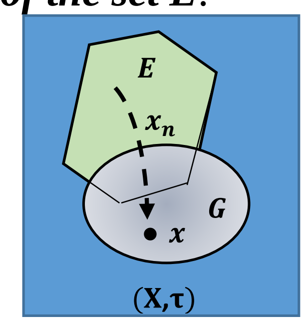
5.3.22 Remark:

1. The converse of theorem 5.3.21 is not true. An example of a non-Hausdorff space in which every convergent sequence has not unique limit was given in example 5.3.19 .
2. A relationship between the limit points of sets and the limit points of sequences of points is given in the following theorem.

5.3.23 Theorem:

If $\langle x_n \rangle$ is a sequence of distinct points of a subset E of a topological space (X,τ) which converges to a point $x \in X$ then x is a limit point of the set E.

Proof:



If x belongs to an open set G,then there exists an integer k such that $x_n \in G$ for all $n > k$. Since the points x_n are distinct, at most one of them equals x and so $E \cap G/\{x\} \neq \emptyset$.□

5.3.24 Remark:

The converse of theorem 5.3.23 is not true, even in a Hausdorff space .as the following example

5.3.25 Example:

Let $X = \{a,b,c\}$,$\tau=\{\emptyset,\{a,b\},\{c\},X\}$. Let x_1=a.x_2=b,x_n=c, ∀n≥3, i.e. $\langle x_n \rangle = \langle a,b,c,c,\ldots \rangle$. It's clear $x_n \to c$ but $c \notin d(\{a,b,c\})$ since $c \in \{c\} \in \tau$, $\{a,b,c\} \cap \{c\}/\{c\} = \emptyset$. Also $a,b \in d(\{a,b,c\})$ but $x_n \nrightarrow a$ and $x_n \nrightarrow b$, since $a,b \in \{a,b,c\}$ and $x_n \notin \{a,b\}, \forall n \geq 3$.

5.3.26 Remark:

A relationship between continuity of functions and convergent sequences of points is given in the following theorem.

5.3.27 Theorem:

If f is a continuous mapping of the topological space (X,τ) into the topological space (X^*,τ^*) and $\langle x_n \rangle$ is a sequence of points of X which converges to the point $x \in X$ then the sequence $\langle f(x_n)\rangle$converges to the point $f(x) \in X^*$.

Proof:

If $f(x)$ belongs to the open set G^* in X^*,then $f^{-1}(G^*)$ is an open set in X containing x since f is continuous. There must then exist an integer k such that $x_n \in f^{-1}(G^*)$ whenever $n > k$. Thus we have $f(x_n) \in G^*$whenever $n > k$, and so $f(x_n) \to f(x)$.□

5.3.28 Remark:

The converse of theorem is also not true, even in a Hausdorff space. That is, a mapping f for which $x_n \to x$ implies $f(x_n) \to f(x)$ may not be continuous as the following example:

5.3.29 Example:

Let $\mathbb{R}$ be the set of real numbers and $\tau = \{\emptyset\} \cup \{G \subseteq X: G^c$ is countable$\}$.Let $X^* = [0,1]$,$\tau^* = \{G \cap [0,1]: G \in \tau\}$ be the relative topology and let $f: (\mathbb{R},\tau) \longrightarrow (X^*,\tau^*)$ be a function defined by

$$f(x) = \begin{cases} x & x\in[0,1] \\ 0 & x\notin[0,1] \end{cases}.$$

Then f is not continuous since $(0,1) \in \tau^*$but $f^{-1}\big((0,1)\big)$=$(0,1) \notin \tau$,where $\mathbb{R}/(0,1)$ is not countable. If $x_n \to x$ in X and iff $x_n = x$, $\forall n \in k$, k is positive integers iff $f(x_n) = f(x)$, $\forall n \in k$ iff $f(x_n) \to f(x)$.

5.3.30 Remark:

The failure of the converses of the preceding three theorems 5.3.21,5.3.23 and 5.3.27 to hold shows that the notion of limit for sequences of points is not completely satisfactory, even if the space satisfies the axiom $[T_2]$.The Axioms of Countability we will introduce another axiom for the open sets of a topological space with which we may prove these converses.

5.4 Axioms of Countability

5.4.1 Definition:

A topological space (X,τ) is a ***first axiom space*** iff it satisfies the following ***first axiom of countability***:

$[\boldsymbol{C_I}]$ For every point $x \in X$, there exists a countable family $\{B_n(x)\}$ of open sets containing x such that whenever x belongs to an open set G, $B_n(x) \subseteq G$for some n.

5.4.2 Example:

Let (X,d) be a metric space and $p \in X$then the countable class of open balls $\{B_1(p), B_{\frac{1}{2}}(p), \dots\}$ with center p is a local base at p.Hence every metric space satisfies the first axiom of countability.

5.4.3 Example:

Let $(\mathbb{R},\tau)$ be the usual topology and $p \in \mathbb{R}$ then the countable class of open sets $\left\{B_n(p) = \left(p - \frac{1}{n}, p + \frac{1}{n}\right): n \in \mathbb{N}\right\}$ is a local base at p.Hence the usual topology satisfies the first axiom of countability.

5.4.4 Example:

Let (X,τ) be any discrete topology. The singleton set $\{p\}$ is open and is contained in every open set G containing $p \in X$.Hence every discrete space satisfies $[\boldsymbol{C_I}]$.

5.4.5 Example:

Let $(\mathbb{R},\tau)$ be the co-finite topology dose not satisfy the first axiom of countability.

Solution:

Suppose that $(\mathbb{R},\tau)$ satisfy $[C_I]$ then $1 \in \mathbb{R}$ possesses a countable open local base $\mathcal{B}_1 = \{B_n: n \in \mathbb{N}\}$.Since each B_n is open then B_n^c is closed and hence is finite , the set $A = \cup \{B_n^c: n \in \mathbb{N}\}$ is the countable union of finite sets and is therefore countable. But $\mathbb{R}$ is not countable then there exists a point $p \in \mathbb{R}$ different from 1 which does not belong to A ,i.e. $p \in A^c = (\cup \{B_n^c: n \in \mathbb{N}\})^c = \cap \{B_n^{cc}: n \in \mathbb{N}\} = \cap \{B_n: n \in \mathbb{N}\}$, hence $p \in B_n, \forall n \in \mathbb{N}$.On the other hand $\{p\}^c$ is open set since it is the complement of a finite set, and $\{p\}^c$contains 1 since p is different from 1. Since $\mathcal{B}_1$is a local base there exists a member $B_{n_0} \in \mathcal{B}_1$ such that $B_{n_0} \subset \{p\}^c$.Hence $p \notin B_{n_0}$.But this is contradicts the statement that $p \in B_n, \forall n \in \mathbb{N}$. So

$(\mathbb{R},\tau)$ does not satisfy the first axiom of countability.

5.4.6 Remark:

If (X,τ) is a topological space satisfy $[C_I]$,i.e. for every $x \in X$ $\exists\{B_n(x)\}$ countable base at x then we arranged the base in decreasing order as following

$$B_1^*(x) = B_1(x)$$
$$B_2^*(x) = B_1^*(x) \cap B_2(x)$$
$$B_3^*(x) = B_2^*(x) \cap B_3(x)$$
$$\vdots$$
$$B_n^*(x) = B_{n-1}^*(x) \cap B_n(x).$$

We get $\{B_n^*(x)\}$ a countable base s.t. $B_n^*(x) = \cap\{\, B_k(x) : k \le n\}$.Also we can arrange the base as increasing order by replace the intersection with union.

Exercise:

Prove that $[C_I]$ is a hereditary property.

5.4.7 Theorem:

$[C_I]$ is a topological property.

Proof:

Let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be A homeomorphism from a topological space (X,τ) which satisfy $[C_I]$ to the topological space (X^*,τ^*),we want to show that (X^*,τ^*) satisfy $[C_I]$.

Let $x^* \in X^*$.Since *f* is onto $\exists x \in X$, s.t. $f(x) = x^*$.Since X satisfy $[C_I]$ then $\exists\{B_n(x)\}$ countable base at x ,so the family $\{f(B_n(x))\}$ is a base since f is open function and countable since *f* is one to one ,so (X^*,τ^*) satisfy $[C_I]$.□

5.4.8 Remark:

In the next three important theorems, we will show the converse of theorems 5.3.21,5.3.23 and 5.3.27 is true in spaces which satisfy the first axiom of countability.

5.4.9 Theorem:

A topological space (X,τ) satisfying the first axiom of countability is a Hausdorff space iff every convergent sequence has a unique limit.

Proof:

$\Longrightarrow$

In theorem 5.3.21 in T_2 –Space every convergent sequence has a unique limit.

$\Longleftarrow$

Assume that every convergent sequence has a unique limit, we want to prove that (X,τ) is T_2 –Space.

If not $\exists x,y \in X. x \neq y$ such that every open set containing x has a nonempty intersection with every open set containing y. Since X satisfy $[C_I]$ then $\exists\{B_n(x)\}$ and $\{B_n(y)\}$ are monotone decreasing countable open bases at x and y respectively with , $B_n(x) \cap B_n(y) \neq \emptyset, \forall n$, so we choose a point $x_n \in B_n(x) \cap B_n(y), \forall n$. If G_x and G_y are arbitrary open sets containing x and y respectively, there must exist some integer k such that $B_n(x) \subseteq G_x$ and $B_n(y) \subseteq G_y$ for all $n > k$ by the definition of a monotone decreasing base. Hence $x_n \to x$ and $x_n \to y$, so that we have a convergent sequence without a unique limit and this is contradiction .so (X,τ) is T_2 –Space.□

5.4.10 Theorem:

If x is a point and E a subset of a T_1 –Space (X,τ) satisfying the first axiom of countability , then x is a limit point of E iff there exists a sequence of distinct points in E converging to x.

Proof:

$\Longrightarrow$

In theorem 5.3.23 we proved the limit point of convergent sequence in E is a limit point of E.

$\Longleftarrow$

Let (X,τ) is T_1 *–Space* and satisfy $[C_I]$ .Let E be a subset of X and $x \in X$ s.t. $x \in d(E)$.Since X satisfy $[C_1]$ then $\exists\{B_n(x)\}$ a monotone decreasing countable open base at x. Since x belongs to the open set $B_n(x)$, the set $B_n(x) \cap E/\{x\}$ must be infinite by theorem 5.2.9. By induction we may choose a point x_n in this set different from each previously chosen x_n with $k < n$. Clearly, $x_n \to x$ since the sets $\{B_n(x)\}$ form a monotone decreasing base at x.□

5.4.11 Theorem:

If f is a mapping of the first axiom space (X,τ) into the topological space (X^*,τ^*) , then f is continuous at $x \in X$ iff for every sequence $\langle x_n \rangle$ of points in X converging to x we have the sequence $\langle f(x_n) \rangle$ converges to the point $f(x) \in X^*$.

Proof:

$\Longrightarrow$

In theorem 5.3.27 we proved if f is continuous and $x_n \to x$ then $f(x_n) \to f(x)$.

$\Longleftarrow$

We want to prove that f is continuous at $x \in X$, if not then $\exists G^* \in \tau^*, f(x) \in G^*$,s.t. $f(G) \nsubseteq G^*$,i.e. $f(G) \cap G^{*c} \neq \emptyset$ for any open set G containing x. Let $\{B_n(x)\}$ be a monotone decreasing countable open base at x (since (X,τ) satisfy $[C_I]$).Then $f(B_n(x)) \cap G^{*c}) \neq \emptyset, \forall n$ and we may pick $x_n^* \in f(B_n(x)) \cap G^{*c}$. Since $x_n^* \in f(B_n(x))$ we may choose a point $x_n \in B_n(x)$ such that $f(x_n) = x_n^*$. We now have $x_n \to x$ since the sets $\{B_n(x)\}$ form a monotone decreasing base at x. The sequence $\langle f(x_n)\rangle = \langle x_n^*\rangle$ cannot converge to $f(x)$, however, since $x_n^* \in G^{*c}, \forall n$. □

5.4.12 Definition:

A topological space(X,τ) is a ***second axiom space*** iff it satisfies the following ***second axiom of countability***:

$[\boldsymbol{C_{II}}]$ There exists a countable base for the topology τ.

5.4.13 Remark:

1. The property $[C_I]$ is local (i.e. there exist a base at each point) but $[C_{II}]$ is global (i.e. there exist a base for every points in a space X).

2. Every topological space satisfy $[C_{II}]$ satisfy $[C_I]$ but the converse is not true as the following examples:

5.4.14 Example:

The discrete topology on any uncountable set, has no countable base (i.e. not satisfy $[C_{II}]$) .Since each set consisting of exactly one point must belong to any base, even though there is a countable open base at each point x obtained by letting $\{B_n(x)\} = \{x\}$, i.e. satisfy $[C_I]$.

5.4.15 Example:

Let (ℝ,τ) be the discrete topology on ℝ .A class $\mathcal{B}$ is a base for a discrete topology iff it contains all singleton $\{p\}$ subset of ℝ, but ℝ is non- countable ,so the discrete topology does not satisfy $[C_{II}]$ but satisfy $[C_I]$.

5.4.15 Example:

The class $\mathcal{B}$ of open intervals (a,b) with rational endpoints ,i.e. $a,b \in \mathbb{Q}$ is countable and is a base for the usual topology on the real line ℝ.Thus (ℝ,τ) satisfies $[C_{II}]$.

Exercise:

Prove that $[C_{II}]$ is a topological property.

5.4.17 Theorem:

$[\boldsymbol{C_{II}}]$ ***is a hereditary property.***

Proof:

Let (Y,τ_Y) be a subspace of a topological space (X,τ) which satisfy $[C_{II}]$. We want to prove that (Y,τ_Y) satisfy$[C_{II}]$.

Since (X,τ) satisfy$[C_{II}]$ then $\exists\{B_n\}$ countable base for X then family $\{B_n^* = B_n \cap Y\}$ is a countable base for Y,so (Y,τ_Y) satisfy$[C_{II}]$.□

5.4.18 Remark:

The relationship between compact and countably compact sets is made clearer by application of the following theorem due to Lindelöf. Indeed, it shows that the two notions are equivalent in second axiom
T_1 – Spaces.

5.4.19 Theorem:

In a second axiom space, every open covering of a subset is reducible to a countable subcovering.

Proof:

Suppose $\mathcal{A}$ *is an open covering of the subset* E of the second axiom space X which has $\mathcal{B}$ as a countable base.

Since $\mathcal{A}$ is an open covering of E then $E = \cup\{G: G \in \mathcal{A}\}$, i.e. $\forall p \in E, \exists G_p \in \mathcal{A}$ such that $p \in G_p$.

Since $\mathcal{B}$ is an open a countable base for X then $\forall p \in E, \exists B_p \in \mathcal{B}$ such that $p \in B_p \subset G_p$.

Hence $E = \cup\{B_p: p \in E\}$. But $\{B_p: p \in E\} \subset \mathcal{B}$,so it is countable ,hence $\{B_p: p \in E\} = \{B_n: n \in N\}$,where N is a countable index set. For each $n \in N$ choose one set $G_n \in \mathcal{A}$ such that $B_n \subset G_n$.Then $E \subset \{B_n: n \in N\} \subset \{G_n: n \in N\}$ and so $\{G_n: n \in N\}$ is a countable subcover of $\mathcal{A}$.□

5.4.20 Theorem:

In a second axiom space, we can find a countable subbase foe every base.

Proof:

Let $\mathcal{A}$ be a base for X. Since (X,τ) satisfy$[C_{II}]$ then X has a countable base $\mathcal{B} = \{B_n: n \in N\}$.Since $\mathcal{A}$ is also a base for X then for each $n \in \mathbb{N}$, $B_n = \cup\{G, G \in \mathcal{A}_n\}$ with $\mathcal{A}_n \subset \mathcal{A}$. So $\mathcal{A}_n$ is an open cover of B_n and by theorem 5.4.19 , $\mathcal{A}_n$ reducible to a countable over $\mathcal{A}_n^*$,i.e. for each $n \in \mathbb{N}$, $B_n = \cup\{G, G \in \mathcal{A}_n^*\}$ with $\mathcal{A}_n^* \subset \mathcal{A}$ and $\mathcal{A}_n^*$ countable. But $\mathcal{A}^* = \{G, G \in \mathcal{A}_n^*, n \in \mathbb{N}\}$ is a base for X since $\mathcal{B}$ is. Furthermore $\mathcal{A}^* \subset \mathcal{A}$, $\mathcal{A}^*$ is countable.□

5.4.21 Definition:

A topological space (X,τ) is called a ***Lindelöf space*** iff every open cover of X is reducible to a countable subcover.

5.4.22 Remark:

1. From definition of Lindelöf we get every compact space is a Lindelöf space (since every finite subcover is countable).

2. Every second countable space is a Lindelöf space.

5.4.23 Theorem:

The Lindelöf space is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a Lindelöf space (X,τ) to the topological space (X*,τ*),we want to prove that (X*,τ*) is a Lindelöf space.

Let $\{G_\lambda^*\}$ be an open cover for X*. Since f is continuous then $\{f^{-1}(G_\lambda^*)\}$ is an open cover for X. Since (X,τ) is a Lindelöf space then there exists a countable subcover $\{f^{-1}(G_n^*)\}_{n\in\mathbb{N}}$ foe X, i.e. $X = \bigcup_{n\in\mathbb{N}} f^{-1}(G_n^*)$, so $X^* = f(X) = f(\bigcup_{n\in\mathbb{N}} f^{-1}(G_n^*)) = \bigcup_{n\in\mathbb{N}} ff^{-1}(G_n^*) = \bigcup_{n\in\mathbb{N}} G_n^*$,(since f is 1-1 and onto).Then (X*,τ*) is a Lindelöf space.□

5.4.24 Remark:

The following example show that the Lindelöf space is not a hereditary property.

5.4.25 Example:

Let $X = \mathbb{R}$ the set of real number and let $\tau = \{G: G \subseteq \mathbb{R}, 0 \notin G \text{ or } R/\{1,2\} \subseteq G\}$ then every open cover for X there exists a finite subcover for X, i.e. X is compact, so X is Lindelöf space. Let $X^* = \mathbb{R}/\{0\}$, τ^* the relative topology on X^*. We have the cover $\{\{r\}: r \in \mathbb{R}/\{0\}\}$ is an open cover for X*but not have a countable subcover for X*,i.e. X*is not a Lindelöf space. So the Lindelöf property is not a hereditary property.

5.4.26 Theorem:

Every topological space satisfy $[C_{II}]$ ***is separable.***

Proof:

Let (X,τ) be a topological space satisfy $[C_{II}]$ then there exists a countable base $\mathcal{B} = \{B_n: n \in \mathbb{N}\}$ for X. Let $x_n \in B_n, \forall n \in \mathbb{N}$ then the set $D = \{x_n: n \in \mathbb{N}\} \subseteq X$ is also countable. We shall prove that D is dense.

Let $x \in D^c$ and let G be an open set contain x then $\exists B_n \in \mathcal{B}$ s.t. $x \in B_n \subseteq G$. Since $D \cap B_n \neq \emptyset$ then $D \cap G/\{x\} \neq \emptyset$, so $x \in d(D)$, i.e. $\overline{D} = X$ so (X,τ) is separable.□

5.4.27 Remark:

1. The converse of theorem 5.4.26 is not true in general , since the lower limit topology on $\mathbb{R}$ is separable topological space which does not satisfy the second axiom of countability.

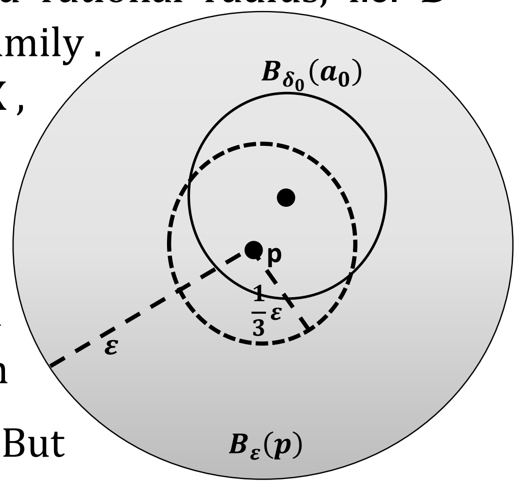
2. In metric space the converse of theorem 5.4.26 is true as the following theorem:

5.4.28 Theorem:

Every seperable metric space is second countable **($[C_{II}]$).**

Proof:

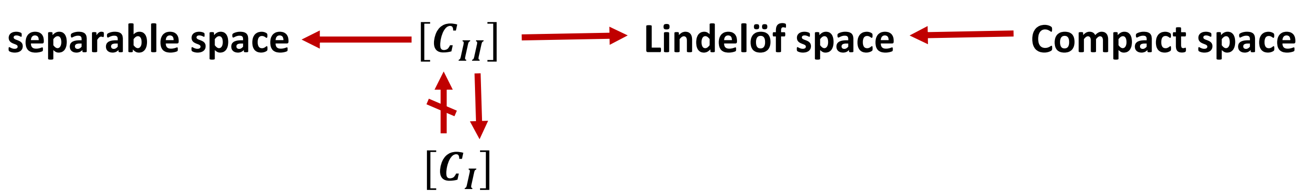
Since X is separable then X contain a countable dense subset A. Let $\mathcal{B}$ be a class of all open balls with centers in A and rational radius, i.e. $\mathcal{B} = \{B_\delta(a) : a \in A, \delta \in \mathbb{Q}\}$. Note that $\mathcal{B}$ is a countable family .



We claim that $\mathcal{B}$ is a base for the topology on X , i.e. for every open set $G \subset X$ and every $p \in G$, $\exists B_\delta(a) \in \mathcal{B}$ s.t. $p \in B_\delta(a) \subset G$. Since $p \in$G there exists an open ball $B_\varepsilon(p)$ with center p such that $p \in B_\varepsilon(p) \subset G$. Since A is dense in X, $\exists a_0 \in A$ such that $d(p, a_0) < \frac{1}{3}\varepsilon$. Let δ_0 be a rational number such that $\frac{1}{3}\varepsilon < \delta_0 < \frac{2}{3}\varepsilon$. Then $p \in B_{\delta_0}(a_0) \subset B_\varepsilon(p) \subset G$.But $B_{\delta_0}(a_0) \in \mathcal{B}$,and so $\mathcal{B}$ is a countable base for the topology on X.□

5.4.29 Remark:

In the following diagram we denote by arrows the implications which hold in any topological space, while no other implications hold, even in a Hausdorff space.

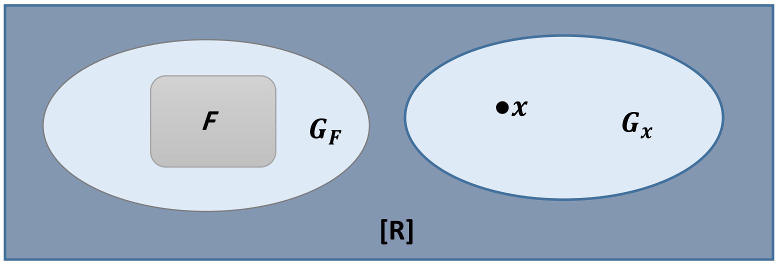


5.5 Regular and Normal Spaces

5.5.1 Definition:

A topological space X is regular iff it satisfies the following axiom of Vietoris:

[R] If F is a closed subset of X and x is a point of X not in F, then there exist two disjoint open sets G_F,G_x , one containing F and the other containing x.



5.5.2 Example:

Let $X = \{a,b,c\}$,$\tau=\{\emptyset,\{a,b\},\{c\},X\}$ then (X,τ) is regular space.

Solution:

The closed sets X,{c},{a,b},∅, so if we take {c} closed set and $a \notin \{c\}$ then $\exists\{c\},\{a,b\} \in \tau$, s.t. $\{c\} \subset \{c\}$, $a \in \{a,b\}$.

5.5.3 Remark:

1. The above example is not T_2 – Space .Since $a,b \in X. a \neq b$ but we can't find disjoint open sets contain a and b.

2. The above example is not T_1 – Space. Since $\{a\}$,$\{b\}$ is not closed sets.

3. So regular space not necessary T_2 – Space and not T_1 – Space. Also T_2 – Space is not regular as the following example:

5.5.4 Example:

Let $X = \mathbb{R}$ the set of real numbers and let $U_x = \{(a,b):x\in(a,b)\}$ and let $U_0 = \{(-p,p)/\{\frac{1}{n}:n\in\mathbb{N}\}:p>0\}$ the family of all open sets form a base for a topology τ on $\mathbb{R}$ then $(\mathbb{R},\tau)$ is T_2 – Space , since if $a,b \in \mathbb{R}. a \neq b$, $a,b \neq 0$ then there exists two open intervals one of them contain a and the other contain b.Since every open interval is an element in U_x and all elements in $U_x$is in τ then it satisfy [T_2].

If $b \neq 0$,$a = 0$, so it's clear if $b > 0$ the interval $(\frac{1}{b},b+1)$ is a neighborhood of b and $\left(-\frac{b}{2},\frac{b}{2}\right)/\{\frac{1}{n}:n\in\mathbb{N}\}$ is a neighborhood of $a = 0$, then the first interval is an element in $U_x$and the second interval is an element in $U_0$and these intervals are disjoint then it satisfy [T_2].

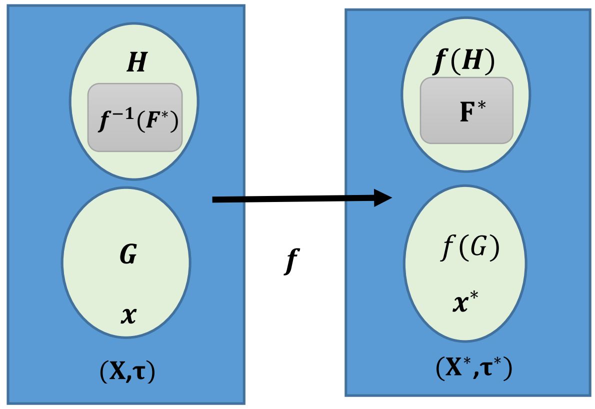
Now if $F=\{\frac{1}{n}:n\in\mathbb{N}\}$, x=0 then $0 \notin$ F and any neighborhood of F intersect with any neighborhood of x=0, so $(\mathbb{R},\tau)$ is not regular.

5.5.5 Remark:

The following theorems shows that the regularity is a topological and hereditary property:

5.5.6 Theorem:

The regularity is a topological property.



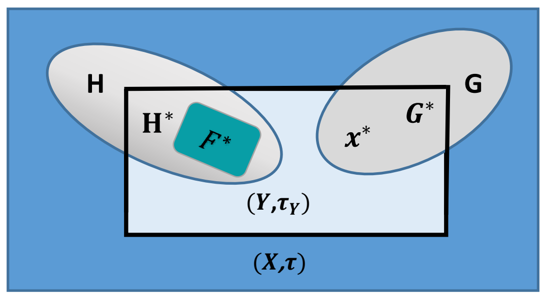
Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a regular space (X,τ) to the topological space (X*,τ*), we want to show that (X*,τ*) is a regular space.

Let F* be a closed set in X*, x* ∈ X^*,$x^* \notin F^*$. Since f is onto then $\exists x \in X$ s.t. $f(x) = x^*$.Since f is continuous then $f^{-1}(F^*)$ is closed X .Since f is onto,1-1 and $x^* \notin F^*$then $x \notin f^{-1}(F^*)$, but (X,τ) is a regular space then ∃G,H∈τ,$G \cap H = \emptyset$ with $x \in G, f^{-1}(F^*) \subseteq H$.Since f is open function then $f(x) \in f(G), F^* \subseteq f(H)$ with $f(G) \cap f(H) = \emptyset$, so (X*,τ*) is a regular space.□

5.5.7 Theorem:

The regularity is a hereditary property.



Proof:

Let (Y,τ_Y) be a subspace of a regular space (X,τ) topological space, we want to prove that (Y,τ_Y) is a regular space.

Let F* be a closed set in Y, x* ∈ Y,$x^* \notin F^*$then $F^* = F \cap Y$,were F is a closed set in X. Since x* ∈ $Y \subset X$, $x^* \notin F^*$ then $x^* \notin F$. Since (X,τ) is a regular space then ∃G , H ∈τ,$G \cap H = \emptyset$ s.t. $x^* \in G$,$F \subseteq H$. Now $G^* = G \cap Y$,$x^* \in G^*$(since $x^* \in G$,$x^* \in Y$) ,$H^* = H \cap Y$,$F^* \subseteq H^*$(since $F \subseteq H$) and $G^* \cap H^*$=$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \emptyset \cap Y = \emptyset$.So (Y,τ_Y) is a regular space.□

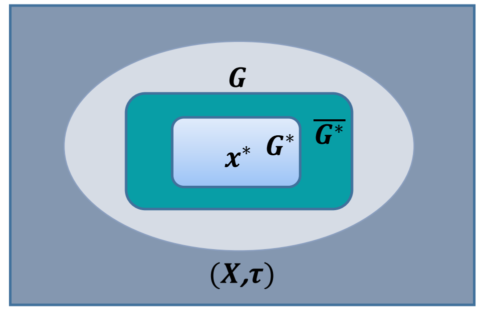
5.5.8 Theorem:

A topological space (X,τ) is regular iff for every point $x \in X$ and open set G containing x there exists an open set G^* such that $x^* \in G^*$and $\overline{G^*} \subseteq G$.

Proof:

$\Longrightarrow$

Suppose (X,τ) is regular, and the point x belongs to the open set G. Then $F = X/G$ is a closed set which does not contain x. By [R], there exist two open sets G_F and G_x such that $F \subseteq G_F$, $x \in G_x$, and $G_F \cap G_x = \emptyset$. Since $G_x \subseteq G_F^c$,$\overline{G_x} \subseteq \overline{G_F^c} = G_F^c \subseteq F^c = G$. Thus, $x \in G_x$ and $\overline{G_x} \subseteq G$ and G_x is the desired set.



$\Leftarrow$

Now suppose the condition holds and x is a point not in the closed set F. Then x belongs to the open set F^c, and by hypothesis there must exist an open set G^*such that $x \in G^*$ and $\overline{G^*} \subseteq F^c$. Clearly G^* and $\overline{G^*}^c$are disjoint open sets containing x and F , respectively.□

<u>5.5.9 Definition:</u>

A topological space (X,τ) is **T_3 – Space** if it regular and T_1 – Space, i.e.

$$\boldsymbol{T_3 \equiv [R]\&[T_1]} \ .$$

<u>5.5.10 Remark:</u>

The following theorem shows that every T_3 – Space is T_2 – Space but the converse is not true as example 5.5.4.

<u>5.5.11 Theorem:</u>

Every T_3 – Space is Hausdorff space (T_2 – Space).

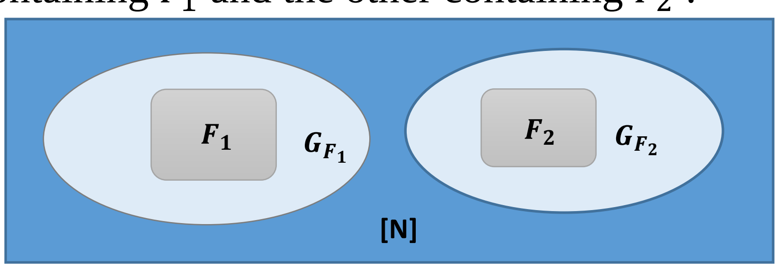
<u>Proof:</u>

Let (X,τ) be a T_3 – Space, we want to prove that (X,τ) is Hausdorff space. Let x,y∈X, $x \neq y$,since X is T_1 – Space then $\{x\}$ is closed set and since $x \neq y$, $y \notin \{x\}$ then by $[R]$, $\exists G$,H∈τ , $G \cap H = \emptyset$ and $\{x\} \subseteq G$,$y \in H$. Hence x and y belong respectively to disjoint open sets G and H.

<u>5.5.12 Definition:</u>

A topological space (X,τ) is ***normal*** iff it satisfies the following axiom of Urysohn:

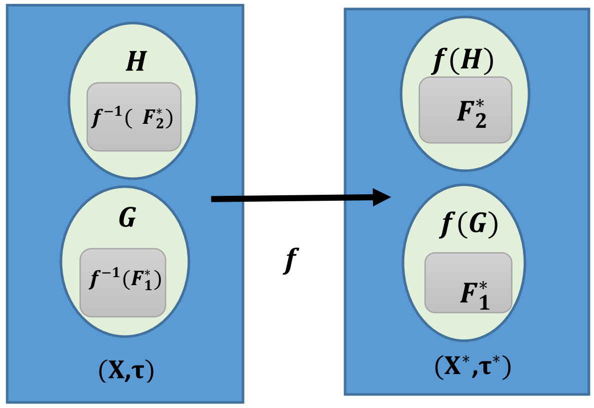
[N] If F_1 and F_2 are two disjoint closed subsets of X, then there exist two disjoint open sets, one containing F_1 and the other containing F_2 .



5.5.13 Theorem:

The normality is a topological property.

Proof:



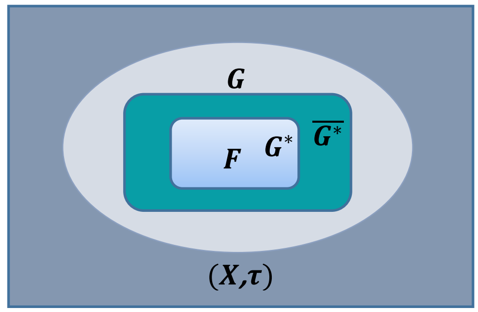
Let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a normal space (X,τ) to the topological space (X^*,τ^*), we want to show that (X^*,τ^*) is a normal space.

Let F_1^* , F_2^* be a disjoint closed sets in X^*. Since f is continuous then $f^{-1}(F_1^*)$,$f^{-1}(F_2^*)$ are closed in X. Since f is onto,1-1 and $F_1^* \cap F_2^* = \emptyset$ then $f^{-1}(F_1^*) \cap f^{-1}(F_2^*) = \emptyset$, Since (X,τ) is normal then $\exists G,H \in \tau$ s.t. $f^{-1}(F_1^*) \subseteq G, f^{-1}(F_2^*) \subseteq H$ and $G \cap H = \emptyset$. Since f is an open function then $F_1^* \subseteq f(G)$, $F_2^* \subseteq f(H)$ and $f(G) \cap f(H) = \emptyset$.So (X^*,τ^*) is a normal space.

5.5.14 Theorem:

A topological space (X,τ) is normal iff for any closed set F and open set G containing F, there exists an open set G^* such that $F \in G^*$and $\overline{G^*} \subseteq G$.

Proof:



$\Longrightarrow$

Suppose (X,τ) is normal and the closed set F is contained in the open set G. Then $K = X/G$ is a closed set which is disjoint from F. By [N], there exist two disjoint open sets G_F and G_K such that $F \subseteq G_F$ and $K \subseteq G_K$.Since $G_F \subseteq G_K^c$, we have $\overline{G_F} \subseteq \overline{G_K^c} = G_K^c \subseteq K^c = G$.Thus G_F is the desired set.

$\Longleftarrow$

Now suppose the condition holds, and let F_1 and F_2 be disjoint closed subsets of X. Then F_1 is contained in the open set $F_2^* = X/F_2$, and, by hypothesis, there exists an open set G^* such that $F_1 \subseteq G^*$ and $\overline{G^*} \subseteq F_2^*$.Clearly, G^* and $X/\overline{G^*}$ are the desired disjoint open sets containing F_1 and F_2, respectively.□

5.5.15 Definition:

A topological space (X,τ) is $\mathbf{T_4}$ **– Space** if it normal and T_1 – Space, i.e.

$$\mathbf{T_4} \equiv [N]\&[\mathbf{T_1}].$$

5.5.16 Example:

Let $X = \{a,b.c\}, \tau = \{\{a\},\{b\},\{a,b\},X,\emptyset\}$ then (X,τ) is normal space.

Solution:

Since the closed sets are {b,c},{a,c},{{c},∅,X are non-empty intersection ,i.e. if F_1 , F_2 are closed disjoint then $F_1 = \emptyset$,$F_2 = X$, so $\exists \emptyset, X \in \tau$, s. t. $F_1 \subseteq \emptyset$,$F_2 \subseteq X$, then (X,τ) is normal space. Also (X,τ) is not regular, since if F={a,c} is closed set and $x = b \notin F$ then every open set contain F intersect with every open set contain x. Also (X,τ) is not T_2 – Space.

5.5.17 Remark:

Example 5.5.16 show that the normal space need not be regular space .The following theorem 5.5.18 show that the T_4 – Space is T_3 – Space.

5.5.18 Theorem:

***Every* T_4 – Space is T_3 – Space**.

Proof:

Let (X,τ) be a T_4 – Space , let F be closed set , $x \in X$, $x \notin F$. Since (X,τ) is T_1 – Space then $F_1 = \{x\}$ is closed set. Since (X,τ) is T_4 – Space then ∃G , H ∈τ , F⊆G , F_1⊆H , G∩H=∅, i. e. x ∈H, F ∈ G , so (X,τ) is T_3 – Space.□

5.5.19 Remark:

The following theorem 5.5.20 gives a relation between normal and T_2 – Space. Also theorems 5.5.20, 5.5.21 give two sufficient conditions for a topological space to be normal.

5.5.20 Theorem:

Every compact Hausdorff space is normal.

Proof:

Let (X,τ) be a compact Hausdorff space and let F , F^* be two disjoint, closed subsets of the compact Hausdorff space X**.** F and F^* are compact since they are closed subsets of a compact space X.

By $[T_2]$,$\forall x \in F$,$\forall y \in F^*$, $\exists G_x, G_y^* \in \tau$,$G_x \cap G_y^* = \emptyset$, s.t. $x \in G_x$ & $y \in G_y^*$. For each fixed point $x \in F$ the collection $\{G_y^* : y \in F^*\}$ forms an open covering of the compact set F^*. There must be a finite subcovering, which we denote by $\{G_{y_i}^* : i = 1,2,..,n\}$. If we let $G_x^* = \bigcup_{i=1}^{n} G_{y_i}^*$ and the finite intersection $G_x = \bigcap_{i=1}^{n} G_x^i$ then G_x and G_x^* are disjoint open sets containing x and F^*, respectively. Now the collection $\{G_x : x \in F\}$ forms an open covering of the compact set F. There must be a finite subcovering, which we denote by $\{G_{x_i} : i = 1,2,..,m\}$. If we let $G = \bigcup_{i=1}^{m} G_{x_i}$ and the finite intersection $G^* = \bigcap_{i=1}^{m} G_{x_i}^*$ then G and G^* are two disjoint open sets containing F and F^* respectively.□

5.5.21 Theorem:

Every regular Lindelöf space is normal.

Proof:

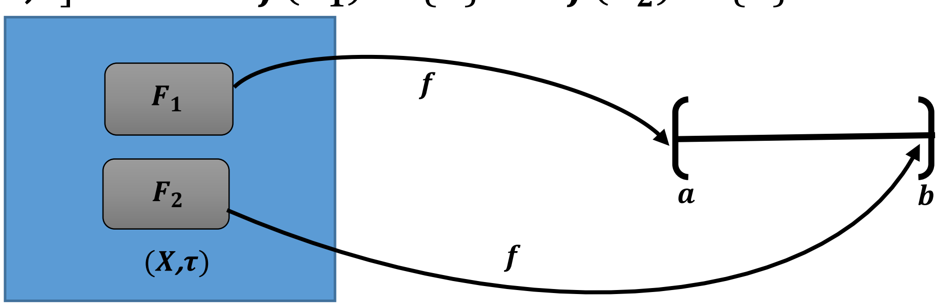
Let F and F^* be two disjoint closed subsets of the regular Lindelöf space (X,τ). Then F and F^* are Lindelöf since every closed subset of a Lindelöf space is Lindelöf space. By [R] **,** $\forall x \in F, \exists G_x \in \tau$, s.t. $x \in G_x \subseteq \overline{G_x} \subseteq F^{*\ c}$. The collection $\{G_x : x \in F\}$ forms an open covering of the Lindelöf set F.There must be a countable subcovering, which we denote by $\{G_i\}_{i=1}^{n}$. Similarly, for each point $x \in F^*$ there must exist an open set $\exists G_x^* \in \tau$, s.t. $x \in G_x^* \subseteq \overline{G_x^*} \subseteq F^{\ c}$.The collection $\{G_x^* : x \in F^*\}$ forms an open covering of the Lindelöf set F^*. There must be a countable subcovering, which we denote by $\{G_i^*\}_{i=1}^{n}$ **.** The reader may show that the sets $G = \bigcup_{n\in\mathbb{N}}[G_n / \bigcup_{i\le n} \overline{G_i^*}]$ and $G^* = \bigcup_{n\in\mathbb{N}}[G_n^* / \bigcup_{i\le n} \overline{G_\iota}]$ are disjoint open sets containing F and F^*, respectively.□

5.5.22 Remark:

Another characterization of normality relates that concept to the number of real-valued continuous functions defined on the space.

5.5.23 Lemma (Urysohn's Lemma):

A topological space (X,τ) is normal iff for every two disjoint closed subsets F_1 and F_2 of X and closed interval [a, b] of reals, there exists a continuous mapping $f: X \to [a,b]$ such that $f(F_1) = \{a\}$ and $f(F_2) = \{b\}$.



5.5.24 Definition:

A topological space (X,τ) is ***completely normal*** iff it satisfies the following axiom of Tietze:

[CN] If A and B are two separated subsets of X, then there exist two disjoint open sets, one containing A and the other containing B.

5.5.25 Definition:

A topological space (X,τ) is **T_5 – Space** if it completely normal space and also T_1 – Space, i.e.

$$\mathbf{T_5 \equiv [CN]\&[T_1].}$$

5.5.26 Example:

Let $X = \{a,b,c\}$,τ=$\{\emptyset,\{a,b\},\{c\},X\}$ then (X,τ) is completely normal.

Solution:

Since every set in τ is open and closed set, so if $A,B \in \tau$ then $\bar{A} \cap B = A \cap \bar{B} = A \cap B = \emptyset$ then A and B are separable and $A \subseteq A$,B⊆B so (X,τ) is completely normal .Also in example 5.5.2 we show that (X,τ) is regular space not T_1 – Space and not T_2 – Space.

5.5.27 Remark:

Since disjoint closed sets are separated, then every completely normal space is normal, and hence every T_5 – Space is a T_4 – Space but the converse is not true. Also the following example show that T_5 – Space does not transfer by continuity.

5.5.28 Example:

Let $X = X^* = \{a,b,c\}$ and let τ be the discrete topology and $\tau^*=\{\emptyset,\{a\},\{b,c\},X^*\}$ and let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be the identity function , i.e. $f(x) = x, \forall x \in X$.

Since (X,τ) is the discrete topology then f is continuous function and since the discrete topology is T_1 – Space and normal then (X,τ) is T_5 – Space. Since (X^*,τ^*) is not T_1 – Space then it's not T_5 – Space.

5.5.29 Theorem:

The completely normal space ([CN]) is topological property.

Proof:

Let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a topological space (X,τ) satisfy [CN] to the topological space (X^*,τ^*),we want to show that (X^*,τ^*) satisfy [CN].

Let A^*,B^*be a separable sets in X^*. Since f is continuous and 1-1 then $f^{-1}(A^*),f^{-1}(B^*)$ are separated subset of X. Since (X,τ) satisfy [CN] then $\exists G,H \in \tau$, $G \cap H = \emptyset$, s.t. $f^{-1}(A^*) \subseteq G, f^{-1}(B^*) \subseteq H$.Since f is open ,1-1 and $G,H \in \tau$ then $A^* \subseteq f(G), B^* \subseteq f(H), f(G) \cap f(H) = \emptyset, f(G), f(H) \in \tau^*$,so (X^*,τ^*) satisfy [CN].□

5.5.30 Theorem:

A topological space(X,τ) is completely normal iff every subspace of X is normal.

Proof:

$\Longrightarrow$

Suppose (X,τ) is completely normal and let (X^*,τ^*) be a subspace of (X,τ), we want to prove that (X^*,τ^*) is normal space.

Let F_1^* and F_2^* be disjoint (relatively) closed subsets of X^*, so $F_1^* = \overline{F_1^*}$,$F_2^* = \overline{F_2^*}$. Since F_1^* and F_2^* are closed subsets of X^* then $\exists F_1$,F_2 closed subset of X such that $\overline{F_1^*}$=$\overline{F_1} \cap X^*$, $\overline{F_2^*} = \overline{F_2} \cap X^*$.Now

$F_1^* \cap \overline{F_2} = \overline{F_1^*} \cap \overline{F_2} = \overline{F_1} \cap X^* \cap \overline{F_2} = \overline{F_1} \cap X^* \cap X^* \cap \overline{F_2} = \overline{F_1^*} \cap \overline{F_2^*}$=$F_1^* \cap F_2^* = \emptyset$. And similarly, $\overline{F_1} \cap F_2^* = \emptyset$. Hence F_1^* and F_2^* are separated subsets of X**.** By[CN], there exist disjoint open sets G_1 and G_2 containing F_1^* and F_2^* respectively. Then the sets $X^* \cap G_1$ and $X^* \cap G_2$ are disjoint (relatively) open subsets of X^* which contain F_1^* and F_2^*, respectively, so X^* is normal.

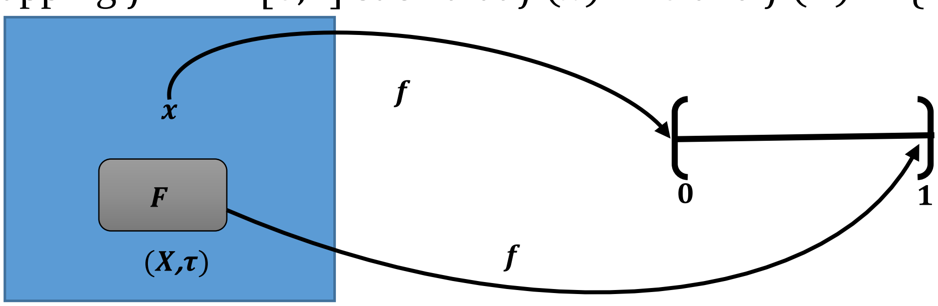
$\Leftarrow$

Now let us suppose that every subspace of X is normal, and let A and B be separated subsets of X. Consider the open set $[\bar{A} \cap \bar{B}]^c = X^*$ as a subspace of X. By hypothesis, X^* is normal. The sets $X^* \cap \bar{A}$ and $X^* \cap \bar{B}$ will be disjoint, relatively closed subsets of X^* and so there must exist two disjoint relatively open sets G_A and G_B containing $X^* \cap \bar{A}$ and $X^* \cap \bar{B}$ respectively. Since X^* is an open subset of X, G_A and G_B are actually open subsets of X .Thus we have $A \subseteq X^* \cap \bar{A} \subseteq G_A$ and $B \subseteq X^* \cap \bar{B} \subseteq G_B$, so that X is completely normal.□

5.5.31 Definition:

A topological space (X,τ) is ***completely regular*** iff it satisfies the following axiom:

[CR] If F is a closed subset of X**,** and x is a point of X not in F**,** then there exists a continuous mapping $f: X \rightarrow [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.



5.5.32 Definition:

A topological space (X,τ) is A ***Tichonov Space*** if it completely regular space and also T_1 – Space, i.e.

$$\mathbf{T}_{3\frac{1}{3}} \equiv [\mathbf{CR}]\&[\mathbf{T_1}].$$

5.5.33 Theorem:

The completely regular space is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a completely regular space (X,τ) to the topological space (X*,τ*), we want to show that (X*,τ*) is compeletly regular space.

Let F^* be a closed subset of $X$and $x \in X$, $x \notin F^*$.Since f is continuous then $F = f^{-1}(F^*)$. Since f is onto then $\exists x \in X$, s.t.$f(x)$=x^*.Since f is 1-1 and $x^* \notin F^*$ then $x \notin F$. Since (X,τ) is completely regular then $\exists\ g : X \to [0,1]$, s. t. $g(x) = 0$ and $g(F) = \{1\}$ then the composition $g \circ f^{-1}$ is continuous (since g and f^{-1} are continuous functions).So $g \circ f^{-1}$: X* → [0,1] and $(g \circ f^{-1})(F^*) = g(f^{-1}(F^*)) = g(F) = \{1\}$ and $(g \circ f^{-1})(x) = g(f^{-1}(x^*)) = g(x) = 0$. So (X*,τ*) is compeletly regular space.□

5.5.34 Theorem:

The completely regular space is a hereditary property.

Proof:

Let (Y,τ_Y) be a subspace of a regular space (X,τ) topological space, we want to prove that (Y,τ_Y) is a regular space.

Let F* be a closed set in Y, x* ∈ Y,$x^* \notin F^*$then $F^* = F \cap Y$,were F is a closed set in X. Since x* ∈ $Y \subset X$, $x^* \notin F^*$ then $x^* \notin F$. Since (X,τ) is completely regular space then $\exists\ f : X \to [0,1]$, $s.\ t.\ f(x) = 0\ and\ f(F) = \{1\}$. Let $\exists\ f^*\ Y \to [0,1]$ defined as $f^*(x) = f(x)$,$\forall x \in Y$, i. e. f^*=f$|_Y$ is continuous and satisfy $f^*(x) = o$, since $x \in Y$ and $f^*(F^*) = \{1\}$, since $F^* = F \cap Y$,so (Y,τ_Y) is a regular space.□

5.5.35 Theorem:

Every completely regular space is regular.

Proof:

Let (X,τ) be a completely regular space. Let F be a closed subset of $X$and $x \in X$, $x \notin F$ then $\exists\ f : X \to [0,1]$, continuous function such that $f(x) = 0$ and $f(F) = \{1\}$. Since ℝ is a T_2 – Space and [0,1] ⊆ ℝ is also a T_2 – Space then $\exists G$,H∈τ , G∩H=∅ and $0 \in G$, 1∈H. Since f is continuous function then $f^{-1}(G)$,g^{-1}(H) are disjoint open subset of X and $x \in f^{-1}(0) \in f^{-1}(G)$, $F \subseteq f^{-1}(1) \subseteq f^{-1}(G)$.So (X,τ) is regular space.□

5.5.36 Remark:

Theorem 5.5.35 every [CR] is [R] ,every Tichonov space is a T_3 – Space, and every T_4 – Space is a Tichonov space by Urysohn's Lemma. Because of these facts, we might be inclined to call a Tichonov space a $T_{3\frac{1}{3}}$-space.

T_4 –Space ⟶ Tichonov space ⟶ $T_{3\frac{1}{3}}$-space

On the other hand, since a normal space need not be regular, it also need not be completely regular. The following implication does hold, however

5.5.37 Theorem:

A normal space is completely regular iff it is regular.

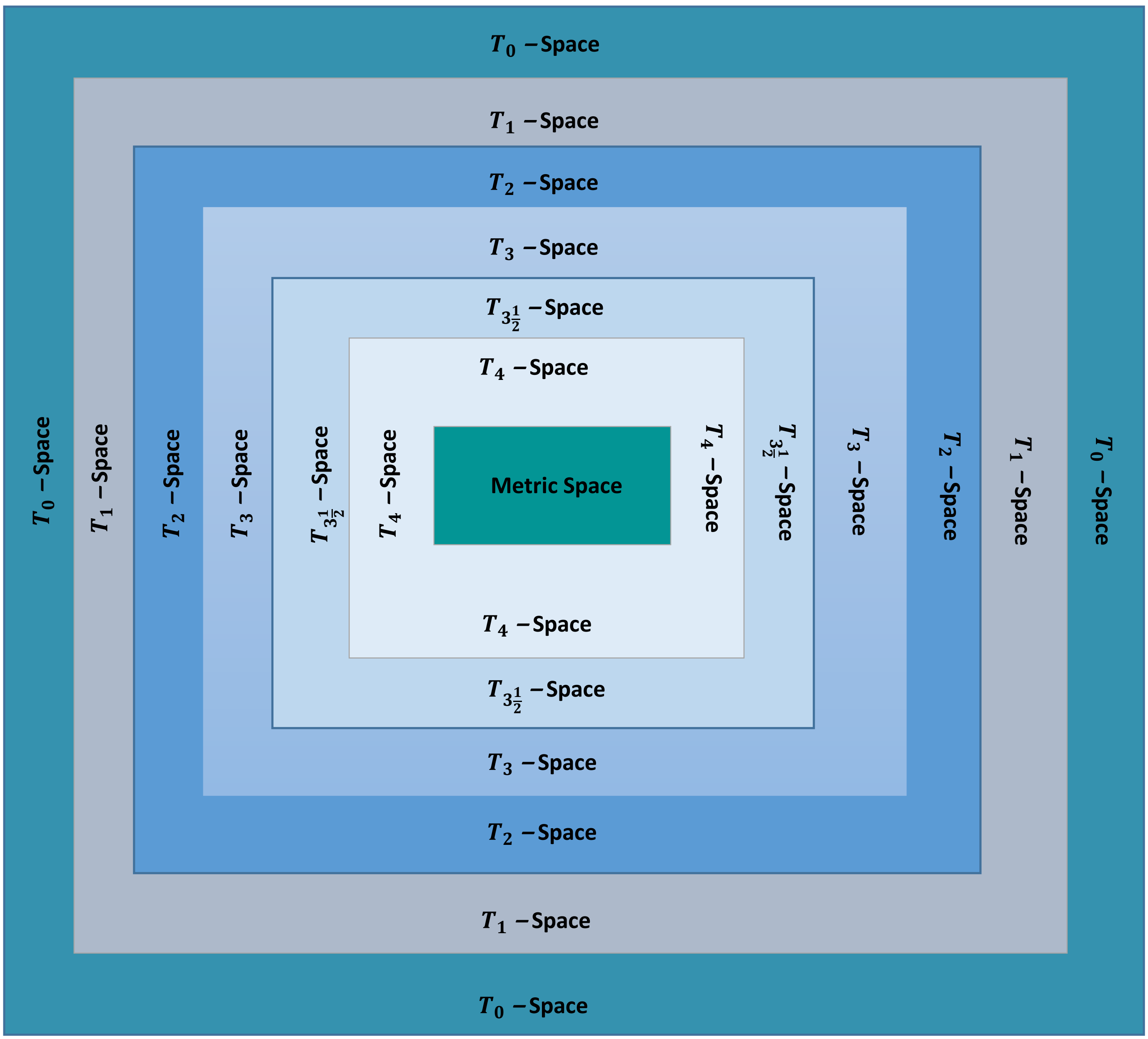
Proof:

$\Longrightarrow$

By theorem 5.5.18 a norm space is regular if it is completely regular.

$\Longleftarrow$

We need to show that any normal, regular space (X,τ) is completely regular. Suppose F is a closed subset of X not containing the point x**,** so that x belongs to the open set F^c**.** By theorem 5.5.14, there exists an open set G such that $x \in G$ and $\bar{G} \subseteq F^c$**.**Since F and $\bar{G}$ are disjoint closed sets in the normal space X, by Urysohn's Lemma there exists a continuous mapping $f : X \to [0,1]$ such that $f(F) = \{1\}$ and $f(\bar{G}) = \{0\}$.. Since $x \in G$, $f(x) = 0$, and so (X,τ) is completely regular.□



T_0 −Space
T_1 −Space
T_2 −Space
T_3 −Space
$T_{3\frac{1}{2}}$ −Space
T_4 −Space
Metric Space