Chapter Two

Creating New Topological Spaces

2.1 The Subspace Topology

Let (X,τ) be a topological space, A be a proper subset of X. Let $\tau^* = \{G^* = G \cap A : G \in \tau\}$,i.e. $G^* \in \tau^* \Leftrightarrow \exists G \in \tau , G^* = G \cap A$. The following theorem shows that τ^* is a topology on A called the ***Relative Topology*** **(or *Induced Topology*)** and (A,τ^*) is called the ***Subspace Topology*** of topological space (X,τ).

2.1.1 Theorem:

Let (X,τ) be a topological space A be a proper subset of X. Then $\tau^* = \{G^* = G \cap A : G \in \tau\}$ is a topology on A.

Proof:

1) $\emptyset = \emptyset \cap A \Rightarrow \emptyset \in \tau^*$

$A = X \cap A \Rightarrow A \in \tau^*$.

2) Let $G_1^*, G_2^* \in \tau^*$ then $\exists G_1, G_2 \in \tau$ s. t. $G_1^* = G_1 \cap A, G_2^* = G_2 \cap A$ then

$G_1^* \cap G_2^* = (G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A \in \tau^*$ since $(G_1 \cap G_2) \in \tau$.

3) Let $\{G_i^* : i \in I\} \subseteq \tau^*$ then $\exists G_i \in \tau$ s. t. $G_i^* = G_i \cap A, \forall i \in I$. So

$\cup_i G_i^* = \cup_i (G_i \cap A) = \cup_i G_i \cap A \in \tau^*$, since $\cup_i G_i \in \tau$.

So τ^* is a topology on A.□

2.1.2 Example:

Let (X,τ) be a topological space where $X = \{a,b,c,d,e\}$, $\tau = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$ .Find τ_A, τ_B, τ_C , $A = \{a,d\}$, $B = \{a,b,c\}$, $C = \{a\}$.

Solution:

$X \cap A = A$, $\emptyset \cap A = \emptyset$, $\{a\} \cap A = \{a\}$, $\{c,d\} \cap A = \{d\}$, $\{a,c,d\} \cap A = A$, $\{b,c,d,e\} \cap A = \{d\}$

So $\tau_A = \{A, \emptyset, \{a\}, \{d\}\}$. Similar $\tau_B = \{B, \emptyset, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$, $\tau_C = \{C, \emptyset\}$.

2.1.3 Remark:

In example 2.1.2 , τ_A is the discrete topology on A , τ_C is the indiscrete topology on C but τ is not discrete or indiscrete topology on X. Also we can find {d}$\in\tau_A$ but {d}$\notin\tau$.

2.1.4 Example:

The subspace of discrete topology (indiscrete topology) is also a discrete topology (indiscrete topology).

2.1.5 Example:

Let (X, τ) be a co-finite topology and let $A \neq \emptyset$ be a subset of X the τ_A is the discrete topology.

Solution:

Let p be any point in A then the set $X\backslash\{A\backslash\{p\}\}$ is open in X and their intersect with A is $\{p\}$ i.e. $A \cap (X\backslash\{A\backslash\{p\}\}) = \{p\}$ is open in A .Since p be any point in A then the subspace topology on A is the discrete topology.

2.1.6 Example:

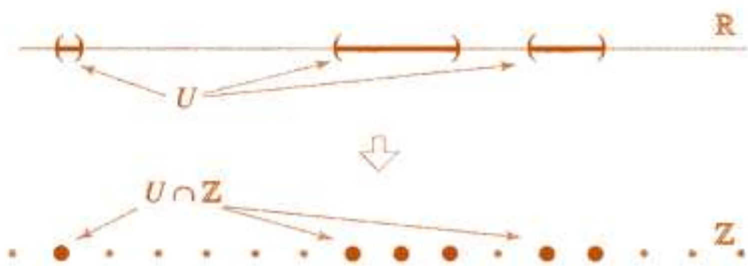
Let $(\mathbb{R}, D)$ be the usual topology on $\mathbb{R}$ then the subspace topology $(\mathbb{N}, D_{\mathbb{N}})$ is the discrete topology.

Solution:

Let $n \in \mathbb{N}$ then $(n-\frac{1}{2}, n+\frac{1}{2})$ is an open interval contain n and $\mathbb{N} \cap (n-\frac{1}{2}, n+\frac{1}{2})$ ={n}.So every {n} contain a natural number in the subspace $(\mathbb{N}, D_{\mathbb{N}})$,so every subset of $\mathbb{N}$ is an open set i.e. $D_{\mathbb{N}}$ is the discrete topology.

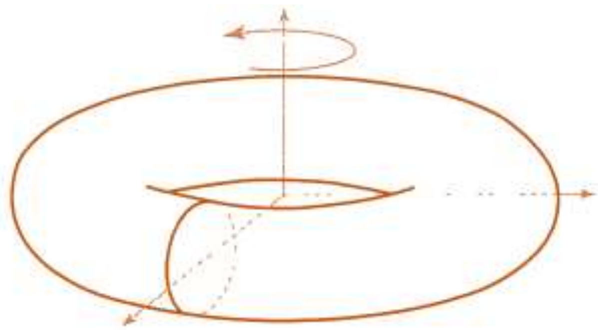
2.1.7 Example:

Let $(\mathbb{R}, D)$ be the usual topology on $\mathbb{R}$ then the subspace topology $(\mathbb{Z}, D_{\mathbb{Z}})$ is the discrete topology.



2.1.8 Example:

In $\mathbb{R}^3$, let C be the circle of radius 1 in the xy-plane with center at the point (2,0,0).Consider the subspace of $\mathbb{R}^3$swept out as C is rotated about the z-axis the resulting space is called the torus and denoted by T which is a subspace of $\mathbb{R}^3$.



2.1.9 Theorem:

Let (A,τ_A) be a subspace of (X,τ) then the subset E of A is closed in (A,τ_A) iff there exist a closed set F in (X,τ) such that $E = F \cap A$.

Proof:

$\Rightarrow$

Let E be a closed in (A,τ_A) the E^c is an open set in (A,τ_A) .By definition of subspace $\exists G \in \tau$ s.t. $E^c = A \cap G = A \backslash E$. So

$$E = A \backslash E^c = A \backslash (A \cap G) = A \cap (A \cap G)^c = A \cap G^c.$$

Put $E^c = F$ which is the closed set we want to find.

$\Leftarrow$

Assume there exist a closed set F in (X,τ) such that $E = F \cap A$ we want to prove that E is closed in (A,τ_A) i.e. E^c is an open set in (A,τ_A)

$$E^c = A \backslash E = A \backslash (A \cap F) = A \cap (A \cap F)^c = A \cap (A^c \cup F^c) = (A \cap A^c) \cup (A \cap F^c) = A \cap F^c.$$

So E^c is an open set in (A,τ_A).□

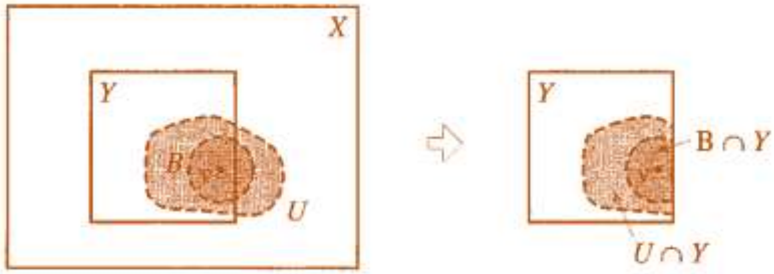
2.1.10 Corollary:

If A is a non-empty open (closed) subset of (X,τ) then the subset B of A is open (closed) in (A,τ_A) iff B an open set F in (X,τ).

2.1.11 Theorem:

Let (Y,τ_Y) be a subspace of (X,τ). If $\mathcal{B} = \{B_i\}_{i\in I}$ is a base for (X,τ) then $\mathcal{B}^* = \{B_i \cap Y\}_{i\in I}$ is a base for (Y,τ_Y).

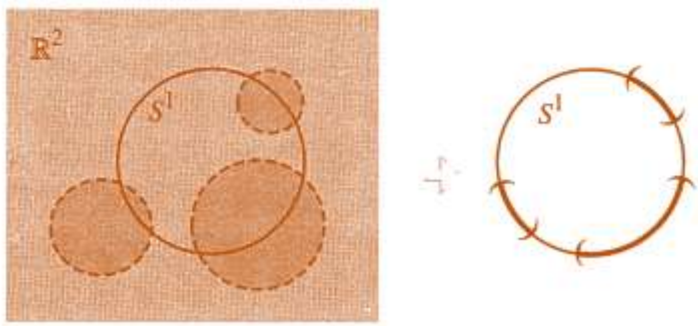
Proof:



Assume $\mathcal{B} = \{B_i\}_{i\in I}$ is a base for (X, τ) then $\forall U \in \tau, y \in U \Rightarrow \exists B \in \mathcal{B}, y \in B \subseteq U$. From definition of subspace the family $\{B_i \cap Y\}_{i\in I}$ is open in (Y, τ_Y).If $y \in Y$ then $y \in B \cap Y \subseteq U \cap Y$ where $U \cap Y \in \tau_Y$ then $\{B_i \cap Y\}_{i\in I}$ is a base for (Y, τ_Y).□

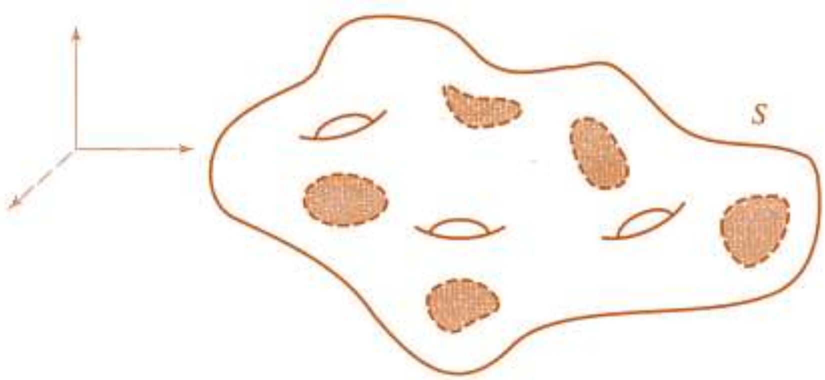
2.1.12 Example:

Let the circle $S^1 \subseteq \mathbb{R}^2$ with the usual topology. Since the class of open balls form a basis for the usual topology on $\mathbb{R}^2$then their intersection with S^1 are class of open intervals in the circle consisting of all points between two angles in the circle .This class form a base for the usual topology on S^1.



2.1.13 Example:

If S is a surface in $\mathbb{R}^3$ then the collection of open patches in S obtained by intersecting open balls in $\mathbb{R}^3$with S is a basis for the standard topology on S.



2.1.14 Remark:

The following theorem gives the relation between the limit and interior points and the closure of sets in subspaces and spaces .we denote $d(A)$,A°,$\bar{A}$ for limit ,interior ,closure for a set A in subspace.

2.1.15 Theorem:

Let (Y,τ_Y) be a subspace of (X,τ). If $A \subseteq Y$ then :

1) $d(A_Y) = d(A) \cap Y$.
2) $A^{\circ} = A_Y^{\circ} \cap Y^{\circ}$, $A^{\circ} \cap Y = A_Y^{\circ}$.
3) $\overline{A_Y} = \bar{A} \cap Y$.

Proof:

1) Assume $x \in d(A_Y)$ then $\forall U \in \tau_Y$,$x \in U$,U∩A≠∅ then $\exists W \in \tau$,$x \in U$,U=W∩Y. So for any $W \in \tau$ s.t. $x \in W$ we find W∩Y≠∅ therefore we get (W∩Y)∩A=W∩A≠∅ i.e. $x \in d(A)$,SO

$$d(A_Y) \subseteq d(A) \qquad \ldots\ldots. \qquad (1)$$

Let $x \in d(A)$ then $\forall U \in \tau$,$x \in U$,U∩A≠∅.Its clear that W=U∩Y $\in \tau_Y$ is an open set in (Y,τ_Y),SO $W \cap A = (U \cap Y) \cap A = Y \cap (U \cap A \neq \emptyset$ *i.e.* $x \in d(A_Y)$

$$d(A) \subseteq d(A_Y) \qquad \ldots\ldots.. \qquad (2)$$

From (1) and (2) we get $d(A_Y) = d(A) \cap Y$.

2) Let $p \in A^{\circ}$them $\exists H \in \tau$ s.t. p∈H⊆A⊆Y ,so $p \in Y \cap A \subseteq A$, p∈$Y^{\circ} \Rightarrow p \in A_Y^{\circ}$, $p \in Y^{\circ} \Rightarrow p \in Y^{\circ} \cap A_Y^{\circ}$,so

$$A^{\circ} \subseteq A_Y^{\circ} \cap Y^{\circ} \qquad \ldots\ldots\ldots \qquad (1)$$

Let $x \in Y^{\circ} \cap A_Y^{\circ} \Rightarrow \exists H_1,H_2 \in \tau$ s.t. $x \in H_2 \subseteq Y, x \in Y \cap H_1 \subseteq A$, SO

$x \in H_1 \cap H_2 \subseteq A \Rightarrow x \in A^{\circ}$,so

$$A_Y^{\circ} \cap Y^{\circ} \subseteq A^{\circ} \qquad \ldots\ldots\ldots \qquad (2)$$

From (1) and (2) we get $A^{\circ} = A_Y^{\circ} \cap Y^{\circ}$.

3) $\overline{d(A_Y)} = d(A_Y) \cup A = (d(A) \cap Y) \cup A$, $A \subseteq Y$

$= (d(A) \cap Y) \cup (A \cap Y) = (d(A) \cap A) \cup Y = \bar{A} \cap Y$.□

2.1.16 Example:

Show that if $d(A) = \emptyset$ in a topological space (X, τ) then τ_A is the discrete topology.

Solution:

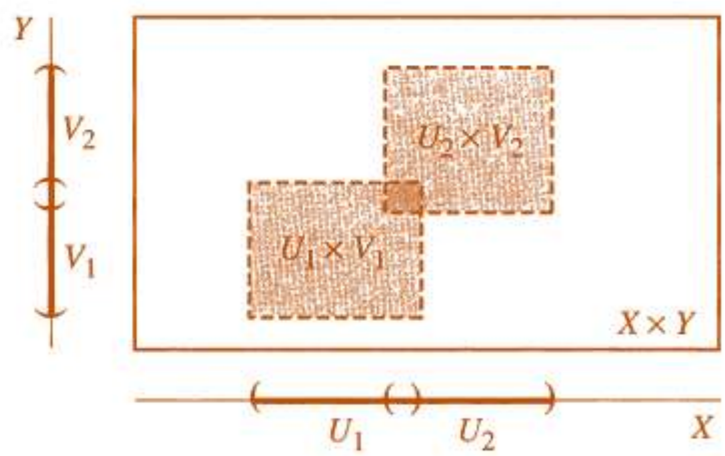
In order to prove that τ_A is the discrete topology we shall show that every subset of A is closed.

If $B \subseteq A$ then $d(B) \subseteq d(A)$,so $d(B) \subseteq \emptyset$ (since $d(A) = \emptyset$),so B is a closed set in X and then B is closed in A (since $B = B \cap A$).

$$(AUB) \cap (C \cup B) = (AUC) \cap B$$

2.2 The Product Topology

Given two topological spaces X and Y ,we would like to generate a natural topology on the product $X \times Y$. Our first inclination might be to take as the topology on $X \times Y$ the collection $\mathcal{C}$ of sets of the form $U \times V$ where U is open in X and V is open in Y. But $\mathcal{C}$ is not a topology since the union of two sets $U_1 \times V_1$ and $U_2 \times V_2$ need not be in the form $U \times V$ for some $U \subset X$ and $V \subset Y$. However, if we use $\mathcal{C}$ as a basis, rather than as the whole topology, we can proceed.



2.2.1 Definition:

Let (X,τ_X) and (Y,τ_Y) be topological spaces and $X \times Y$ be their product. The ***product topology*** on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

2.2.2 Remark:

We shall verify that $\mathcal{B}$ actually is a basis for a topology on the product, $X \times Y$.

2.2.3 Theorem:

The collection $\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is a basis for a topology on $X \times Y$.

Proof:

1- Every point (x, y) is in $X \times Y$, and $X \times Y \in \mathcal{B}$. Therefore, the first condition for a basis is satisfied.

2- Assume that (x, y) is in the intersection of two elements of $\mathcal{B}$. That is, $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$ where U_1 and U_2 are open sets in X, and V_1 and

V_2 are open sets in Y. Let $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Then U_3 is open in X, and V_3 is open in Y, and therefore $U_3 \times V_3 \epsilon \mathcal{B}$. Also,

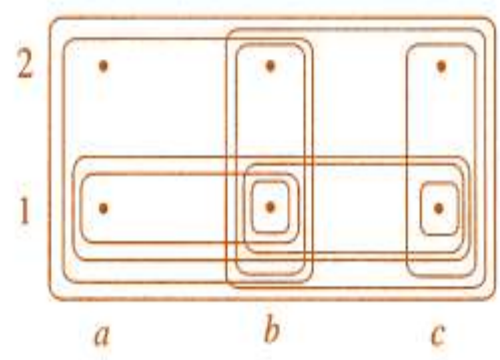
$$U_3 \times V_3 = (U_1 \cap U_2) \times (V_1 \cap V_2) = (U_1 \times V_1) \cap (U_2 \times V_2)$$

and thus $(x, y) \epsilon U_3 \times V_3 \subset (U_1 \times V_1) \cap (U_2 \times V_2)$. It follows that the second condition for a basis is satisfied.

Therefore $\mathcal{B}$ is a basis for a topology on $X \times Y$.□

2.2.4 Example:

Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$ with topologies $\{ø,\{b\},\{c\},\{a,b\},\{b,c\}, X\}$ and $\{ø, \{1\}, Y\}$,respectively. A basis for the product topology on $X \times Y$. Each nonempty open set in the product topology on $X \times Y$ is a union of the basis elements.



2.2.5 Remark:

As with open sets, products of closed sets are closed sets in the product topology. But here too, this does not account for all of the closed sets because there are closed sets in the product topology that cannot be expressed as a product of closed sets. For instance, the set $\{(a, 2), (c, 1), (c, 2)\}$ is a closed set in the product topology in Example 2.2.4, but it is not a product of closed sets.

2.2.6 Remark:

In Definition 2.2.1, the basis B that we use to define the product topology is relatively large since we obtain it by pairing up every open set U in X with every open set V in Y. Fortunately, as the next theorem indicates, we can find a smaller basis for the product topology by using bases for the topologies on X and Y, rather than using the whole topologies themselves.

2.2.7 Theorem:

If $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, then

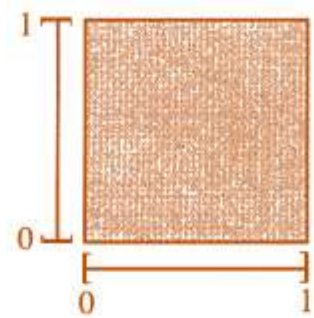
$$\mathcal{B} = \{C \times D : C \in \mathcal{B}_X \text{ and } D \in \mathcal{B}_Y\}$$

is a basis that generates the product topology on $X \times Y$.

Proof:

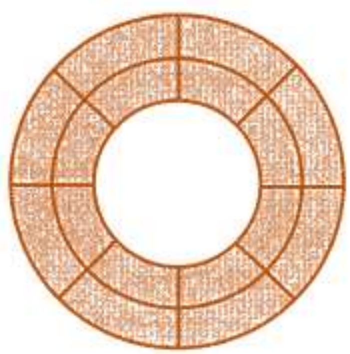
Each set $C \times D \in \mathcal{B}$ is an open set in the product topology; therefore, by definition 1.6.1, it suffices to show that for every open set W in $X \times Y$ and every point $(x, y) \in W$, there is a set $C \times D \in \mathcal{B}$ such that $(x, y) \in C \times D \subset W$. But since W is open in X, we know that there are open sets U in X and V in Y such that $(x, y) \in U \times V \subset W$. So $x \in U$ and $y \in V$. Since U is open in X, there is a basis element $C \in \mathcal{B}_X$ such that $x \in C \subset U$. Similarly, since V is open in Y, there is a basis element $D \in \mathcal{B}_Y$ such that $y \in D \subset V$. Thus $(x, y) \in C \times D \subset W$. Hence, by definition 1.6.1, it follows that $\mathcal{B} = \{C \times D : C \in \mathcal{B}_X \text{ and } D \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$.□

2.2.8 Example:



Let I = [0, 1] have the slandered topology as a subspace of ℝ. The product space $I \times I$ is called the unit square. The product topology on $I \times I$ is the same as the standard topology on $I \times I$ as a subspace of $\mathbb{R}^3$.

2.2.9 Example:

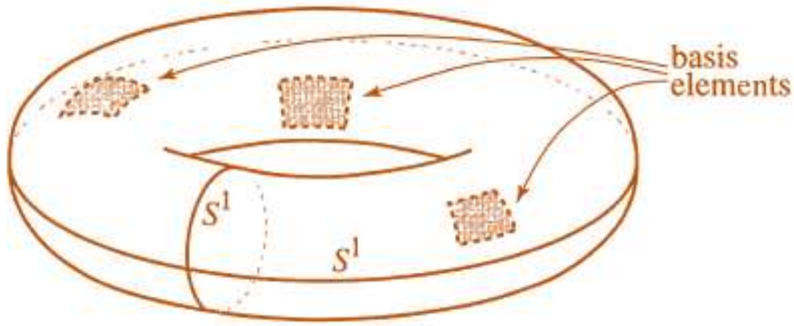


Let S^1 be the circle, and let I = [0, 1] have the standard topology.Then $S^1 \times I$ can think of it as a circle with intervals perpendicular at each point of the circle. Seen this way, it is a circle's worth of intervals. Or it can be thought of as an interval with perpendicular circles at each point. Thus it is an interval's worth of circles. The resulting topological space is called the ***annulus***.

The product space $S^1 \times (0, 1)$ is the annulus with the inner most and outermost circles removed. We refer to it as the ***open annulus***.

2.2.10 Example:

Consider the product space $S^1 \times S^1$, where S^1 is the circle. For each point in the first S^1. there is a circle corresponding to the second S^1.Since each S^1 has a topology generated by open intervals in the circle, it follows by Theorem 2.2.7 that $S^1 \times S^1$ has a basis consisting of rectangular open patches. The resulting space resembles the torus introduced in Example 2.1.8; in fact, they are topologically equivalent.



2.2.11 Example:

Let D be the disk as a subspace of the plane. The product space $S^1 \times D$ is called the ***solid torus***. If we think of the torus as the surface of a doughnut, then the solid torus is the whole doughnut itself.

2.2.12 Remark:

Let A and B be subsets of topological spaces X and Y, respectively. We now have two natural ways to put a topology on $A \times B$. On the one hand, we can view $A \times B$ as a subspace of the product $X \times Y$. On the other hand, we can view $A \times B$ as the product of subspaces, $A \subset X$ and $B \subset Y$. The next theorem indicates that both approaches result in the same topology.

2.2.13 Theorem:

Let (X,τ_X) and (Y,τ_Y) be topological spaces, and assume that $A \subset X$ and $B \subset Y$. Then the topology on $A \times B$ as a subspace of the product $X \times Y$ is the same as the product topology on $A \times B$, where A has the subspace topology inherited from X, and B has the subspace topology inherited from Y.

Proof: Left as exercise.

2.2.14 Remark:

The approach used to define a product of two spaces extends to a product $X_1 \times \cdots \times X_n$ of *n* topological spaces. It is straightforward to see that the collection $\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \text{ open in } X_i \text{ for each } i\}$ is a basis for a topology on $X_1 \times \cdots \times X_n$.The resulting topology is called the ***product topology*** on $X_1 \times \cdots \times X_n$. We have an analog to Theorem 2.2.7 for this case. Specifically, if $\mathcal{B}_i$ is a basis for X_i for each $i = 1, \cdots, n$, then the collection

$$\mathcal{B}' = \{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i \text{ for } i = 1, \cdots, n\}$$

is a basis for $X_1 \times \cdots \times X_n$.

2.2.15 Remark:

We note that the standard topology on $\mathbb{R}^n$ is the topology generated by the basis of open balls defined by the Euclidean distance formula on We also pointed that the same topology results from taking a basis made up of products of open intervals in $\mathbb{R}$ It follows that the standard topology on $\mathbb{R}^n$ is the same as the product topology that results from taking the product of *n* copies of $\mathbb{R}$ with the standard topology.

2.2.16 Example:

The ***n-torus***, T^n is the topological space obtained by taking the product of *n* copies of the circle, S^1.

2.2.17 Remark:

The next theorem indicates that the interior of a product is the product of the interiors.

2.2.13 Theorem:

Let A and B be subsets of topological spaces X and Y, respectively. Then

$$(A \times B)^{\circ} = A^{\circ} \times B^{\circ}.$$

Proof: $\Rightarrow$

Since A° is an open set contained in A, and B° is an open set contained in B, it follows that $A^{\circ} \times B^{\circ}$ is an open set in the product topology and is contained in $A \times B$. Thus $A^{\circ} \times B^{\circ} \subset (A \times B)^{\circ}$

$\Leftarrow$

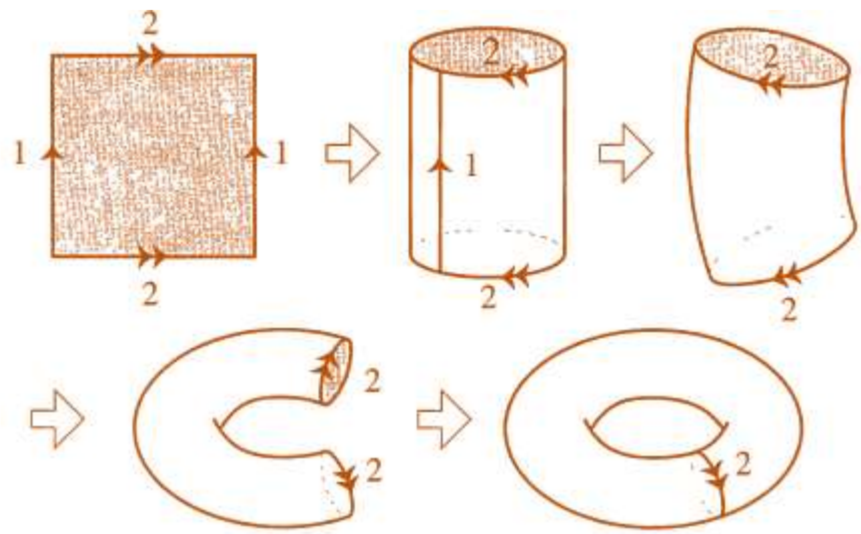
Now suppose $(x, y) \in (A \times B)^{\circ}$. We will prove that $(x, y) \in A^{\circ} \times B^{\circ}$. Since $(x, y) \in (A \times B)^{\circ}$, it follows that (x, y) is contained in an open set contained in $A \times B$ and therefore is also contained in a basis element contained in $A \times B$. So there exists a U and V open in X and Y, respectively, such that $(x, y) \in U \times V \subset A \times B$. Thus, x is in an open set U contained in A, and y is in an open set V contained in B, implying that $x \in A^{\circ}$ and $y \in B^{\circ}$. Therefore

$(x, y) \in A^{\circ} \times B^{\circ}$. It follows that $(A \times B)^{\circ} \subset A^{\circ} \times B^{\circ}$.

Since we have both $A^{\circ} \times B^{\circ} \subset (A \times B)^{\circ}$ and $(A \times B)^{\circ} \subset A^{\circ} \times B^{\circ}$ then $(A \times B)^{\circ} = A^{\circ} \times B^{\circ}$.□

2.3 The Quotient Topology

The concept of a quotient topology allows us to construct a variety of additional topological spaces from the ones that we have already introduced. Put simply, we create a topological model that mimics the process of gluing together or collapsing parts of one or more objects. One of the most well-known examples is the torus, as obtained from a square sheet by gluing together the opposite edges.



2.3.1 Definition:

Let X be a topological space and A be a set (that is not necessarily a subset of X). Let $p: X \to A$ be a surjective map. Define a subset U of A to be open in A if and only if $p^{-1}(U)$ is open in X. The resultant collection of open sets in A is called the ***quotient topology induced by p***, and the function p is called a ***quotient map***. The topological space A is called a ***quotient space***.

2.3.2 Theorem:

Let $p: X \to A$ be a quotient map. The quotient topology on A induced by p is a topology.

Proof:

We verify each of the three conditions for a topology.

1- The set $p^{-1}(\emptyset) = \emptyset$, which is open in X. The set $p^{-1}(A) = X$, which is open in X. So $\emptyset$ and A are open in the quotient topology.

2- Suppose each of the sets U_i , $i = 1, \cdots, n$, is open in the quotient topology on A. Then $p^{-1}(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} p^{-1}(U_i)$, which is a finite intersection of open sets in X, and therefore is open in X. Hence, $\bigcap_{i=1}^{n} U_i$ is open in the quotient topology, and it follows that the finite intersection of open sets in the quotient topology is an open set in the quotient topology.

3- Suppose each of the sets in the collection $\{U_i\}_{i\in I}$ is open in the quotient topology on A. Then $p^{-1}(\bigcup_i U_i) = \bigcup_i p^{-1}(U_i)$, which is a union of open sets in X, and therefore is open in X. Thus, $\bigcup_i U_i$ is open in the quotient topology, implying that the arbitrary union of open sets in the quotient topology is an open set in the quotient topology.

Hence, the quotient topology is a topology on A.

2.3.3 Example:

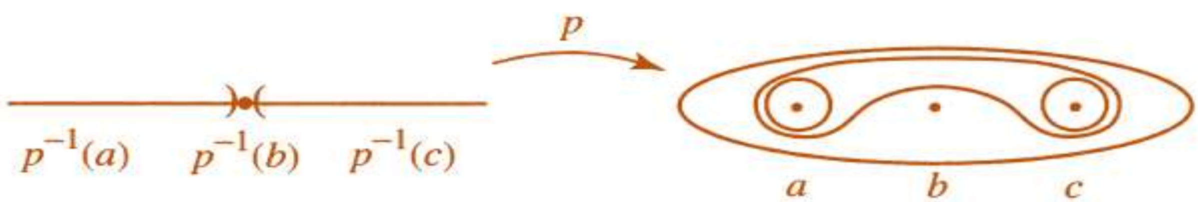
Give $\mathbb{R}$ the standard topology, and define $p: \mathbb{R} \to \{a,b,c\}$ by

$$p(\mathrm{x}) = \begin{cases} a & if\ x < 0 \\ b & if\ x = 0 \\ c & if\ x > 0 \end{cases}$$

The resulting quotient topology on $\{a,b,c\}$ is $\{\{a\},\{c\},\{a,c\},\{a,b,c\}\}$. The

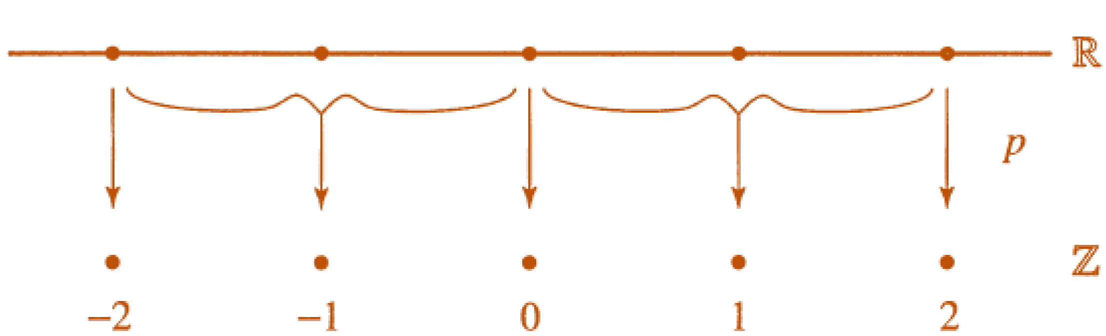
subsets $\{a\}$, $\{c\}$, and $\{a,c\}$ are all open since their preimages are open in $\mathbb{R}$.

But $\{b\}$ is not open since its preimage is $\{0\}$, which is not open in $\mathbb{R}$.



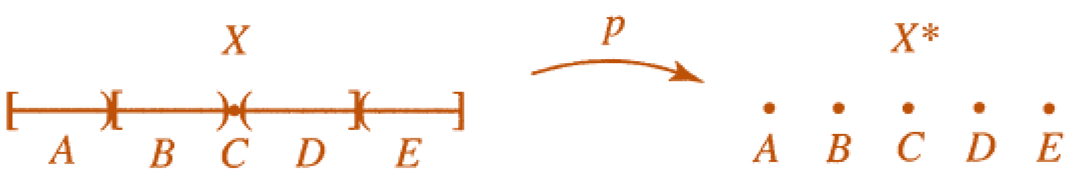
2.3.4 Example:

Let $\mathbb{R}$ have the standard topology, and define $p: \mathbb{R} \to \mathbb{Z}$ by $p(x) = x$ if x is an integer, and $p(x) = n$ if $x \in (n-1, n+1)$ and n is an odd integer. So p is the identity on the integers, and p maps non integer values to the nearest odd integer. In the resulting quotient topology on $\mathbb{Z}$, if n is an odd integer, then $\{n\}$ is an open set since $p^{-1}(\{n\}) = (n-1, n+1)$, an open set in $\mathbb{R}$. If n is an even integer, then $\{n\}$ is not an open set since $p^{-1}(\{n\})$ is not open in $\mathbb{R}$. In the quotient topology, the smallest open set containing an even integer n is the set $\{n - 1, n, n+1\}$. It follows that the quotient topology induced by p on Z is the digital line topology.



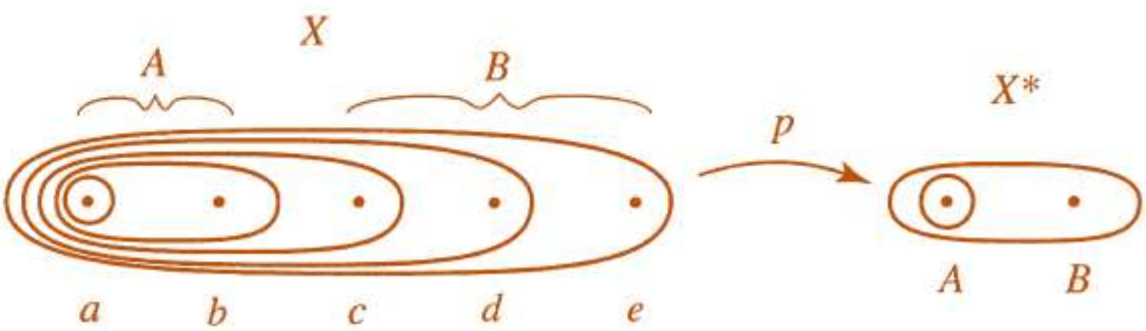
2.3.5 Remark:

Let (X,τ) be a topological space. We are particularly interested in quotient spaces defined on partitions of X. Specifically, let X^* be a collection of mutually disjoint subsets of X whose union is X, and let $p: X \rightarrow X^*$ be the surjective map that takes each point in X to the corresponding element of X^* that contains it. Then p induces a quotient topology on X^*. We think of the process of going from the topology on X to the quotient topology on as taking each subset S in the partition and identifying all of the points in S with one another, thereby collapsing S to a single point in the quotient space. A set U of points in is open in the quotient topology on exactly when the union of the subsets of X, corresponding to the points in U, is an open subset in X.



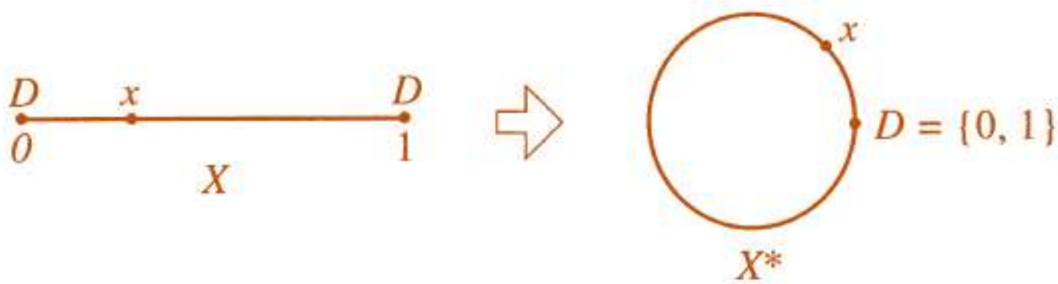
2.3.6 Example:

Let $X = \{a, b, c, d, e)$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. With $A = \{a , b\}$ and $B = \{c , d , e\}$, let X^* be the partition of X given by $X^* = \{A, B\}$. Note that X^* is a two-point set. Since $\{a, b\}$ is open in X and $\{c, d, e\}$ is not, the only open sets in the quotient topology on are $\emptyset$, $\{A\}$, and X^* itself.



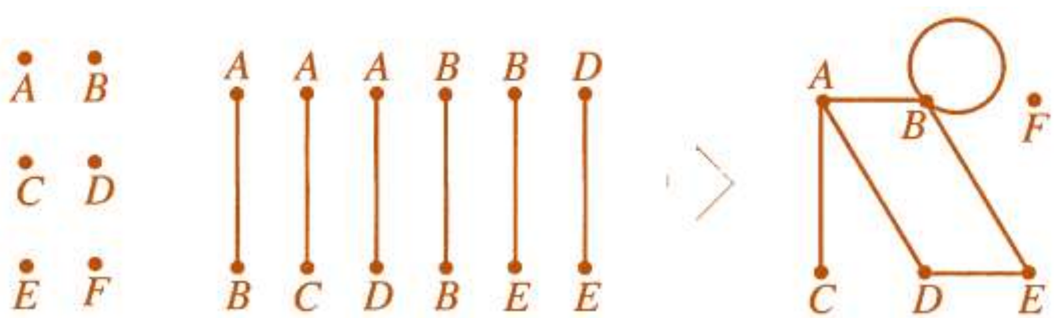
2.3.7 Example:

Let $X = [0, 1]$, and consider the partition X^* that is made up of the single-point sets $\{x\}$, for $0 < x < 1$, and the double-point set $D = \{0, 1\}$. Then, in the quotient topology on we think of *D* as a single point, as if we had glued the two endpoints of [0, 1] together. A subset of X^* that does not contain *D* is a collection of single-point subsets, and it is open in X^* exactly when the union of those single-point sets is an open subset of (0, 1). A subset of X^* that contains *D* is open in X^* when the union of all the sets making up the subset is an open subset of [0, 1]. Such an open subset must contain 0 and 1, and therefore must contain intervals [0, *a*) and (*b*, 1], which are open in the subspace topology on [0, 1]. The resulting space is topologically equivalent to the circle, S^1.



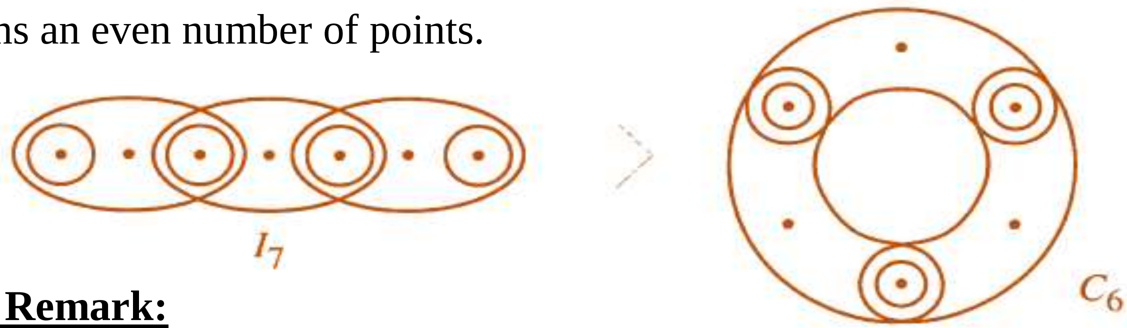
2.3.8 Example:

In the previous example 2.3.7, we glued the endpoints of an interval together to obtain a single point. That is an example of a more general construction that results in a space known as a topological graph. Specifically, a ***topological graph*** *G* is a quotient space constructed by taking a finite set of points, called the **vertices** of *G*, along with a finite set of mutually disjoint closed bounded intervals in ℝ. and gluing the endpoints of the intervals to the vertices in some fashion. The glued intervals are called the ***edges*** of G.



2.3.9 Example:

In Example 2.3.7 we obtained a circle by identifying endpoints of an interval in the real line. We describe a similar process here, using the digital line, that yields spaces we call digital circles. Specifically, a ***digital interval*** is a subset $\{m, m+1, \cdots, n\}$ of $\mathbb{Z}$ with the subspace topology inherited from the digital line topology. Let I_n be the digital interval in the form $\{1, 2, \cdots . n - 1, n\}$. If $n \geq 5$ is an odd integer, then the topological space C_{n-1} resulting from identifying the endpoints 1 and n in I_n is called a ***digital circle***. The digital circle C_{n-1} is a quotient space of the digital interval I_n.The following Figure we illustrate I_7 and C_6 along with a basis for each. By definition, a digital circle contains an even number of points.



2.3.10 Remark:

The following examples 2.3.11 and 2.3.12 gives two different quotient spaces defined on $I \times I$.

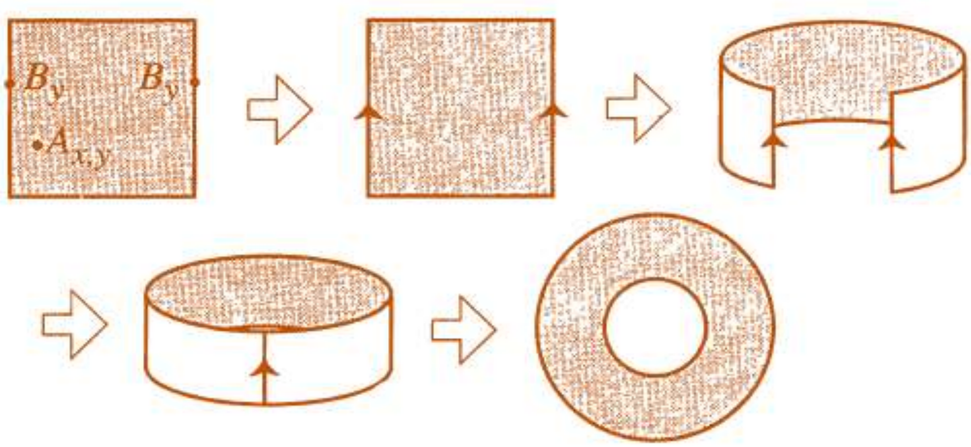
2.3.11 Example:

Define a partition on $I \times I$ by taking subsets of the following form:

i) $A_{x,y} = \{(x,y)\}$ for every x and y such that $0 < x < 1$ and $0 \leq y \leq 1$.

ii) $B_y = \{(0,y),(1,y)\}$ for every y such that $0 < y < 1$.

In the quotient topology, the subsets B_y cause the left and right edges of the square to be glued. The result is a space that is topologically equivalent to the ***annulus***.



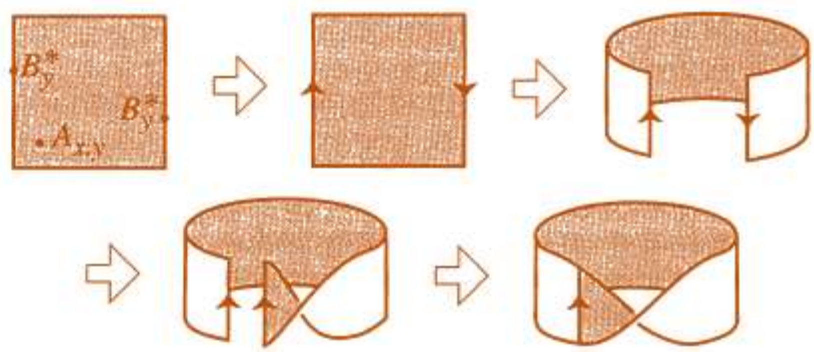
2.3.12 Example:

Define a partition on $I \times I$ by taking subsets of the following form:

i) $A_{x,y} = \{(x,y)\}$ for every x and y such that $0 < x < 1$ and $0 \leq y \leq 1$.

ii) $B_y^* = \{(0,y),(1,1\text{-}y)\}$ for every y such that $0 < y < 1$.

Here the subsets B_y^* also cause the left and right edges of the square to be glued. But in order to accomplish the gluing, we need to perform a half twist so that the identified points on the edges can be properly brought together. The result is the well-known ***Möbius band***.



2.3.13 Example:

Define a partition of $I \times I$ by taking subsets of the following form:

i) $A_{x,y} = \{(x,y)\}$for every x and y such that $0 < x < 1$ and $0 < y < 1$.

ii) $B_y = \{(0,y),(1,y)\}$ for every y such that $0 < y < 1$.

iii) $C_y = \{(x,0),(x,1)\}$ for every x such that $0 < x < 1$.

iv) $D = \{(0,0),(0,1),(1,0),(1,1)\}$.

In the quotient topology, the two-point subsets in (ii) cause the gluing of the left edge of the square to the right edge, and the two-point subsets in (iii) cause the gluing of the top edge of the square to the bottom edge. Furthermore, the four-point subset causes the gluing of the four corners of the square to a single point. The topological space we obtain is therefore the result of taking a square and gluing together its opposite edges. Such a construction results in a ***torus***.