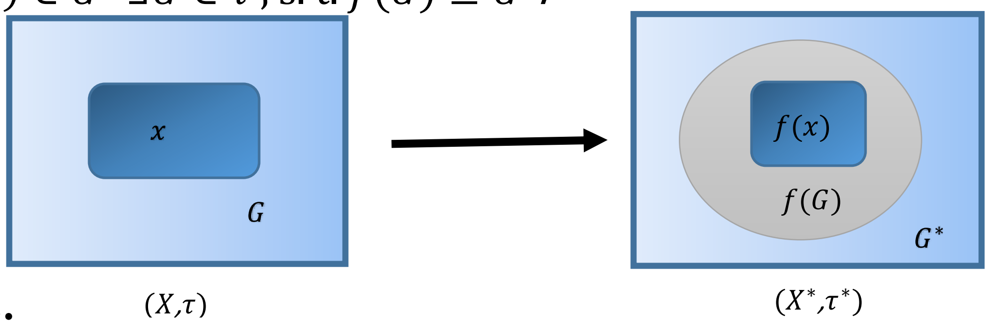
Chapter Four

Continuity and Topological Equivalence

4.1 Continuous Functions

4.1.1 Definition:

A function f mapping a topological space (X,τ) into a topological space(X^*,τ^*) will be said to be ***continuous at a point*** $\boldsymbol{x \in X}$ iff for every open set G^* containing $f(x)$ there is an open set G containing x such that $f(G) \subseteq G^*$, i.e. $\forall G^* \in \tau^*, f(x) \in G^*\ \exists G \in \tau$, s. t. $f(G) \subseteq G^*$.

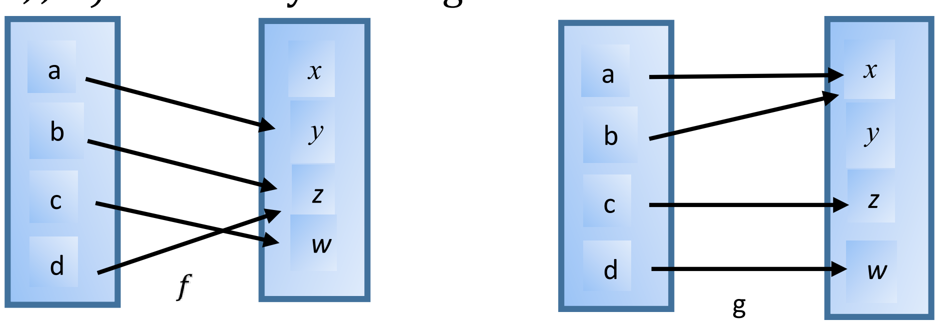


4.1.2 Remark:

We say that f is continuous on a set $E \subseteq X$ iff it is continuous at each point of E.

4.1.3 Example:

Let $X = \{a,b,c,d\}$ and $X^* = \{x,y,z,w\}$ have the topologies $\tau = \{X,\emptyset,\{a\},\{a,b\},\{a,b,c\}\}$,$\tau^* = \{X^*,\emptyset,\{x\},\{y\},\{x,y\},\{y,z,w\}\}$ respectively consider the functions f,g: $(X,\tau) \longrightarrow (X,,\tau^*)$ defined by the diagrams below:



The function f is continuous but the function g is not continuous on X.

Solution:

Take $a \in X$,$f(a)$=y the open sets in X^*contain y are X^*,$\{y\}$,$\{x,y\}$and$\{y,z,w\}$, so

$\exists$ X $\in \tau$, s. t. $f(X) \subseteq X^*$,
$\exists\{a\} \in \tau$, s. t. $f(\{a\}) \subseteq \{y\}$,
$\exists\{a\} \in \tau$, s. t. $f(\{a\}) \subseteq \{x , y\}$,
$\exists$ X $\in \tau$, s. t. $f(X) \subseteq \{y,z,w\}$

Thus the function f is continuous at a similar we can show that f is continuous at b , c *and* d ,so f is continuous on X but the function g is not continuous on X since it's not continuous on c, i.e. $g(c) = z, z \in \{y,z,w\} \in \tau^*, \nexists G \in \tau\ s.t.\ g(G) = \{y,z,w\}$.

<u>4.1.4 Theorem:</u>

If $f:(X,\tau) \longrightarrow (X^*, \tau^*)$ then the following conditions are each equivalent to the continuity of f on X:

1) ***The inverse of every open set in X^* is an open set in X.***

2)***The inverse of every closed set in X^* is a closed set in X.***

3) $f(\bar{E}) \subseteq \overline{f(E)}$ ***for every $E \subseteq X$.***

<u>Proof:</u>

Continuity ⟺ (1)

Suppose that f is continuous on X, and G^* is an open set in X^*.If x is any point of $f^{-1}(G^*)$ then f is continuous at x, and there must exist an open set G containing x such that $f(G) \subseteq G^*$. Thus G is contained in $f^{-1}(G^*)$, and hence $f^{-1}(G^*)$, is an open set in X**.** Conversely, if the inverses of open sets are open, we may choose the set $f^{-1}(G^*)$, let $x \in X$ and let G^*be an open set in X^* contain $f(x)$,i.e. $f(x) \in G^*$, so $x \in f^{-1}(G^*)$ which is an open set in X satisfy $ff^{-1}(G^*) \subseteq G^*$. Then $f$is continues at x and x is arbitrary so $f$is continues on X.

(1) ⟺ (2)

Suppose that the inverses of open sets are open and let F^*be a closed set in X^*,so $F^{*C}$is an open set in X^* then by (1) , $f^{-1}(F^{*C}) = (f^{-1}(F^*))^c$is open in X ,i.e. $f^{-1}(F^*)$ is closed set in X. Conversely, assume the inverses of closed sets are closed and let G^*be an open set in X^*,so $G^{*C}$is a closed set in X^* then by (2) , $f^{-1}(G^{*C}) = (f^{-1}(G^*))^c$ is closed in X ,i.e. $f^{-1}(G^*)$ is an open set in X.

(2) ⟺ (3)

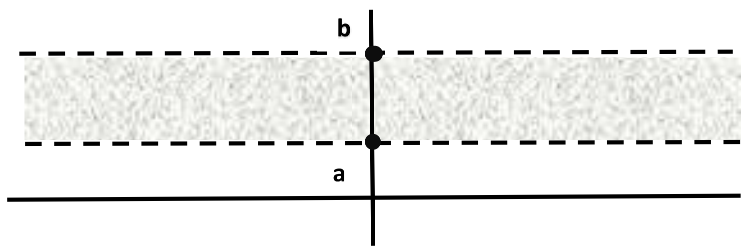
Suppose that the inverses of closed sets are closed, and $E \subseteq X$.Since $E \subseteq f^{-1}(f(E))$ for any function, $E \subseteq f^{-1}(\overline{f(E)})$.But $f^{-1}(\overline{f(E)})$ is the inverse under a continuous mapping of a closed set and hence is a closed set containing E. Therefore, $\bar{E} \subseteq f^{-1}(\overline{f(E)})$ and so $f(\bar{E}) \subseteq f(f^{-1}(\overline{f(E)})) \subseteq \overline{f(E)}$. Conversely, suppose the condition (3) holds for all subsets $E \subseteq X$, and F^*be a closed set in X^*, $f\left(\overline{f^{-1}(F^*)}\right) \subseteq \overline{ff^{-1}(F^*)} \subseteq \overline{F^*} = F^*$also $\overline{f^{-1}(F^*)} \subseteq f^{-1}(F^*)$, i.e. $f^{-1}(F^*) = \overline{f^{-1}(F^*)}$, so . $f^{-1}(F^*)$ is closed in X, i.e. the inverse of every closed set is a closed set. □

4.1.5 Example:

Consider (X,τ) any discrete topology and $(X^*, ,\tau^*)$ any topological space then every function $f:(X,\tau) \longrightarrow (X^*, ,\tau^*)$ is continuous, since if H is any open subset of X^*its invers $f^{-1}(H)$ is open subset of X (every subset of a discrete topology is open).

4.1.6 Example:

The projection map $f:(\mathbb{R}^2,\tau) \longrightarrow (\mathbb{R}, ,\tau^*)$ defined by $f(x,y) = x$ is continuous relative to the relative topology. Since the inverse of any open interval (a,b) is an infinite open strip then by theorem 4.1.4 the inverse of every open subset of $\mathbb{R}$ is an open in $\mathbb{R}^2$,i.e. f is continuous.



4.1.7 Example:

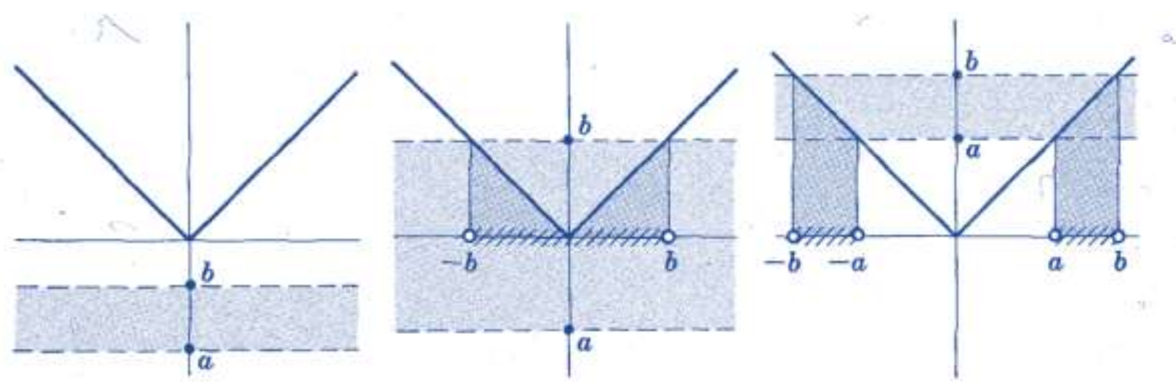
The absolute value function $f:(\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau)$, *i.e.* $f(x) = |x|$ for every $x \in \mathbb{R}$ is continuous.

Solution:

Since if $G =$ (a,b) is an open interval in $\mathbb{R}$ then

$$f^{-1}(G) = \begin{cases} \emptyset & if\ a < b \leq 0 \\ (-b,b) & if\ a < o < b \\ (-b,\text{-a}) \cup (\text{a},b) & if\ 0 \leq a < b \end{cases}$$

In each case $f^{-1}(G)$ is open, hence f is continuous.



$f^{-1}[A] = \emptyset$ $\quad$ $f^{-1}[A] = (-b, b)$ $\quad$ $f^{-1}[A] = (-b, -a) \cup (a, b)$

4.1.8 Example:

Let $f:(X,\tau) \longrightarrow (X^*, ,\tau^*)$ be a constant function ,i.e. . $f(x) = c \in X^*$ for every $x \in X$. Then f is continuous relative to any topology τ on X and any topology τ^*on X^*.

Solution:

We need to show that the inverse image of any τ^* −open subset of Y is a τ −open subset of X. Let $G^* \in \tau^*$.Now $f(x) = c$ for every $x \in X$,so

$$f^{-1}(G^*) = \begin{cases} X & if\ c \in G^* \\ \emptyset & if\ c \notin G^* \end{cases}$$

In either case $f^{-1}(G^*)$ is an open subset of X since X and $\emptyset$ belong to every topology τ on X.

4.1.9 Example:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be any function. If (X^*,τ^*) is any indiscrete space then f is continuous for any τ.

Solution:

We want to show that the inverse image of every open subset of $X^*$is an open subset of X. Since $(X^*, ,\tau^*)$ is an indiscrete space, $X^*$and $\emptyset$ are the only open subset of X^*.But $f^{-1}(X^*) = X$, $f^{-1}(\emptyset) = \emptyset$ and X, $\emptyset \in \tau$ on X. Hence f is continuous for any τ.

4.1.10 Example:

Let $(\mathbb{R},\tau)$ be the real topology and let f,g,h: $(\mathbb{R},\tau) \longrightarrow (\mathbb{R}, ,\tau)$ be functions defined on $\mathbb{R}$ as $f(x) = x + 2$, g(x)=2x and $h(x) = x^2$.Show that the all functions f,g and h are continuous.

Solution:

Since if $G =$ (a,b) is an open interval in $\mathbb{R}$ then

$$f^{-1}((a,b)) = (a - 2, b - 2)$$

$$g^{-1}((a,b)) = (\tfrac{a}{2},\tfrac{b}{2})$$

$$h^{-1}((a,b)) = \begin{cases} (-\sqrt{b},-\sqrt{a}) \cup (\sqrt{a},\sqrt{b}) & if\ a \geq 0, \\ (-\sqrt{b},\sqrt{b}) & if\ a < 0\ and b > 0, \\ \emptyset & if\ b \leq 0. \end{cases}$$

In each case the preimage of an arbitrary G is an open set. Thus each function is continuous.

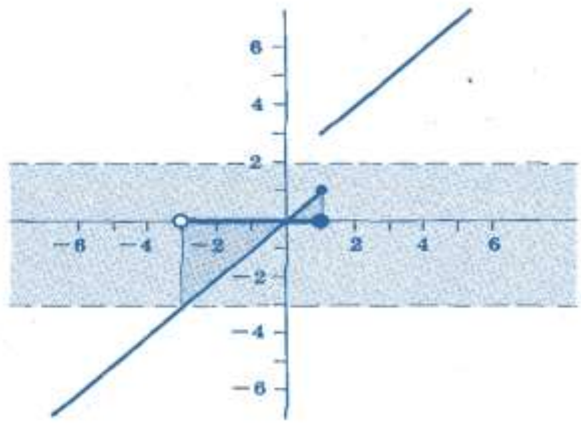
4.1.11 Example:

Let τ be the usual topology on $\mathbb{R}$ and let τ^* be the upper limit topology on $\mathbb{R}$ which generated by the open – closed intervals $(a,\text{b}]$.Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & if\ x \leq 1 \\ x+2 & if\ x > 1 \end{cases}$$

a) Show that $f: (\mathbb{R},\tau) \longrightarrow (\mathbb{R}, ,\tau)$ is not continuous.

b) Show that $f: (\mathbb{R},\tau^*) \longrightarrow (\mathbb{R}, ,\tau^*)$ is continuous.



Solution:

a) Let $A = (-3,2) \in \tau$ then $f^{-1}(\text{A}) = (-3,1] \notin \tau$.

So f is not continuous.

b) Let $A = (a,\text{b}] \in \tau^*$then

$$f^{-1}(\text{A}) = \begin{cases} (a,\text{b}] & if\ a < b \leq 1 \\ (a,1] & if\ a < 1 < b \leq 3 \\ (a,\text{b-2}] & if\ a < 1 < 3 < b \\ \emptyset & if\ 1 \leq a < b \leq 3 \\ (1,b-2] & if\ 1 \leq a < 3 < b \\ (a-2,\text{b-2}] & if\ 3 \leq a < b \end{cases}$$

In each case the$f^{-1}(\text{A})$ is a τ^*- open set. Hence f is τ^*continuous.

4.1.12 Example:

Let τ^* be the usual topology on $\mathbb{R}$ and let τ be the co-finite topology on $\mathbb{R}$. If $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x$, $\forall x \in \mathbb{R}$ then f is not continuous.

Solution:

Since if $G = (\text{a,b}) \in \tau$ then $f^{-1}((\text{a,b})) = (a,b) \in \tau^*$,since $(a,b)^C = (-\infty,\text{a}] \cup [\text{b},\infty)$ is finite, so f is not continuous.

4.1.13 Example:

Show that the identity function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is continuous iff τ is finer than τ^*, i.e. $\tau^* \subset \tau$.

Solution:

The identity function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is continuous iff $\forall G \in \tau^* \Longrightarrow f^{-1}(\text{G}) \in \tau$. But $f^{-1}(\text{G}) = \text{G}$,so f is continuous iff $\forall G \in \tau^* \Longrightarrow G \in \tau$, i.e. $\tau^* \subset \tau$.

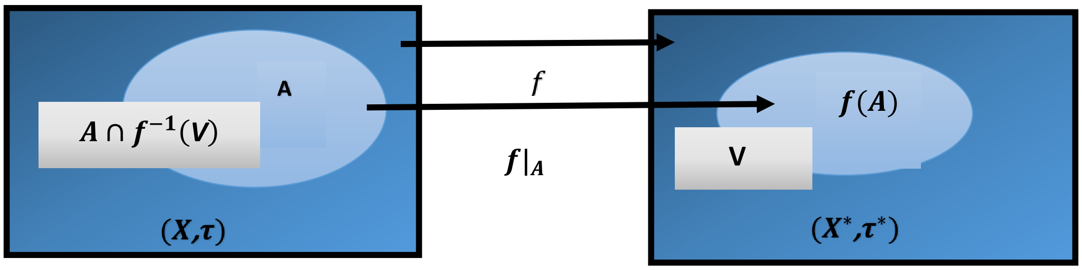
4.1.14 Example:

Let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be continuous then Prove that $f|_A: (X,\tau_A) \longrightarrow (X^*,\tau^*{}_A)$ is continuous, where $A \subset X$ and $f|_A$ is restriction of f to A.

Solution:

If $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is a function and $A \subset X$ then the restriction function $f|_A:(X,\tau_A) \longrightarrow (X^*,{\tau^*}_A)$ is defined as $f|_A(x) = f(x), \forall x \in A$.

Let $V \in \tau^*$, since f is continuous then $f^{-1}(\mathrm{V}) \in \tau$ then $A \cap f^{-1}(V) \in \tau_A$.Since ${f|_A}^{-1}(V) = A \cap f^{-1}(V)$ then $f|_A$ is continuous function.

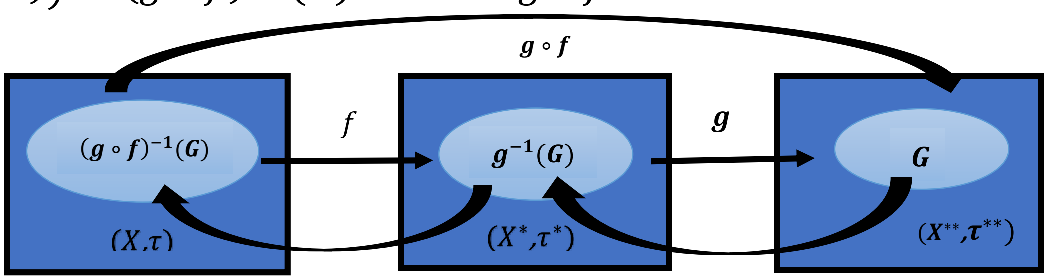


4.1.15 Corollary:

Let the functions $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ and $f:(X^*,\tau^*) \longrightarrow (X^{**},\tau^{**})$ be continuous then the composition $g \circ f:(X,\tau) \longrightarrow (X^{**},\tau^{**})$.

Proof:

Let $G \in \tau^{**}$ then $g^{-1}(G) \in \tau^*$since g is continuous .But f is also continuous, so $f^{-1}\left(g^{-1}(G)\right) = (g \circ f)^{-1}(G) \in \tau$ then $g \circ f$ is continuous.□



4.1.16 Theorem:

A function $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is continuous iff the inverse of each member of a base $\mathcal{B}$ for X^* is an open subset of X.

Proof: $\Longrightarrow$

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a continuous function and let $\mathcal{B}$ be a base for the topology τ^*, i.e. $\mathcal{B} \subset \tau^*$. Now for every $B \in \mathcal{B}$ we have $f^{-1}(B) \in \tau$ so $f^{-1}(B)$ is an open subset of X function.

$\Longleftarrow$

Let $G \in \tau$, since $\mathcal{B}$ is a base for τ^*then $G = \cup_i B_i, B_i \in \mathcal{B}$, so $f^{-1}(G) = f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$ and since $f^{-1}(B_i) \in \tau$ then $f^{-1}(G)$ is union of open sets and therefore its open ,so f is continuous.□

4.1.17 Theorem:

Let $\mathcal{S}$ be a subbase for a topological space (X^*,τ^*).Then a function $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is continuous iff the inverse of each member of $\mathcal{S}$ is an open subset of X.

Proof: $\Longrightarrow$

Suppose $f^{-1}(S) \in \tau$ for every $S \in \mathcal{S}$.We want to show that f is continuous, i.e. if $G \in \tau^*$ then $f^{-1}(G) \in \tau$.Let $G \in \tau^*$then by definition of subbase

$$G = \bigcup_i \left(S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{n_i}}\right), \text{ where } S_{i_k} \in \mathcal{S}$$

Hence, $f^{-1}(G) = f^{-1}(\bigcup_i \left(S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{n_i}}\right)) = \bigcup_i f^{-1}\left(S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{n_i}}\right)$

$$= \bigcup_i \left(f^{-1}(S_{i_1}) \cap f^{-1}(S_{i_2}) \cap \ldots \cap f^{-1}(S_{i_{n_i}})\right)$$

But $S_{i_k} \in \mathcal{S} \Longrightarrow f^{-1}(S_{i_k}) \in \tau$.Hence $f^{-1}(G) \in \tau$ since it is the union of finite intersections of open sets.herefore f is continuous.

$\Longleftarrow$

If f is contiuous then the inverse of all open sets, including the member of $\mathcal{S}$ are open. □

4.1.18 Example:

Let f be a function from a topological space (X,τ) into the unit interval [0,1]. Show that if $f^{-1}((a,1])$ and $f^{-1}([0,b))$ are open subsets of X for all $0 < a$,b<1, then f is continuous.

Solution:

Since the intervals $(a,1]$ and $[0,b)$ form a subbase for the unit interval [0,1] then by theorem 4.1.17 , f is continues.

4.1.19 Theorem:

Let $\{\tau_i\}$ be a collection of topologies on a set X. If a function $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is continuous with respect to each τ_i, then f is continuous with respect to the intersection topology $\tau = \bigcap_i \tau_i$.

Proof:

Let G be an open subset of X^*then by hypothesis $f^{-1}(G)$ belongs to each τ_i. Hence $f^{-1}(G)$ belongs to the intersection, i.e. $f^{-1}(G) \in \bigcap_i \tau_i = \tau$ and so f is continuous with respect to the intersection topology τ.□

4.1.20 Theorem:

A function $f:(X,\tau) \longrightarrow (X^*, ,\tau^*)$ be a continuous at a point $a_0 \in X$ if for every sequence $\langle a_n \rangle$ in X converges to a_0 the sequence $\langle f(a_n) \rangle$ in X^*converges to f(a_0), i.e. $a_n \longrightarrow a_0 \Rightarrow f(a_n) \longrightarrow$ f(a_0).

4.1.21 Remark:

The following theorems show that some characteristics transfer by continuity.

4.1.22 Theorem:

If $\boldsymbol{f}: (\boldsymbol{X},\boldsymbol{\tau}) \longrightarrow (\boldsymbol{X}^{*}, \boldsymbol{\tau}^{*})$ ***is a continuous function then*** $\boldsymbol{f}$ ***maps every connected subset of X onto a connected subset of*** $\boldsymbol{X}^{*}$***.***

Proof:

Let E be a connected subset of X and suppose that $E^{*} = f(E)$ is not connected then there exists a separation $E^{*} = A^{*}|B^{*}$,where $A^{*}$and $B^{*}$are nonempty disjoint sets which are both and closed subsets of E^{*}.Let $A = f^{-1}(A^{*}) \cap E$ and $B = f^{-1}(B^{*}) \cap E$.

Since f is continuous function and A^{*}, $B^{*}$are both and closed subsets of E^{*}then by theorem 4.1.4 , A and B are nonempty disjoint sets which are both and closed subsets of E. Thus E has a separation $E = A|B$, i.e. E is not connected and this is contradiction , so $E^{*} = f(E)$ is connected. □

4.1.23 Theorem:

If $\boldsymbol{f}: (\boldsymbol{X},\boldsymbol{\tau}) \longrightarrow (\boldsymbol{X}^{*},\boldsymbol{\tau}^{*})$ ***is a continuous function then*** $\boldsymbol{f}$ ***maps every compact subset of X onto a compact subset of*** $\boldsymbol{X}^{*}$***.***

Proof:

Let E be a compact subset of X and suppose that $\{G_i^{*}\}$ be an open cover of $f(E)$, i.e. $f(E) \subseteq \bigcup_i G_i^{*}$.Since $E \subseteq f^{-1}\big(f(E)\big) \subseteq f^{-1}(\bigcup_i G_i^{*}) = \bigcup_i f^{-1}(G_i^{*})$. Since f is continuous function and by theorem 4.1.4 we get $\{f^{-1}(G_i^{*})\}$ is an open covering of E. But E is compact then there exists a finite subcover $\{f^{-1}(G_i^{*})\}_{i=1}^{n}$ of $\{f^{-1}(G_i^{*})\}$ for E, i.e. $E \subseteq \bigcup_{i=1}^{n} f^{-1}(G_i^{*})$, so $f(E) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(G_i^{*})) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(G_i^{*})) \subseteq \bigcup_{i=1}^{n} G_i^{*}$.Then $f(E)$ is compact. □

4.1.24 Theorem:

If $\boldsymbol{f}: (\boldsymbol{X},\boldsymbol{\tau}) \longrightarrow (\boldsymbol{X}^{*},\boldsymbol{\tau}^{*})$ ***is a continuous function then*** $\boldsymbol{f}$ ***maps every sequentially compact subset of X onto a sequentially compact subset of*** $\boldsymbol{X}^{*}$***.***

Proof:

Let $f: (X,\tau) \longrightarrow (X^{*}, \tau^{*})$ be a continuous function and let E be a sequentially compact subset of X. We want to show that $f(E)$ is a sequentially compact subset of X^{*}.

Let $\langle b_1, b_2, ...\rangle$ be a sequence in $f(E)$ then $\exists a_1, a_2, ... \in E$ s.t. $f(a_n) = b_n, \forall n \in \mathbb{N}$. But E is a sequentially compact subset of X, so the sequence $\langle a_1, a_2, \dots\rangle$ contains a subsequence $\langle a_{i_1}, a_{i_2}, \dots\rangle$ which converges to a point $a_0 \in E$. Since f is continuous then $\langle f(a_{i_1}), f(a_{i_2}), \dots\rangle = \langle b_{i_1}, b_{i_2}, ...\rangle$ converges to $f(a_0) \in f(E)$. Thus $f(E)$ is sequentially compact. □

4.1.25 Example:

Show that :

a) A continuous image of a countably compact set need not be countably compact.

b) A continuous image of a locally compact set need not be locally compact.

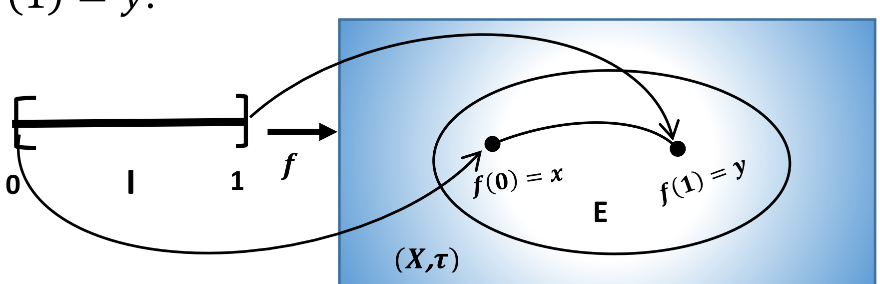
Solution:

a) Let τ be the topology on $\mathbb{N}$, the set of positive integers generated by sets $\{\{1,2\},\{3,4\},\{5,6\},\ldots\}$ by example 3.7.8, X is countably compact. Let $(\mathbb{N}, \tau^*)$ be the discrete topology on $\mathbb{N}$ which is not countably compact. The function $f: (\mathbb{N},\tau) \longrightarrow (\mathbb{N}, \tau^*)$ which maps $2n$ and $2n-1$ onto n for $n \in \mathbb{N}$ is continuous and maps the countably compact space $(\mathbb{N},\tau)$ onto the non – countably compact space $(\mathbb{N}, ,\tau^*)$.

b) Let $(\mathbb{Q},\tau)$ be the discrete topology which is locally compact and $(\mathbb{Q},\tau^*)$ be the usual topology which is not locally compact. Consider $f: (\mathbb{Q},\tau) \longrightarrow (\mathbb{Q}, \tau^*)$ to be the identity function which is continuous.

4.1.26 Definition:

If E is a subset of a topological space (X,τ) and we let $\mathrm{I} = [0, 1]$, then a ***path*** in E joining two points x and y of E is a continuous function $f: I \longrightarrow E$ such that $f(0) = x$ and $f(1) = y$.



4.1.27 Definition:

A subset E of a topological space (X,τ) is said to be ***arcwise connected*** if for any two points $a,b \in E$ there is a path $f: I \longrightarrow E$ from a to b which is contained in E, i.e. $f(I) \subseteq E$.

4.1.28 Remark:

The relationship between connected and arcwise sets connected sets is given in the following diagram, theorem 4.1.29 and example 4.1.30.

arcwise connected $\Rightarrow$ / $\not\Leftarrow$ connected

4.1.29 Theorem:

A rcwise connected sets are connected.

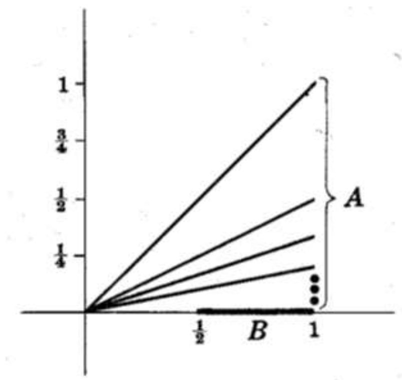
Proof:

Since I is connected, $f(I)$ is connected for any continuous function f. Thus any two points is an arcwise connected space belong to a connected subset $f(I)$ of the space, where f is a path joining the two points. By Corollary 3.1.19, any arcwise connected space must be connected. □

4.1.30 Example:

Consider the following subsets of the plane $\mathbb{R}^2$

$A=\{(x,y): 0\leq x\leq 1 , y=\frac{x}{n} , n\in\mathbb{N}\}$, $B=\{(x,0): \frac{1}{2}\leq x\leq 1\}$.



Here A consists of the points on the line segments joining the origin (0,0) to the points $\left(1,\frac{1}{n}\right)$, $n\in\mathbb{N}$ and B consists of points on the x - axis between $\frac{1}{2}$ and 1. Now A and B are both arcwise connected ,hence each also connected. Also A and B are not separated since each $p\in B$ is a limit point of A and so $A\cup B$ is connected.But $A\cup B$ is not arcewise connected since there is no path from any point in A to any point in B.

4.1.31 Theorem:

If $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ is a continuous function then f maps every arcwise connected subset of X onto an arcwise connected subset of X^*.

Proof:

Suppose E is an arcwise connected subset of X, and $x^*$and $y^*$are any two points of $f(E)$.There must exist points x and y in A such that $f(x)= x^*$and $f(y)=y^*$.Since E is arcwise connected, there exists a path g in E joining x and y , i.e. a continuous function g from I into E such that $g(0)=x$ and $g(1)=y$.By Corollary 4.1.15, we have $f\circ g$ is a continues function from I into $f(E)$ such that $(f\circ g)(0)= x^*$and $(f\circ g)(1)= y^*$.Thus $f\circ g$ is a path in $f(E)$ joining$x^*$and y^* and $f(E)$ **)** must be arcwise connected.□

4.1.32 Remark:

Although very few properties of sets are preserved by continuous transformations, many of the important properties are preserved if we put additional restrictions on the function. The following is an example of a property that is preserved if we merely add the restriction of one-to-oneness.

4.1.33 Definition:

A subset E of a topological space (X,τ) is ***dense-in-itself*** if every point of E is a limit point of E, i.e. $E \subseteq d(E)$.

4.1.34 Theorem:

If f is a one-to-one continuous function of (X,τ) into (X^*,τ^*) then f maps every dense-in-itself subset of X onto a dense-in-itself subset of X^*.

Proof:

Suppose E is a dense – in –itself subset of X. We want to show that $f(E)$ is dense – in –itself , i.e. $f(E) \subseteq d(f(E))$.

$Let\ x^* \in f(E)$, G^*open in X^*,s.t.$x^* \in G^*$ then $\exists x \in E$, s. t. $f(x) = x^*$.Now $x \in f^{-1}(\{x^*\}) \subseteq f^{-1}(G^*)$ and $f^{-1}(G^*)$ is an open set since f is continuous. But E is dense-in-itself, so $x \in E \subseteq d(E)$.Thus x is a limit point of the set E which is contained in the open set $f^{-1}(G^*)$, and so, by the definition of limit point,$E \cap f^{-1}(G^*)/\{x\} \neq \emptyset$. Since this set is nonempty, let us choose a point $z \in E \cap f^{-1}(G^*)/\{x\}$. Since z is in this intersection, it is in each part. Thus, z ∈ E , and so $f(z) \in f(E)$, while $z \in f^{-1}(G^*)$, and so $f(z) \in f(f^{-1}(G^*)) \subseteq G^*$. Finally, $z \neq x$, and so $f(z) \neq f(x) = x^*$ since f is one-to-one. This shows that $f(z) \in f(E) \cap G^*/\{x^*\}$, and so $f(E) \cap G^*/\{x^*\} \neq \emptyset$, as desired. □

Exercise:

Show that if D is a dense-in-itself set, $\overline{D}$ is dense-in -itself, and any set E such that $D \subseteq E \subseteq d(E)$ is also dense-in-itself. Furthermore, the union of any family of dense-in-itself sets is dense-in- itself.

4.1.35 Definition:

Let E be a subset of a topological space (X,τ), ***the nucleus*** of E is defined to be the union of all dense-in-itself subsets of E and is clearly the largest set contain in E and dense-in-itself.

4.1.36 Definition:

A subset E of a topological space (X,τ) is whose nucleus is empty is called ***scattered.***

4.1.37 Definition:

A subset E of a topological space (X,τ) is called ***perfect*** if it's both closed and dense-in-itself (i.e. $E = d(E)$).

4.1.38 Theorem:

If f is a one-to-one continuous function of (X,τ) into (X^*,τ^*) then f maps every scattered subset of X onto a scattered subset of X^*.

Proof:

Suppose E is a scattered subset of X. We want to show that $f(E)$ is scattered. Since E is scattered then their nucleus is empty set ,i.e, $\bigcup_i G_i = \emptyset$, where $\forall i, G_i \subseteq E$ is dense-in-itself .Since f is one-to-one and continuous then by theorem 4.1.34 we get $\forall i, f(G_i)$ is dense-in-itself . Since f is one-to-one and $\bigcup_i G_i = \emptyset$ then $\bigcup_i f(G_i) = \emptyset$, so the nucleus of $f(E)$ is empty set, i.e. $f(E)$ is scattered. □

4.2 Open and Closed Functions

4.2.1 Definition:

A function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is called an ***open function*** if the image of every open set is open.

4.2.2 Definition:

A function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is called a ***closed function*** if the image of every closed set is closed.

4.2.3 Remark:

In general, functions which are open(closed) need not be closed (open) even if they are continuous as the following example:

4.2.4 Example:

Let (X,τ) be any topological space and let (X^*,τ^*) be the space for which $X^*=\{a,b,c\}$ and $\tau^* = \{\emptyset,\{a\},\{a,c\},X^*\}$. The function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ defined by $f(x) = a$, $\forall x \in X$ is a continuous open map which is not closed. Since the image of every open set G in X is $\{a\}$open in X^*but the image of every closed set F in X is $\{a\}$which is not closed in X^*.

If $g: (X,\tau) \longrightarrow (X^*,\tau^*)$ defined by $g(x) = b, \forall x \in X$ is a continuous closed map which is not open. Since the image of every open set G in X is $\{b\}$which is not open in X^*but the image of every closed set F in X is $\{b\}$which is closed in X^*.

4.2.5 Example:

Give an example of a real function $f: (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau)$ such that f is continuous and closed, but not open.

Solution:

Let $f: (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau^*)$ be a constant function, $f(x) = 1$, $\forall x \in \mathbb{R}$. Then $f(A) = \{1\}$ for any $A \subseteq \mathbb{R}$. Hence if A is open then $f(A) = \{1\}$ is not open, so f is not open function and if if A is closed then $f(A) = \{1\}$ is closed, so f is closed function (since singleton sets are closed in the usual topology).Also by example 4.1.8, f is continuous on $\mathbb{R}$.

4.2.6 Example:

Let the real function $f: (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau)$ be defined by $f(x) = x$, $\forall x \in \mathbb{R}$. Show that f is not open.

Solution:

Let $A = (-1,1)$ be an open set. Note that $f(A) = [0,1)$,which is not open hence f is not an open function.

4.2.7 Example:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a function from any topological space (X,τ) to the discrete topology (X^*,τ^*) then f is open function.

Solution:

Let $G \in \tau$ then $f(G) \subseteq X^*$, since X^* discrete topology then $f(G) \in \tau^*$,i.e f is open function.

4.2.8 Remark:

1. Let (X,τ),(X^*,τ) be the discrete topologies then the function $f:(X,\tau) \longrightarrow (X^*,\tau)$ is continuous , open and closed function.
2. Let (X,τ) be the discrete topologies and (X,τ^*) be the indiscrete topology , X contain more than one point then the function $f:(X,\tau) \longrightarrow (X^*,\tau)$ is continuous function not open and not closed function.
3. Let (X,τ) be the indiscrete topologies and (X,τ^*) be the discrete topology , X contain more than one point then the function $f:(X,\tau) \longrightarrow (X^*,\tau)$ is open and closed function not continuous.

4.2.9 Example:

Let the functions $f:(\mathrm{X},\tau) \longrightarrow (\mathrm{X}^*,\tau^*)$ and $\mathrm{g}:(\mathrm{X}^*,\tau^*) \longrightarrow (\mathrm{X}^{**},\tau^{**})$ be open functions then the composition $g \circ f:(\mathrm{X},\tau) \longrightarrow (\mathrm{X}^{**},\tau^{**})$ is an open function.

Solution:

Let $G \in \tau$ then $f(G) \in \tau^*$since f is an open function and $g\big(f(G)\big) \in \tau^{**}$ since g is an open function then $g \circ f$ is an open function.

4.2.10 Theorem:

A function $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is open iff $f(E^\circ) \subseteq f(E)^\circ$ for every $E \subseteq X$.

Proof:

Suppose f is open and $E \subseteq X$.Since $E^\circ$is an open set and f is an open function, then $f(E^\circ)$ is an open set in X^*. Since $E^\circ \subseteq E$, $f(E^\circ) \subseteq f(E)$. Thus $f(E^\circ)$ is an open set contained in $f(E)$, and hence $f(E^\circ) \subseteq f(E)^\circ$.

Conversely, if G is an open set in X and $f(G^\circ) \subseteq f(G)^\circ$ for all $E \subseteq X$ then $f(G) = f(G^\circ) \subseteq f(G)^\circ$, and so $f(G)$ an open set in X^*.□

4.2.11 Theorem:

A function $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ is closed iff $\overline{f(E)}\subseteq f(\bar{E})$ for every $E\subseteq X$.

Proof:

Suppose f is closed and $E\subseteq X$.Since $\bar{E}$ is closed set and f is closed function, then $f(\bar{E})$ is a closed set in X^*. Since $E\subseteq\bar{E}$,$f(E)\subseteq f(\bar{E})$. Thus$f(\bar{E})$ is a closed set contain $f(E)$, and hence $\overline{f(E)}\subseteq f(\bar{E})$.

Conversely, if F is a closed set in X and $\overline{f(F)}\subseteq f(\bar{F})$ for all $F\subseteq X$ then $\overline{f(F)}\subseteq f(\bar{F})=f(F)$, and so $f(F)$ closed set in X^*.□

4.2.12 Theorem:

Let $\mathcal{B}$ be a base for a topological space (X,τ).Show that if function $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ has the property that $f(B)$ is open for every $B\in\mathcal{B}$ then f is an open function.

Proof:

We want to show that the image of every open subset of X is open in X^*. Let $G\subseteq X$ be open. By definition of a base $G=\bigcup_i B_i$ where $B_i\in\mathcal{B}$.Now $f(G)=f(\bigcup_i B_i)=\bigcup_i f(B_i)$. By hypothesis, each $f(B_i)$ is open in $X^*$and so $f(G)$ a union of open sets in X^*, hence f is an open function.□

4.3 Homeomorphisms

4.3.1 Definition:

Let $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ be function from a topological space (X,τ) to the topological space (X^*,τ^*), f is said to be a ***homeomorphism*** if it satisfy the following:

1. f is one to one.
2. f is onto.
3. f is an open function (i.e. f^{-1} is a continuous function)
4. f is a continuous function.

4.3.2 Remark:

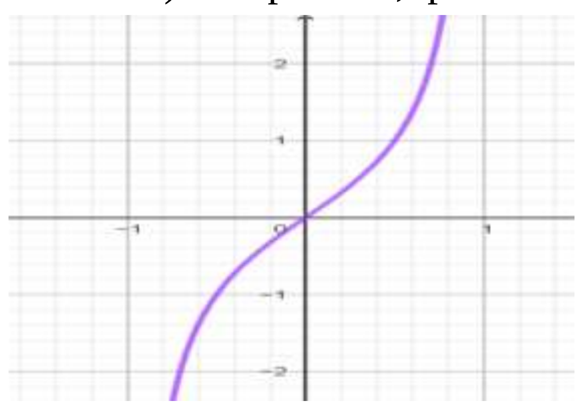
If there exists a homeomorphism between (X,τ) and (X^*,τ^*),we say that X and $X^*$are ***homotopic*** or ***topologically equivalent*** denote by $X\cong X^*$.

4.3.3 Definition:

A property p of sets is called ***topological*** or a ***topological invariant*** if whenever a topological space (X,τ) has p then every space homeomorphic to (X,τ) also has p.

4.3.4 Example:

Let $X=\{a,b,c\}$,$X^*=\{1,2,3\}$,$\tau=\{X,\emptyset,\{a\},\{c\},\{a,c\}\}$ and $\tau^*=\{X^*,\emptyset,\{1\},\{3\},\{1,3\}\}$. Define $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ by $f(a)=1$,$f(b)=2$,$f(c)=3$. The function f is a homeomorphism since it is a bijection (1-1 and onto) on points ,open and continuous function.



4.3.5 Example:

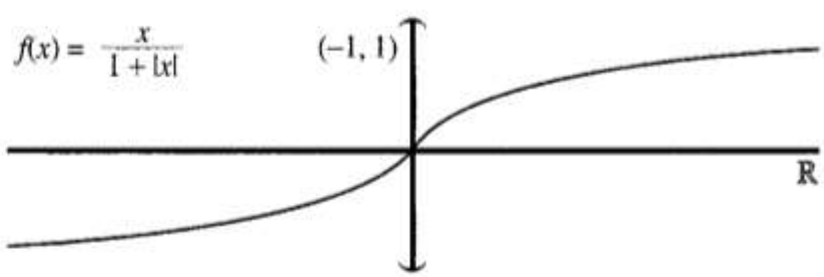
Show that $X=(-1,1)\cong\mathbb{R}$.

Solution:

Define $f:(-1,1)\longrightarrow\mathbb{R}$ by $f(x)=tan\frac{1}{2}\pi x$.
f is one to one, onto, continuous function and open function. Hence $(-1,1)\cong\mathbb{R}$.

4.3.6 Remark:

1. We can use function $f:\mathbb{R}\longrightarrow(-1,1)$ by $f(x)=\frac{x}{1+|x|}$.From the graph of f is shown f is one to one, onto, continuous function and open function. Hence $(-1,1)\cong\mathbb{R}$.



2. Example 4.3.5 shows that the length and boundness is not homeomorphism since $\mathbb{R}$ is unbounded but $(-1,1)$ is bounded and its length is 2.

<u>4.3.7 Remark:</u>

Not only $(-1,1)$ is homomorphic to $\mathbb{R}$, but every nonempty open interval (a,b) is as well. Now consider the following collections of intervals with the usual topology (assume a and b are arbitrary real numbers with $a < b$):

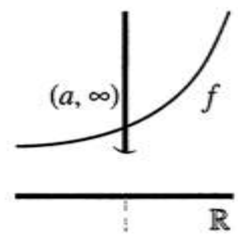
1) Open intervals (a,b),(-∞,a),(a,∞),$\mathbb{R}$.

2) Closed bounded intervals [a,b].

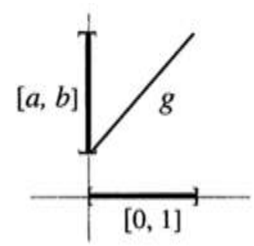
3) Half – open intervals and closed unbounded intervals $[a,b)$,(a,b],(-∞,a],[a,∞).

Each of the collections 1),2) and 3) all of the spaces are topologically equivalent.

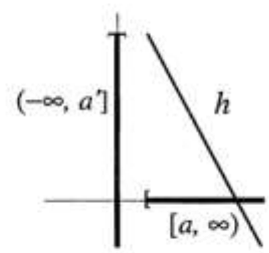
The function $f: \mathbb{R} \longrightarrow (a,\infty)$ defined by $f(x) = e^x + a$ is a homeomorphism. Thus $\mathbb{R}$ is homeomorphism to every interval (a,∞). Since topological equivalence is an equivalence relation, it also follow that every interval (a,∞) is homeomorphic to every other intervals in the form (a',∞).



The linear function $g : [0,1] \longrightarrow [a,b]$ given by $g(x) = (b-a)\,x + a$ is a homeomorphisms between $[0,1]$ and $[a,b]$. Therefore every interval $[a,b]$ is homeomorphic to $[0,1]$ and consequently every interval $[a,b]$ is homeomorphic to every other closed interval $[a',b']$ with $a' < b'$.



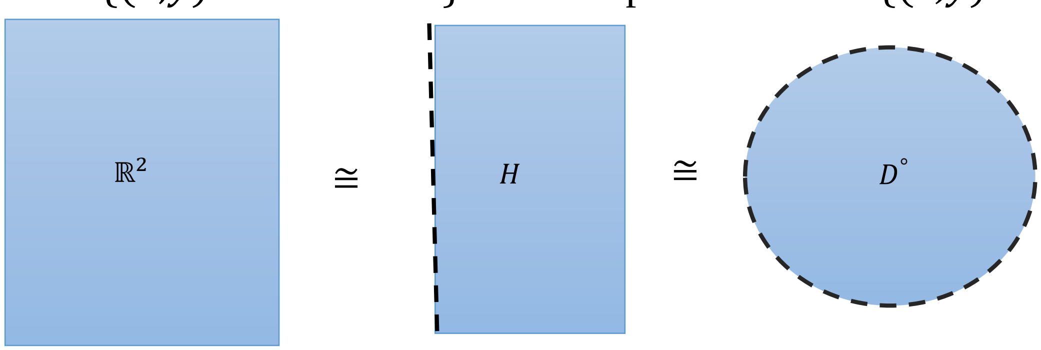
The function $h : [a,\infty) \longrightarrow (-\infty,a']$ given by $h(x) = -x + a' + a$ is a homeomorphism between intervals $[a,\infty)$ and $(-\infty,a']$. Thus if I_1 and I_2 are intervals of either form $[a,\infty)$ or $(-\infty,a']$.Then I_1 and I_2 are homotopic.



<u>4.3.8 Example:</u>

The usual topology on each ,the plane $\mathbb{R}^2$is topologically equivalent to the open right half plane $H = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and the open disk $D^{\circ} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

<u>Solution:</u>



The function $f: \mathbb{R}^2 \longrightarrow H$, defined by f(x,y)=(e^x,y) is a homeomorphism between $\mathbb{R}^2$ and H. It maps $\mathbb{R}^2$ to H, sending vertical lines to vertical lines as followings:

1) The left half plane is mapped to the strip in H where $0 < x < 1$.

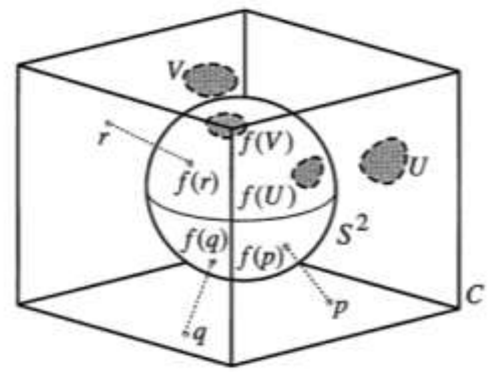
2) The y-axis is mapped to the line $x = 1$.

3) The right half plane is mapped to the region in H where $x > 1$.

The function $g: \mathbb{R}^2 \longrightarrow D^\circ$, defined by f(r,θ)=($\frac{r}{1+r}$,θ) is a homeomorphism between $\mathbb{R}^2$ and D°.It contracts the whole plane radially inwards to coincidence with the open disk D°.

4.3.9 Example:

The surface of cube C is homeomorphic to the sphere S^2. If we regard each as centered at the origin origin in 3- space the function $f: C \longrightarrow S^2$defined by $f(p) = \frac{p}{|p|}$ is a homeomorphism. f maps points in C bijectively to points in $S^2$and maps the collection of the open sets in C bijectively to the collection of open sets in S^2.



4.3.10 Example:

Let X be the set of positive real numbers ,i.e. $X = (0,\infty)$. The function $f: X \longrightarrow X$ defined by $f(x) = \frac{1}{x}$ is a homeomorphism from X to X.

4.3.11 Remark:

In example 4.3.10 if we take the cushy sequence $\langle a_n \rangle = \langle 1,\frac{1}{2},\frac{1}{3},\ldots \rangle$ then the corresponds$\langle f(a_n) \rangle = \langle f(1) = 1, f(\frac{1}{2})=2, f(\frac{1}{3})=3, \ldots \rangle$ under the homeomorphism is not a cushy sequence, hence the property of being a cushy sequence is not topological.

4.3.12 Example:

Show that area is not a topological property.

Solution:

1. The open disk $D = \{(r,\theta): r < 1\}$ with radius 1 is homeomorphism to the open $D^\circ = \{(r,\theta): r < 2\}$ with radius 2.The function $f: D \longrightarrow D^\circ$ defined by $f\big((r,\theta)\big)=(2r,\theta)$ is a homeomorphism. Here (r,θ) denotes the polar coordinates of a point in the plane $\mathbb{R}^2$ the area of D is $r^2\pi \neq 4r^2\pi$ the area of D°.

4.3.13 Remark:

1. From remarks 4.3.6 and 4.3.11and example 4.3.12 show that the length, boundness, area and cushy sequence are not homeomorphism.

2. Let (X,τ) and (X*,τ*) be discrete topological spaces then from examples 4.1.5 and 4.2.7 every bijective (one to one and onto) functions f: (X,τ) ⟶ (X*,τ*) are homeomorphism.

4.3.14 Example:

Let f: (X,τ) ⟶ (X*,τ*) be a one to one and open function , let $A \subset X$, and let $f(A) = B$.Show that the function f_A: (A,τ_A) ⟶ (B,τ_B^*) is also one to one and open function. Here f_A denote the restriction of f to A and τ_A and τ_B^* are relative topologies.

Solution:

If f is one to one then every restriction of f is also one to one, hence we need only show that f_A is open.

Let $H \subset A$ be τ_A – open. Then by definition of the relative topology, $H^* = A \cap G$ where $G \in \tau$. Since f is one to one $f(A \cap G) = f(A) \cap f(G)$, and so

$$f_A(H) = f(H) = f(A \cap G) = f(A) \cap f(G) = B \cap f(G).$$

Since f is open and $G \in \tau$, $f(G) \in \tau_B^*$ then $B \cap f(G) = \tau_B^*$ and so $f_A$is open.

4.3.15 Example:

Let f: (X,τ) ⟶ (X*,τ*) be a homomorphism and let (A,τ_A) be any subspace of (X,τ). Show that f_A: (A,τ_A) ⟶ (B,τ_B^*) is also a homomorphism where f_A is the restriction of f to A ,$f(A) = B$, and τ_B^* is the relative topology on B.

Solution:

Since f is one to one and onto, f_A: (A,τ_A) ⟶ (B,τ_B^*),where $f(A) = B$ is also one to one and onto. Hence we need only show that $f_A$is continuous and open function. By example 4.3.14 f_A is open and the restriction of any continuous function is also continuous hence f_A is a homeomorphism.

4.3.16 Theorem:

The perfect property is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a topological space (X,τ) to the topological space (X*,τ*) and $let\ E\ be\ a\ perfect$ (closed and dense in itself) subset of X, we want to prove that $f(E)$ is perfect subset of X*.

By theorem 4.1.34 $f(E)$ is dense itself. Since E is closed subset of X then $E^c$is open in X. Since f is open function then $f(E^c)$ is open set in X*.Since f is bijective then $f(E^c) = f(E)^{\ c}$, so $f(E)$ is closed in X*, i.e. perfect set in X*.□

4.3.17 Theorem:

The locally compact set property is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a topological space (X,τ) to the topological space (X*,τ*) and let E be a locally compact set in X we want to prove that $f(E)$ is a locally compact subset of X*.

Let $x^* \in f(E)$, since f is onto then $\exists x \in E,$ s.t. $f(x) = x^*$.Since E is locally compact set in X then there exists a compact neighborhood G for x. Since f is open function and G is compact then $f(G)$ is a compact neighborhood for x^* in $f(E)$, so $f(E)$ is a locally compact subset of X*.□

4.3.18 Definition:

A subset E of a topological space is ***isolated*** iff no point of E is a limit point of E that is, if $E \cap d(E) = \emptyset$.

4.3.19 Example:

Let $X = \{a$,b,c,d,$e\}$ and τ = {∅,X,{a},{a,b},{a,c,d},{a,b,c,d},{a,b,e}} then $E = \{c,e\}$ is isolated set since $d(E) = \{d\}$ and $E \cap d(E) = \emptyset$.

4.3.20 Theorem:

The isolated property is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a topological space (X,τ) to the topological space (X*,τ*) and let E is isolated set in X we want to prove that $f(E)$ is isolated subset of X*.

Let $x^* \in f(E)$, since f is onto then $\exists x \in E$, s.t. $f(x) = x^*$.Since E is isolated then $x \notin d(E)$ then there exists an open set G containing x such that $G/\{x\} \cap E = \emptyset$. But f is a homeomorphism, and so f(G) is an open set in X*which contains $f(x) = x^*$. From the fact that f is one-to-one it follows that $f(E) \cap f(G)/\{x^*\} = \emptyset$, i.e $x^* \notin d(f(E))$.□

4.3.21 Theorem:

The countably compact property is a topological property.

Proof:

Let f: (X,τ) ⟶ (X^*,$τ^*$) be a homeomorphism from a topological space (X,τ) to the topological space (X^*,$τ^*$) and let E is countably compact set in X we want to prove that $f(E)$ is countably compact subset of X^*.

Assume that A^* be infinite subset of $f(E)$. Since f is bijective then there exists an infinite subset of E such that $f(A) = A^*$.Since A is countably compact set then it has a limit point x in E ($x \in E$,$x \in d(E)$).

Since f is open and one to one function then $x^* = f(x) \in f(E)$,$x^* \in d(f(A))$,so $A^* = f(A)$ has a limit point in $f(E)$, i.e. $f(E)$ is countably compact. □

4.3.22 Theorem:

The locally connected property is a topological property.

Proof:

Let f: (X,τ) ⟶ (X^*,$τ^*$) be a homeomorphism from a topological space (X,τ) to the topological space (X^*,$τ^*$) and let E is locally connected set in X we want to prove that $f(E)$ is locally connected subset of X^*.

Let $x^* \in f(E)$ and G^*open subset of $f(E)$ contain x^*.Since f is onto then $\exists x \in E$, s.t. $f(x) = x^*$, so $x \in f^{-1}(G^*)$. Since f is continuous then $f^{-1}(G^*)$ is open subset of E. Since $E = f^{-1}(f(E)) \subseteq f^{-1}(G^*)$, by theorem 4.1.4.

Since E is locally connected and $x \in f^{-1}(G^*) \subseteq E$ then there exists an open connected G such that $x \in G \subseteq f^{-1}(G^*)$, so by theorem 4.1.4 we get $f(x) \in f(G) \subseteq f(f^{-1}(G^*)) \subset G^*$.Since f is onto and $f(G)$ is connected by theorem 4.1.22,so $f(E)$ is locally connected. □

4.4 Hereditary Properties

4.4.1 Definition:

A property P of a topological space (X,τ) is said to be ***hereditary*** iff every subspace of X also possesses property P.

4.4.2 Example:

A property of being a topological space a discrete topological spaces is a hereditary property.

Solution:

Let (Y,τ_Y) be a subspace of a discrete topological space (X,τ) we want to show that (Y,τ_Y) is also a discrete topological space.

Let $A^* \subset Y \subset X$ and let $A^* = A \cap Y$. Now $A \subset X$ and X is a discrete topology then $A \in \tau$ .Since (Y,τ_Y) is a subspace of (X,τ) then $A^* \in \tau_Y$, i.e. (Y,τ_Y) is a discrete topological space.

4.4.3 Example:

A property of being a topological space an indiscrete topological spaces is a hereditary property.

Solution:

Let (Y,τ_Y) be a subspace of an indiscrete topological space (X,τ) then (Y,τ_Y) is also an indiscrete topological space, since the only open sets in X are $X,\emptyset$ and their intersect with Y are $Y,\emptyset$.

4.4.4 Definition:

A subset ***E*** of a topological space $(\boldsymbol{X,\tau})$ will be called ***dense*** in ***X*** iff $\bar{E} = X$.

4.4.5 Example:

Consider the topology $\tau = \{\emptyset$,X,{b,c,d,e},{a,b,e},{b,e},{a},X} on $X = \{a$,b,c,d,e} then $\{a,c\}$ is a dense subset of X ,since $\overline{\{a,c\}} = X$ but $\{b,d\}$ is not dense since $\overline{\{b,d\}}=\{b,c,d,e\}$.

4.4.6 Example:

The usual topology $(\mathbb{R},\tau)$ the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$, since $\overline{\mathbb{Q}} = \mathbb{R}$.

4.4.7 Example:

Let (X,τ) be the discrete topology then X is the only dense set in X, Since every A$\subset X$, A is closed and $\bar{A} = A$.

4.4.8 Definition:

A topological space (X,τ) will be called ***separable*** iff it satisfies the following condition:

[S] There exists a countable dense subset of X**.**

4.4.9 Example:

In example 4.4.6 we show that $\mathbb{Q}$ is dense in the usual topology $(\mathbb{R},\tau)$ and since $\mathbb{Q}$ is countable then $\mathbb{R}$ is a separable space.

4.4.10 Example:

Let (X,τ) be the co-finite topology. Show that (X,τ) is separable, i.e. contains a countable dense subset.

Solution:

If X is countable then X is a countable dense subset of (X,τ). On the other hand, suppose X is not countable then X contains a non-finite countable subset A. Since the closed sets in X are the finite sets then the closure of the non-finite set A is the space X, i.e. $\bar{A} = X$. Gut A is countable hence (X,τ) is separable.

4.4.11 Example:

Let $(\mathbb{R},\tau)$ be the discrete topology. Since every subset of $\mathbb{R}$ is both open and closed so the only dense subset of $\mathbb{R}$ is $\mathbb{R}$ itself. But $\mathbb{R}$ is not countable set, hence $(\mathbb{R},\tau)$ is not a separable space.

4.4.12 Example:

A discrete topological space (X,τ) is separable iff X is countable.

Solution:

Since every subset of a discrete topological space (X,τ) is both open and closed then the only subset of X is X itself. Hence X contains a countable dense subset iff X is countable, i.e. X is separable iff X is countable.

4.4.13 Example:

Let τ be the topology on the real line $\mathbb{R}^2$ generated by the half- open rectangles, [a,b)×[c,d)={(x,y):a≤x<b,c≤y $< d$}. Show that $(\mathbb{R}^2,\tau)$ is seperable.

Solution:

Now there are always rational numbers x_0 and y_0 such that $a < x_0 < b$ and $c < y_0 < d$, so the above open rectangle contains the point $p = (x_0\ ,y_0)$ with rational coordinates. Hence the set $A = \mathbb{Q} \times \mathbb{Q}$ consisting of all points in $\mathbb{R}^2$with rational coordinates is dense in $\mathbb{R}^2$.But A is a countable set thus $(\mathbb{R}^2,\tau)$ is seperable.

4.4.14 Theorem:

***The separable property is a topological property*.**

Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a separable topological space (X,τ) to the topological space (X^*,τ^*), we want to prove that X^* is separable space.

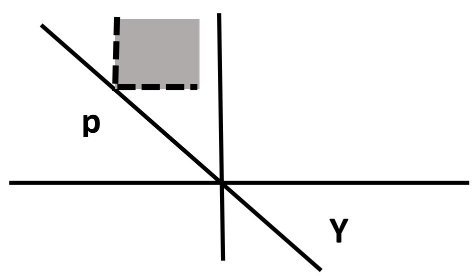
Since X is a separable then there exists a countable A subset of X such that $\bar{A} = X$. Now since f is a homeomorphism then $f(A)$ is countable subset of $X^*$and $X^* = f(X) = f(\bar{A}) \subseteq \overline{f(A)}$.So $X^* = \overline{f(A)}$, i.e. $f(A)$ is dense in X^*, i.e. (X^*,τ^*) is separable space.□

4.4.15 Example:

Show that by a counterexample that a subspace of a separable space need not be separable, i.e. reparability is not a hereditary property.

Solution:

Consider the separable topological space $(\mathbb{R}^2,\tau)$ in example 4.4.13 and let $Y = \{(x,y): x + y = 0\}$ be a subset of $(\mathbb{R}^2,\tau)$ then τ_Y the relative topology is the discrete topology since each singleton $\{p\}$ of Y is τ_Y- open. But an uncountable space is not separable. Thus the reparability of $(\mathbb{R}^2,\tau)$ is not inherited by the subspace (Y,τ_Y).



4.4.16 Example:

Show that by a counterexample that a subspace of compact space need not be compact, i.e. compactness is not a hereditary property.

Solution:

The closed interval [0,1] is compact subset in the usual topology $(\mathbb{R},\tau)$ since its closed and bounded (by Hein Boral theorem) but the subset (0,1) of [0,1] is not compact. Thus compactness is not a hereditary property.