محاضرات مادة التبولوجي /المرحلة الرابعة

الكورس الأول

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Contents

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Chapter One

Topological Spaces

1.1 Topological space

1.1.1 Definition:-

Let X be a non empty set .A class τ of subsets of X is a ***topology*** on X iff τ satisfies the following axioms

1) X and $\emptyset$ are members of τ.
2) The intersection of any finite number of members of τ is a member of τ.
3) The union of any family of members of τ is again in τ.

The pair (X, τ) is called a ***topological space*** and the members of τ are called ***τ-open sets*** or simply **open sets**.

1.1.2 Example:-

If X is any set, then the collection $\{X, \emptyset\}$ of subsets of X also forms a topology on X. This topology is called the ***trivial*** (***indiscrete***) topology on X.

1.1.3 Example:-

If X is any set, then the family of all subsets of X forms a topology on X. This topology is called the ***discrete topology*** on X.

Notice that the discrete topology contains the maximum possible number of open sets since, relative to the discrete topology, every subset of X is open.

1.1.4 Example:-

Let τ be a class of all open sets of a metric space (X, *d*) then τ is a topology on X ,called the ***usual topology*** on X.

1.1.5 Example:-

Let τ be a class of all subsets of X whose complements are finite together with the empty set $\emptyset$. This class τ is a topology on X which is called the co-finite topology.

1.1.6 Example:-

Consider the following classes of subsets of $X = \{a,b,c\}$

$\tau_1 = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$

$\tau_2 = \{X,\emptyset,\{a\},\{b\}\}$

$\tau_3 = \{X,\emptyset,\{a,c\},\{b,c\}\}$

Observe that $\tau_1$is a topology on X since it satisfies the necessary three axioms. But $\tau_2$is not a topology on X since the unions $\{a\} \cup \{b\} = \{a,b\}$ of two members of τ_2 does not belongs to τ_2 ,i.e does not satisfy the axiom 3. Also τ_3 is not a topology on X since the intersection $\{a,c\} \cap \{b,c\} = \{c\}$ of two sets in $\tau_3$does not belongs to τ_3,i.e τ_3 does not satisfy the axiom 2.

1.1.7 Example:-

Let τ be a class of all subsets of N consisting of ∅ ,X and all subsets of ℕ of the form E_n ={1,2,…,n}with $n \in \mathbb{N}$ then the class τ is a topology on X.

1.1.8 Theorem:-

Let $\{\tau_i : i \in I\}$ be a collection of topologies on a set X. Then the intersection $\bigcap_i \tau_i$ is also a topology on X.

Note that the union of two topologies for X need not be a topology on X, for example $\tau_1 = \{X,\emptyset,\{a\}\}$, $\tau_2 = \{X,\emptyset,\{b\}\}$ is two topologies on $X = \{a,b,c\}$ but the union $\tau_1 \cup \tau_2$is not a topology on X.

1.1.9 Definition:-

Let X be a non-empty set and $\tau_1$and τ_2 be two topologies on X. If $\tau_1 \subset \tau_2$ then τ_2 is said to be ***finer*** than τ_1,and $\tau_1$is said to be the ***courser*** than τ_2.

1.1.10 Example:-

Let X be a non-empty set then the discrete topology is finer of all topologies on X and the indiscrete topology is courser of all topologies on X.

Notice that the class $\{T_i\}$ of all topologies on X i partially ordered by class inclusion :

$$\tau_1 \lesssim \tau_2 \quad for \quad \tau_1 \subseteq \tau_2.$$

And we say that two topologies on X are not comparable if neither is coarser than the other.

Exercises:-

1. Let τ be a topology on a set X consisting of four sets ,i.e. $\tau = \{A,\emptyset,B,C\}$, where A and B are non-empty disjoint proper subsets of X .What conditions must A and B satisfy?
2. Determine all of the possible topologies on $X = \{a,b,c\}$.
3. List all topologies on $X = \{a,b,c\}$ which consist of exactly four members.
4. Show that the class τ of all subsets of X whose complements are finite together with the empty set ∅ is a topology on X.
5. Let X be a set and assume $p \epsilon X$.Show that the collection τ consisting of $\emptyset, X$, and all subsets of X containing p, is a topology on X. This topology is called the ***particular point topology*** on X.
6. Let X be a set and assume $p \epsilon X$.Show that the collection τ consisting of ∅,X, and all subsets of X that exclude p, is a topology on X. This topology is called the ***excluded point topology*** on X.
7. Let τ consist of ∅,R, and all intervals (-∞,p) for $p \epsilon \mathbb{R}$ .Prove that τ is a topology on ℝ.

8. Let $f: X \to Y$ be a function fromm a non – empty set X into a topological space (Y,τ_Y) and let $\tau_X\tau$ be the class of intervals of open subsets of Y,i.e. $\tau_X = \{f^{-1}(G): G\epsilon\tau_Y\}$.Show that $\tau_X$is a topology on X.
9. Let τ be a class of all subsets of N consisting of ∅ and all subsets of ℕ of the form E_n ={n,n+1,n+2,…}with $n \in \mathbb{N}$.
   a) Show that τ is a topology on N.
   b) List the open sets containing the positive integer 6.

1.2 limit points

1.2.1 Definition:-

Let A be a subset of a topological space (X,τ).A point $p\epsilon X$ is ***an accumulation point*** or ***a limit point*** of A if every open set G containing *p* contains a point of A different from *p*, i.e.

$$G\ open\ , p\epsilon G \ \rightarrow A \cap (G/\{p\}) \neq \emptyset.$$

The set of accumulation points of A, denoted by d(A) (or A`).

Notice that a limit point *p* of a set A may or may n ot lie in the set A. Notice also that in every topology, the point *p* is not a limit point of the set {x}.

1.2.2 Example:-

Consider $A \subset \mathbb{R}$ with the usual topology on ℝ then :

a) $d(A = \{\frac{1}{n}\epsilon\mathbb{R}: n\epsilon\mathbb{Z}^+\}) = \{0\}$.
b) $d([a,b])=d((a,b])=d([a,b))=d((a,b))=[a,b]$.
c) $d(\mathbb{Q}) = \mathbb{R}$.
d) $d(\mathbb{Z}) = \emptyset$.

1.2.3 Example:-

Let $X = \{a,b,c,d,e\}$ and $\tau = \{\emptyset,\{a\},\{b,d\},\{a,b,d\},\{b,c,d,e\}, X\}$ then

$$d(\{a,b,c\})=\{c,d,e\},\ d(\{b,c,d\}) = \{b,c,d,e\}$$

1.2.4 Theorem:-

If A,B and E are subsets of the topological space (X,τ), then the derived set has the following properties:

a) $d(\emptyset) = \emptyset$.
b) If $A \subseteq B$ then $d(A) \subseteq d(B)$.
c) If $x \in d(E)$, then $x \in d(E\backslash\{x\})$.
d) $d(A \cup B) = d(A) \cup d(B)$.

Note that $d(A \cap B) \neq d(A) \cap d(B)$, for example let $X = \{a,b,c\}$ and let $A = \{a,c\}$,B={b,c} ,define the topology τ on X by $\tau = \{X,\emptyset,\{b\},\{a,b\}$ then $d(A \cap B) = d(\{c\}) = \emptyset \neq d(A) \cap d(B) = \{c\} \cap \{a,c\} = \{c\}$.

Exercises: -

1. Let A be a subset of a topological space (X, τ) .When will a point $p \in X$ not be a limit point of A?
2. Let A be any subset of a discrete topological space X. Show that $d(A) = \emptyset$.
3. Consider the topological space $(\mathbb{R}, \tau)$, where τ consists of of $\emptyset$,$\mathbb{R}$, and all open intervals $E_p = (a,\infty), a \in \mathbb{R}$. Find the derived set of
   a) The interval $(4,10]$; **b)** $\mathbb{Z}$ the set of integers.
4. Determine the set of limit points of $[0,1]$ in the complement topology on $\mathbb{R}$.
5. Let τbe the topology on $\mathbb{N}$ which consists of $\emptyset$ and all subsets of $\mathbb{N}$ of the form E_n ={n,n+1,n+2,…}were $n \in \mathbb{N}$.
   a) Find the limit points of the set $A = \{4,13,28,37\}$.
   b) Determine those subsets E of $\mathbb{N}$ for which $d(E) = \mathbb{N}$.
6. Let $\tau_1$and τ_2 be topologies on X such that $\tau_1 \subset \tau_2$ and let A be any subset of X. Show that every τ_2- limit point of A is also a τ_1- limit point of A.

1.3 Closed Sets

1.3.1 Definition:-

Let (X, τ) be a topological space. A subset A of X is ***closed set*** if it contains all its limit points, i.e. $d(A) \subseteq A$.

1.3.2 Example:-

Let X={a,b,c,d} and $\tau = \{\emptyset$,{a},{b,c},{a,b,c},X} then $A =$ {a,d} is a closed set since $d(A) = \{d\} \subseteq A = \{a,d\}$.

1.3.3 Theorem:-

If $x \notin A$, where A is a closed subset of a topological space (X, τ) then there exists an open set G such that $x \in G \subseteq A^c$.

1.3.4 Corollary:-

Let (X, τ) be a topological space. A subset A of X is closed set iff its complement A^c is open.

1.3.5 Example:-

Let $X =$ {a,b,c,d,e} and $\tau = \{\emptyset$,{a},{b,c},{a,b,c},{b,c,d,e},$X\}$ then

1) $\emptyset$,{a},{b,c},{a,b,c},{b,c,d,e},X are open sets.
2) X,{b,c,d,e},{a,d,e},{d,e},{a},$\emptyset$ are closed sets.
3) $\emptyset$,X,{a},{b,c,d,e} are both open and closed sets.
4) {b,c},{a,b,c} are open not closed sets.
5) {d,e},{a,d,e} are closed not open sets.
6) {e},{c},{d},{c,d} are not open and closed sets.

1.3.6 Example:-

In a discrete topology all subsets are both open and closed.

1.3.7 Corollary:-

Let $\mathcal{F}$ be a family of closed subsets in a topological space (X,τ) then it has the following property:

a) *The intersection of any number of members of $\mathcal{F}$ is a member of $\mathcal{F}$ ($X \in \mathcal{F}$)***.***

b) *The union of any finite number of members of $\mathcal{F}$ is a member of $\mathcal{F}$ ($\emptyset \in \mathcal{F}$).*

Note that if A is a closed set then $d(A)$ is also a closed set (since A is closed then $d(A) \subseteq A$, i.e. $d(d(A)) \subseteq d(A)$,so $d(A)$ is a closed set) but the converse is not true for example in the usual topology (ℝ,u) the set (a,b) is an open set but d(a,b)=[a,b] is a closed set.

1.4 The Closure of Sets

1.4.1 Definition:-

Let A be a subset of a topological space (X,τ) the ***closure*** of A ,denote by $\bar{A}$ is the intersection of all closed subsets of X containing A , i.e.

$$\bar{A} = \bigcap_i F_i \ , \ A \subseteq F_i, F_i \text{ is closed set.}$$

Notice that $\bar{A}$ is closed set since its equals to intersection of closed sets (corollary 1.3.7 part a) . Also $\bar{A}$ is the smallest closed set containing A, i.e. if F is any closed set contain A then $\subseteq \bar{A} \subseteq F$.

1.4.2 Example:-

From example 1.3.5 we have $\overline{\{b,c\}}$={b,c,d,e} ∩ X={b,c,d,e} , $\overline{\{d,e\}}$ = {d,e}∩{a,d,e}∩X={d,e} and $\overline{\{a,b\}} = X$.

1.4.3 Exmaple:-

Let A be a subset of the cofinite topological space (X,τ) then

$$\bar{A} = \begin{cases} A & if\ A\ is\ finite \\ X & if\ A\ is\ infinite \end{cases}$$

Notice that the following theorem define the closure sets in terms of its limit points

1.4.4 Theorem:-

Let A be a subset of a topological space (X,τ) the closure of A is the union of A and its set of limit points, i.e.

$$\bar{A} = A \cup d(A).$$

1.4.5 Example:-

Let (ℝ,τ) be the usual topology then $\overline{(a,b)}=\overline{[a,b)}=\overline{(a,b]}=\overline{[a,b]}$=[a,b].

1.4.6 Example:-

Let (ℝ,τ) be the usual topology then

a) If $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \subset \mathbb{R}$ then

$\bar{A} = A \cup d(A) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots, 0\}$.

b) If $\mathbb{Q} \subset \mathbb{R}$ the set of rational numbers then

$$\bar{\mathbb{Q}} = \mathbb{Q} \cup \boldsymbol{d}(\mathbb{Q}) = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}.$$

1.4.7 Theorem (Closure Axioms):-

If A and B are subsets of a topological space (X, τ) then

a) $\bar{\emptyset} = \emptyset$, $\bar{X} = X$.

b) $A \subseteq \bar{A}$.

c) $A = \bar{A}$ *iff A is closed.*

d) $\bar{\bar{A}} = \bar{A}$.

e) $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.

Notice that $\overline{(A \cap B)} \neq \bar{A} \cap \bar{B}$ as the following example:

1.4.8 Example:-

Let $X = \{a, b, c, d, e\}$,$\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. If $A = \{a, c\}$, $B = \{b, c\}$ then $A \cap B = \{c\}$, ,$\bar{A} = X$,$\bar{B} = B$,$\overline{A \cap B} = \{c\}$,So $\overline{A \cap B} = \{c\} \neq \bar{A} \cap \bar{B} = X \cap B = B = \{b, c\}$

1.4.9 Example:-

If E is a subset of a topological space (X, τ),and if $d(F) \subseteq E \subseteq F$ for some subset $F \subseteq X$,show that E is a closed set.

1.4.10 Definition:-

A subset A of a topological space (X, τ) is called ***dense*** in X if $\bar{A} = X$.

1.4.11 Example:-

Let (X, τ) be the indiscrete topology. If $\emptyset \neq A \subseteq X$ then A is dense in X , i.e. $\bar{A} = X$ (since X the only closed set contain A).

1.4.12 Example:-

In discrete topology (X, τ) every proper subset of X is not dense in X ,i.e. $\forall A \subset X, \bar{A} = A$.

1.4.13 Example:-

In topological space $(\mathbb{R}, \tau)$ where $\tau = \{\mathbb{R}, \emptyset, E_a = (a, \infty) : a \in \mathbb{R}\}$ the sets $A = \{2, 4, 6, \dots\}$, $B = \{1, 3, 5, \dots\}$ are dense in $\mathbb{R}$ while the set $C = \{-2, -4, -6, \dots\}$ is not dense in $\mathbb{R}$.

1.4.14 Example:-

The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ in the usual topology $(\mathbb{R}, \tau)$ is dense in $\mathbb{R}$.

Exercises: -

1. Consider the following topology on $X = \{a, b, c, d, e\}$,$\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, e\}\}$

a) List the closed subsets of X.

b) Determine the closure of the sets $\{a\}$,$\{b\}$ and $\{c\}$.

c) Which sets in b) are dense in X.

2. Let τ be the topology on $\mathbb{N}$ which consists of $\emptyset$ and all subsets of $\mathbb{N}$ of the form E_n ={n,n+1,n+2,…}were $n \in \mathbb{N}$.

a) Determine the closed subsets of ($\mathbb{N}$,τ).

b) Determine the closure of the sets {7,24,47,85} and {3,6,9,12,…}.

c) Determine those subsets of $\mathbb{N}$ which are dense in $\mathbb{N}$.

3. Let τ be the topological $\mathbb{R}$ consists of of $\emptyset$,$\mathbb{R}$, and all open infinite intervals $E_p =$ (a,∞),$a \in \mathbb{R}$.

a) Determine the closed subsets of ($\mathbb{R}$,τ).

b) Determine the closure of the sets [3,7),{7,24,47,85},{3,6,9,12,…}.

4. Prove: If F is a closed contain any set A, then $\bar{A} \subset F$.

5. If $A \cap B \neq \emptyset$ prove that $\bar{A} \cap \bar{B} = \overline{A \cap B}$.

6. If F is a closed set ,prove that $\forall A \subseteq X; \overline{F \cap A} \subseteq F \cap \bar{A}$.

7. If U is an open set, prove that $\forall A \subseteq X; U \cap \bar{A} \subseteq \overline{U \cap A}$.

8. If U is an open set and A is dense in X ,prove that $U \subseteq \overline{U \cap A}$.

9. Prove that, A is dense in X iff $A^c \cap (A')^c = \emptyset$.

10. Show that every non-finite subset of an infinite cofinite spae X is dense in X.

1.5 The Interior,Exterior and Boundary points of a Set

1.5.1 Definition:-

Let A be a subset of a topological space (X, τ) the ***interior*** of A ,denote by A° is the union of all open subsets of X contained in A , i.e.

$$A^\circ = \cup_i G_i \;,\; G_i \subseteq A \,, G_i \text{ is an open set.}$$

1.5.2 Example:-

Let $X =$ {a,b,c,d,e} and $\tau =$ {$\emptyset$,{a},{c,d},{a,c,d},{b,c,d,e},X} then {a,b,e}$^\circ = \emptyset \cup \{a\} = \{a\}$ and$\{a,c,d\}^\circ = \emptyset \cup \{a\} \cup \{c,d\} \cup$ {a,c,d}={a,c,d}.

1.5.3 Theorem:-

Let A be a subset of a topological space (X, τ) then $A^\circ = \overline{A^c}^{\,c}$.

1.5.4 Theorem (Interior Axioms):-

If A and B are subsets of a topological space(X, τ) then

a) $X^\circ = X$.

b) *$A^\circ$the largest open set contained in A.*

c) *$A^\circ$is open iff $A^\circ = A$.*

d) $A^\circ \subseteq A$

e) $A^{\circ\circ} = A^\circ$.

f) $(A \cap B)^\circ = A^\circ \cap B^\circ$

Notice that$(A \cup B)^\circ \neq A^\circ \cup B^\circ$ as the following example:

1.5.5 Example:-

In example 1.5.2 A ∪ B = {a,b,e}∪{a,c,d}={a,b,c,d,e} then $A^\circ \cup B^\circ = \{a\} \cup$ {a,c,d}={a,c,d} and $(A \cup B)^\circ = \{a$,b,c,d,e} ,i.e. $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

1.5.6 Definition:-

Let A be a subset of a topological space (X, τ) the ***exterior*** of A ,denote by A^e is the set of all points interior to the complement, i.e. $A^e = A^{c^\circ}$.

1.5.7 Theorem (Exterior Axioms):-

If A and B are subsets of a topological space(X, τ) *then*

a) $X^e = \emptyset$, $\emptyset^e = X$.

b) $A^e \subseteq A^c$

c) $A^e = A^{e^{c^e}}$.

d) $(A \cup B)^e = A^e \cap B^e$

1.5.8 Definition:-

Let A be a subset of a topological space (X, τ) the ***boundary*** of A ,denote by $b(A)$ is the set of all points interior to neither A nor A^c, i.e. $b(A) = \left(A^\circ \cup A^{c^\circ}\right)^c$.

1.5.9 Example:-

Let $X =$ {a,b,c,d,e} ,$\tau =$ {∅,X,{a},{c,d},{a,c,d},{b,c,d,e}} and let $A = \{b$,c,$d\}$ then $A^\circ = \{c,d\}$, $A^e = \{a\}$, $b(A) = \{b,e\}$.

1.5.10 Example:-

Let A be a non-empty proper subset of an indiscrete space X. Then $A^\circ = \emptyset$, $A^e = \emptyset$, $b(A) = X$.

1.5.11 Example:-

Let A be a non-empty proper subset of discrete space X. Then $A^\circ = A$, $A^e = A^c$, $b(A) = \emptyset$.

1.5.12 Example:-

Let (ℝ,τ) be the usual topology then

1) $[a,b]^\circ = [a,b)^\circ = (a,b]^\circ = (a,b)^\circ = (a,b)$, $\mathbb{Q}^\circ = \emptyset$.

2) $[a,b]^e = [a,b)^e = (a,b]^e = (a,b)^e = (-\infty,a) \cup (b,\infty)$, $\mathbb{Q}^e = \emptyset$.

3) $b([a,b]) = b([a,b)) = b((a,b]) = b((a,b)) = \{a,b\}$, $b(\mathbb{Q}) = \mathbb{R}$.

1.5.13 Example:-

The function f which assigns to each set its interior ,i.e. $f(A) = A^\circ$,does not commute with the function g which assigns to each set to its closure ,i.e. $g(A) = \bar{A}$,since if we take ℚ the set of rational numbers as a subset of ℝ with the usual topology. Then

$$(g \circ f)(\mathbb{Q}) = g(f(\mathbb{Q})) = g(\mathbb{Q}^{\circ}) = g(\emptyset) = \overline{\emptyset} = \emptyset.$$
$$(f \circ g)(\mathbb{Q}) = f(g(\mathbb{Q})) = f(\overline{\mathbb{Q}}) = f(\mathbb{R}) = \mathbb{R}^{\circ} = \mathbb{R}.$$

1.5.14 Example:-

Let $(\mathbb{N},\tau)$ be a topological space, $\tau = \{\emptyset, \mathrm{N}, A_n=\{1,2,\ldots,n\}$, $\mathbb{N}$ the set of natural numbers then

1) $\{1,2,4,6\}^{\circ}=\{1,2\}$, $\{1,2,4,6\}^{e}=\emptyset$, $b(\{1,2,4,6\}=\{3,4,5,\ldots\}$.

2) $\{5,7,9,20\}^{\circ}=\emptyset$, $\{5,7,9,20\}^{e}=\{1,2,3,4\}$, $b(5,7,9,20\})=\{5,6,7,\ldots\}$.

1.5.15 Example:-

Let A be a subset of a co-finite topological space (X,τ)then

a) If A is finite then $A^{\circ} = \emptyset$, $A^e = A^c$, $b(A) = A$.

b) If A is infinite then

either $A^c$is finite, i.e. A is open set then $A^{\circ} = A$, $A^e = \emptyset$, $b(A) = A^c$.

nor A is infinite then $A^{\circ} = \emptyset$, $A^e = \emptyset$, $b(A) = X$.

1.5.16 Example:-

Consider the topological space $(\mathbb{R},\tau)$, where τ consists of $\emptyset$, $\mathbb{R}$, and all open intervals $E_a = (a,\infty)$, $a \in \mathbb{R}$ then $[7,\infty)^{\circ}=(7,\infty)$, $[7,\infty)^e = \emptyset$, $b([7,\infty)=(-\infty,7]$.

Exercises: -

1. Let A be a subset of a topological space (X,τ) then prove that:

a) $b(A) = \bar{A} \cap \overline{A^c}$.

b) $b(A)$ is a closed set.

c) $b(A) = b(A^c)$.

d) $b(A) = \bar{A} - A^{\circ}$.

e) $\bar{A} = b(A) \cup A^{\circ}$.

f) $b(A) \cap A^{\circ} = \emptyset$.

g) $b(A) \cap A^e = \emptyset$.

h) $A^{\circ} \cap A^e = \emptyset$.

i) $A^{\circ} \cup A^e \cup b(A) = X$.

2. Let A be a subset of a topological space (X,τ),show that $\bar{A} = A^{\circ} \cup b(A)$.

3. Prove that A is closed and open iff $b(A) = \emptyset$.

4. Prove that in any topological space A subset A is closed iff $b(A) \subseteq A$ and A subset A is open iff $b(A) \subseteq X - A$.

5. Give an example to show that $b(A \cup B) \neq b(A) \cup b(B)$ for any A and B subsets of a topological space (X,τ).

6. Let τ_1 and τ_2 be topologies on X with τ_1 coarser than τ_2 ,i.e. $\tau_1 \subset \tau_2$ and let $A \subset X$.Then

a) The τ_1 −interior of A is subset of the τ_2- interior of A.

b) The τ_2 −boundary of A is subset of the τ_1- boundary of A.

1.6 Bases and subbases

1.6.1 Definition:-

Let (X,τ) be a topological space. A class $\mathcal{B}$ of open subsets of X, i.e. $\mathcal{B} \subset \tau$,is ***a base for the topology*** τ iff every open set $G\epsilon\tau$ is the union of members of $\mathcal{B}$, (equivalently for any point p belonging to an open set G there exists $B \in \mathcal{B}$ with $p \in B \subset G$.

1.6.2 Example:-

The class of open intervals $\mathcal{B} = \{(a,b): a,b\in\mathbb{R}\}$ is a base for the usual topology $(\mathbb{R}, \tau)$.Similarly, the class of open discs form a base for the usual topology $(\mathbb{R}^2, \tau)$.

1.6.3 Example:-

The class $\mathcal{B} = \{\{a\}: a\in X\}$ of all singleton subsets of X is a base for the discrete topology τ on X.

1.6.4 Example:-

Let (X, τ) be a topological space where X={a,b,c,d} ,τ={X,∅,{a,b}, {c,d}} then $\mathcal{B}_1 = \{\{a,b\},[c,d\}\}$, $\mathcal{B}_2 = \{X,\{a,b\},\{c,d\}\}$ are bases for the topology τ while $\mathcal{B}_3 = \{X,\{a,b\}\}$ is not a base for the topology τ ,since {c,d}is an open set but it is not a union of members of $\mathcal{B}_3$.

Note that it is not necessary to include the empty set in a base for a topology, since $\emptyset = \bigcup\{B_\lambda: \lambda \in \emptyset\}$,also it is not every family of subsets of a set ***X*** is a base for a topology for X for example let X={a,b,c} then the class $\mathcal{B}$={{a,b},{b,c}} is not a base for any topology on X ,since {a,b},{b,c} are open sets and their intersection {a,b}∩{b,c}={b} is also an open set but {b}is not a union of members of $\mathcal{B}$.

The following theorem gives the necessary and sufficient conditions for a family of subsets to be a base for a topology.

1.6.5 Theorem:-

Let $\mathcal{B}$ be a class of subsets of a non- empty set X. Then $\mathcal{B}$ is a base for some topology on X iff it possesses the following two properties :

1) $X = \cup \{B: B \in \mathcal{B}\}$.

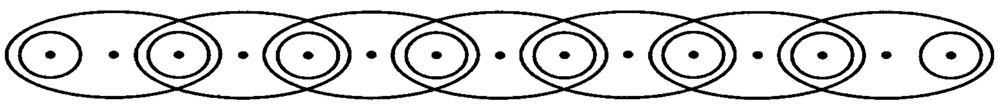
2) For any $B_1,B_2 \in \mathcal{B}$,$B_1 \cap B_2$ is a union of members of $\mathcal{B}$ or equivalently , if $p \in B_1 \cap B_2$ then $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subset B_1 \cap B_2$.

1.6.6 Example:-

Let $\mathcal{B}$ be a class of open –closed intervals in the real line $\mathbb{R}$, i.e. $\mathcal{B}$={(a,b]:a,b∈$\mathbb{R}$,a<b} then $\mathcal{B}$ is a base for a topology τ on $\mathbb{R}$ .This topology τ is called the upper limit topology on $\mathbb{R}$ (this topology is not equals to the usual topology). Similarly, the class of closed – open intervals , $\mathcal{B}^*$={[a,b):a,b∈$\mathbb{R}$,a<b} is a base for a topology τ^* on $\mathbb{R}$ called lower limit topology on $\mathbb{R}$.

1.6.7 Example:-

For each $n \in \mathbb{Z}$,define $B(n) = \begin{cases} \{n\} & if\ n\ is\ odd \\ \{n-1, n, n+1\} & if\ n\ is\ even \end{cases}$ .The collection



The collection $\mathcal{B} = \{B(n): n \in \mathbb{Z}\}$ is a basis for a topology on $\mathbb{Z}$,this topology is called the digital line topology ,also $\mathbb{Z}$ with this topology is the digital line.

1.6.8 Definition:-

Let (X, τ) be a topological space,A class Ψ of open subsets of X, i.e. $\Psi \subset \tau$ is ***a subbase*** for the topology τ on X iff finite intersection of members of Ψ form a base for τ.

1.6.9 Example:-

Let X = {a,b,c,d} ,τ = {∅,X,{a},{a,c},{a,d},{a,c,d}} and let $S = \{\{a,c\},\{a,d\}\}$ so finite intersection of members of S is $\mathscr{B}$ = {{a},{a,c},{a,d},X} which is a base for τ therefore, S is a subbase for τ .

1.6.10 Example:-

Every open interval (a,b) in the real line $\mathbb{R}$ is the intersection of two infinite open intervals (a,∞) and $(-\infty,b)$,i.e. (a,b)=(a,∞)∩(-∞,b). But the open intervals form a base for the usual topology on $\mathbb{R}$, hence the class of all infinite open intervals ($S = \{(a,\infty) , (-\infty,b)$:a,b∈$\mathbb{R}\}$) is a subbase for $\mathbb{R}$.

1.6.11 Example:-

Let (X,τ) be the discrete topology then the family $S = \{\{a,b\}: a,b \in X\}$ is a subbase for the discrete topology.

1.6.12 Example:-

The family S of all infinite open strips is a subbase for $\mathbb{R}^2$.

1.6.13 Remark:-

Let S be any family of subsets of a non-empty set X. S may not be a base for a topology on X. However S is always generates a topology on X in the following sense:

1.6.14 Theorem:-

Any family S of subsets of a non-empty set X is the subbase for a unique topology τ on X. That is, finite intersection of members of S form a base for topology τ on X.

1.6.15 Example:-

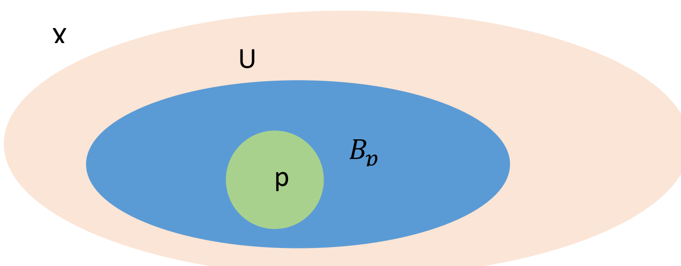
Let $X = \{a,b,c,d\}$ then the family S={{a,b},{b,c},{d}} is a subbase for a topology on X.

1.6.16 Theorem:-

Let S be a class of subsets of a non – empty set X. Then the topology τ on X generated by S is the intersection of all topologies on X which contain S.

1.6.17 Definition:-

Let p be any arbitrary point in a topological space (X,τ). A class $\mathcal{B}_p$ of open sets containing p is called ***a local base at p*** iff for each open set U contained p , $\exists B_p \in \mathcal{B}_p$ with the property $p \in B_p \subset U$.



1.6.18 Example:-

Let $X = \{a,b,c,d\}$ and $T = \{X,\emptyset,\{a\},\{a,b\},\{a,b,c\}\}$ then

$\mathcal{B}_a = \{\{a\}\}$ (or $\mathcal{B}_a = \{\{a\},\{a,b\},\{a,b,c\},X\}$) ,

$\mathcal{B}_b = \{\{a,b\}\}$ (or $\mathcal{B}_b = \{\{a,b\},\{a,b,c\},X\}$) ,

$\mathcal{B}_c = \{\{a,b,c\}\}$ (or $\mathcal{B}_c = \{\{a,b,c\},X\}$) ,

$\mathcal{B}_d = \{X\}$.

1.6.19 Example:-

Consider the topological space $(\mathbb{R}, \tau)$,where τ is the usual topology of open intervals on $\mathbb{R}$. Consider the point $0 \in \mathbb{R}$. The local base of 0 is the

$\mathcal{B}_0 = \{$(a,b):a, b∈ℝ,a<$0 < b\}$. Now if we take any $x \in \mathbb{R}$ then the local base of x is $\mathcal{B}_x = \{$(a,b):a, b∈ℝ,a<$x < b\}$.

1.6.20 Example:-

Consider the topological space $(\mathbb{R}^2, \tau)$ where τ is the usual topology on $\mathbb{R}^2$. Consider the point $p \in \mathbb{R}^2$. Then the class $\mathcal{B}_p$ of all open discs centered at p is a local base at p.

1.6.21 Theorem:-

Let $\mathcal{B}$ be a base for a topology τ on X and let $p \in X$,. Then the members of the base $\mathcal{B}$ which contain p from a local base at the point p.

1.6.22 Theorem:-

A point p in a topological space X is a limit point of $A \subset X$ iff each members of some local base $\mathcal{B}_p$ at p contains a point of A different from p.

1.6.23 Example:-

Consider the lower limit topology τ on the real line $\mathbb{R}$ which has as a base the class of closed-open intervals $[a,b)$, and let $A = (0,1)$. Note that $G = \{1,2)$ is a τ- open set containing $1 \in \mathbb{R}$ for which $G \cap A = \emptyset$ hence 1 is not a limit point of A. On the other hand , $0 \in \mathbb{R}$ is a limit point of A since any open base set $[a,b)$ containing 0 ,i.e. for which $a \leq 0 < b$ contains points of A other than 0.

1.6.24 Example:-

Every point p in a discrete topology has a finite local base.

Exercises: -

1. Let $\mathcal{B} = \{$(a,b):a,$b \in \mathbb{Q}\}$ be the class of open intervals in $\mathbb{R}$ with rational endpoints . Show that
   (1) $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$.
   (2) The topology generated by $\mathcal{B}$is the usual Euclidean topology on $\mathbb{R}$.
2. Let $\mathcal{B} = \{$[a,b]:a,$b \in \mathbb{R}\}$ be the class of all closed intervals in $\mathbb{R}$. Can $\mathcal{B}$ be a basis of some (not necessarily standard) topology on $\mathbb{R}$? Why or why not?
3. Show that the class of closed intervals [a,b], where a and b are rational and a<b is not a base for a topology on the real line $\mathbb{R}$.

4. Show that the class of closed intervals [a,b],where a is rational and b is irrational and a<b is a base for a topology on the real line $\mathbb{R}$.
5. Let $\mathcal{B}$,$\mathcal{B}'$ be two bases for X, satisfy the following conditions:
 (1) For every $B \subset \mathcal{B}$ and every $x \in B$,there exists a $B' \in \mathcal{B}'$s.t. $x \in B' \subset B$.
 (2) For every$B' \subset \mathcal{B}'$and every $x \in B'$,there exists a $B \subset \mathcal{B}$s.t. $x \in B \subset B'$.
 Show that $\mathcal{B}$ and $\mathcal{B}'$generate the same topology on X.
6. Let $\mathcal{B}$and $\mathcal{B}^*$ be bases , respectively ,for topologies τ and τ^* on a set X. Suppose that $B \in \mathcal{B}$ is the union of members of $\mathcal{B}^*$ .Show that τ is coarser than τ^*, i.e. $\tau \subset \tau^*$.
7. Show that the usual topology τ on the real line $\mathbb{R}$ is coarser than the upper limit topology τ^* on $\mathbb{R}$ which has as a base the class of open – closed intervals $(a,b]$.
8. Determine which of the following collection of subsets of $\mathbb{R}$ are bases:
 (1) $\tau_1 = \{(n,n+2) \subset \mathbb{R}: n \in \mathbb{Z}\}$.
 (2) $\tau_2 = \{[a,b) \subset \mathbb{R}: a \leq b\}$.
 (3) $\tau_3 = \{(-x,x) \subset \mathbb{R}: x \in \mathbb{R}\}$.
 (4) $\tau_4 = \{(a,b) \cup \{b+1\} \subset \mathbb{R}: a < b\}$.