Chapter Three

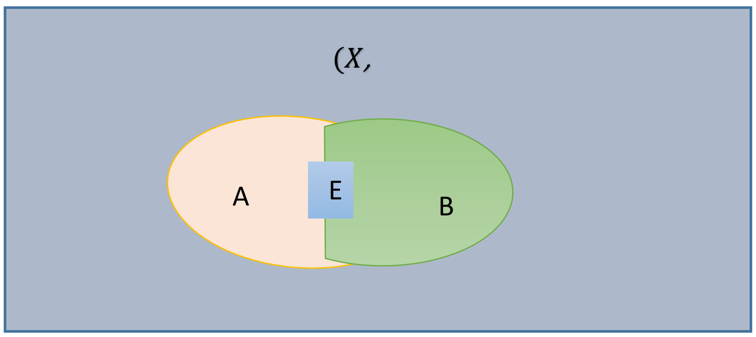
Connected and Compact Spaces

3.1 Connected Sets

3.1.1 Definition:

Two subsets A and B form ***a*** ***separation*** or ***partition*** of a set E in a topological space (X, τ) denote by $E = A|B$ iff they satisfy the followings:

1) $A \neq \emptyset$, $B \neq \emptyset$.
2) $E = A \cup B$.
3) $A \cap B = \emptyset$.
4) $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$.



3.1.2 Remark:

We can replace condition 4) by $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$.

3.1.3 Example:

Let (X, τ) be a topological space where $X = \{a,b,c,d,e\}$, $\tau=\{X,\emptyset,\{c\},\{a,b,c\},\{c,d,e\}\}$, $E = \{a,d,e\}$, $F=\{b,c,e\}$, $A=\{a\}$, $B=\{d,e\}$, $C=\{b\}$ and $D = \{c,e\}$. Show that $E = A|B$ and $F = C \nmid D$.

Solution:

1. $A \neq \emptyset$, $B \neq \emptyset$, 2. $E = A \cup B$, 3. $A \cap B = \emptyset$, 4. $\bar{A} \cap B = \{a,b\} \cap \{d,e\} = \emptyset$, $A \cap \bar{B} = \{a\} \cap \{d,e\} = \emptyset$, so $E = A|B$ but $C \cap \bar{D} = \{b\} \cap X = \{b\} \neq \emptyset$ i.e. $F = C \nmid D$.

3.1.4 Example:

Let $(\mathbb{R}, D)$ be the usual topology on $\mathbb{R}$. If $A = (1,2)$, $B=(2,3)$ & $C=[3,4)$ then the sets A,B are separation since $\bar{A}=[1,2]$, $\bar{B}=[2,3]$ then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$ but C,B are not separation since $3 \in C$ and 3 is a limit point of B i.e. $\bar{B} \cap C = [2,3] \cap [3,4) = \{3\} \neq \emptyset$.

3.1.5 Definition:

Let E be a subset of topological (X, τ) is ***connected*** set if there does not exist a separation for E and E is ***disconnected*** set if there exist a separation for E.

3.1.6 Example:

Consider the two topologies $\tau_1 = \{\{b\},\{a,b\},\{b,c\},X,\emptyset\}$, $\tau_2 = \{\{b\},\{c\},\{a,b\},\{b,c\},X,\emptyset\}$ On the set $X = \{a,b,c\}$ then X is connected in τ_1 and X is disconnected in $\tau_2$since there is $U = \{a,b\}$,$V=\{c\}$ s.t. $X = U|V$.

3.1.7 Example:

If a set X consists of more than one point and it has a discrete topology, then it is disconnected.

Solution:

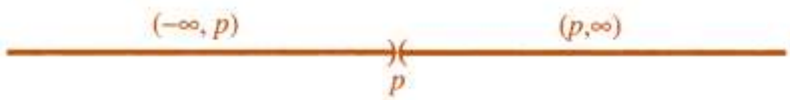
If A is any nonempty proper subset of X then the pair of sets A and X/A is a separation of X.

3.1.8 Example:

If $p \in \mathbb{R}$ then $\mathbb{R}/\{p\}$ is a disconnected topological space.

Solution:

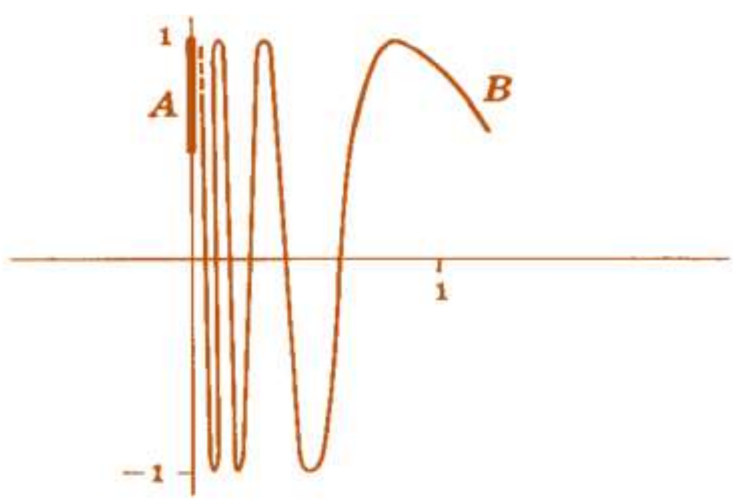
The pair $U = (-\infty,p)$ and $V=(p,\infty)$ is a separation of $\mathbb{R}/\{p\}$.



3.1.9 Example:

Consider the following subsets of the plane $\mathbb{R}^2$is connected

$$A = \{(0,y): \tfrac{1}{2} \leq y \leq 1\}\,,\, B = \{(x,y): y=\sin\left(\tfrac{1}{x}\right), 0 < x \leq 1\}$$



Solution:

Each point in A is a limit point of B then A and B are not separation i.e. they are connected.

3.1.10 Example:

Assume $X = (\text{-}1,0) \cup (0,1)$ is disconnected then there exists $\mathbb{R}$ is disconnected since the pair of sets (-1,0) and (0,1) is a separation of X.

3.1.11 Theorem:

If E is a subset of a subspace (Y,τ_Y) of a topological space (X, τ) then E is $\tau_Y - \text{connected}$ iff it is $\tau - \text{connected}$.

Proof:

In order to have a separation of ***E*** with respect to either topology, we must be able to write ***E*** as the union of two nonempty, disjoint sets. If ***A*** and ***B*** are two nonempty, disjoint sets whose union is ***E*** then $A,B \subseteq E \subseteq Y \subseteq X$.

$$(A \cap \bar{B}) \cup (\bar{A} \cap B) = \big((A \cap Y) \cap \bar{B}\big) \cup \big(\bar{A} \cap (Y \cap B)\big) = (A \cap \overline{B_Y}) \cup (\overline{A_Y} \cap B)$$

Thus if the condition is satisfied with respect to one topology, it is satisfied with respect to the other. □

3.1.12 Theorem:

Let (X, τ) be a topological space . X is disconnected iff there exists a non-empty proper subset of X which is both open and closed.

Proof:

$\Rightarrow$

Suppose $X = G \cup H$ where G and H are non-empty and open then G is a non-empty proper subset of X and since $G = H^c$, G is both open and closed.

$\Leftarrow$

Suppose A is a non-empty proper subset of X which is both open and closed. Then A^c is also non-empty and open and $X = A \cup A^c$. Accordingly, X is disconnected.□

3.1.13 Example:

The indiscrete topology (X, τ) is connected topology since X and Ø are only subsets of X which are both open and closed.

2.1.14 Example:

Let (X,τ) be a co-finite topology where X is infinite is connected space.

Solution:

Assume X is disconnected then there exists A,B are nonempty open subset of X and $A \cap B = \emptyset$ separation for X then A^c,B^c are finite sets and $A^c \cup B^c = X$ this implies that X is finite and this is contradiction since X is infinite ,so X is connected.

2.1.15 Exercise:

Let (X, τ) be a co-finite topology where X is finite is disconnected space.

2.1.16 Example:

In $\mathbb{R}$ with the lower limit topology then $\mathbb{R}$ is disconnected since every intervals [a,b) are open and closed sets.

3.1.17 Theorem:

If C is a connected subset of a topological space (X, τ) which has a separation $X = A|B$ then either $C \subseteq A$ or $C \subseteq B$.

Proof:

Suppose that $X = A|B$ then

$$C = C \cap X = C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$$

$$(C \cap A) \cap (C \cap B) = C \cap (A \cap B) = C \cap \emptyset = \emptyset$$

$$\left((C \cap A) \cap \overline{(C \cap B)}\right) \cup \left(\overline{(C \cap A)} \cap (C \cap B)\right) \subseteq (A \cap \bar{B}) \cup (\bar{A} \cap B) = \emptyset$$

Thus we see that if we assume that both $C \cap A = \emptyset$ and $C \cap B = \emptyset$ we have a separation for $C = (C \cap A)|(C \cap B)$.Hence, either $C \cap A$ is empty so that $C \subseteq B$ or or $C \cap B$ is empty so that $C \subseteq A$.□

3.1.18 Corollary(1):

If C is a connected set in a topological space (X, τ) and $C \subseteq E \subseteq \bar{C}$ then E is a connected set.

Proof:

If E is not a connected set, it must have a separation $E = A|B$ .By theorem 3.1.17 must be contained in A or contained in B. Assume $C \subseteq A$ it follows that $\bar{C} \subseteq \bar{A}$ and hence $\bar{C} \cap B \subseteq \bar{A} \cap B = \emptyset$. On the other hand,$B \subseteq E \subseteq \bar{C}$ and so $B \cap \bar{C} = B$,so that we must have $B = \emptyset$,which contradicts our hypothesis that $E = A|B$.□

3.1.19 Corollary(2):

If every two points of a set E are contained in some connected subset of E, then E is a connected set.

Proof:

If *E* is not connected, it must have a separation $E = A|B$.Since *A* and *B* must be nonempty, let us choose points $a \in A$ and $b \in B$.From the hypothesis we know that *a* and *b* must be contained in some connected subset *C* contained in *E.* By theorem 3.1.17 requires that *C* be either a subset of *A* or a subset of *B.* Since *A* and *B* are disjoint, this is a contradiction then *E* is connected. □

3.1.20 Corollary (3):

The union E of any family $\{C_\lambda\}$ of connected sets having a nonempty intersection ($\bigcap_\lambda C_\lambda \neq \emptyset$) is a connected set.

Proof:

If *E* is not connected, it must have a separation $E = A|B$.By hypothesis, we may choose a point $x \in \bigcap_\lambda C_\lambda$**.** The point x must belong to either *A* or *B.* Let us suppose $x \in A$.Since x belongs to C_λ for every λ, $C_\lambda \cap A \neq \emptyset$ for every λ.By theorem 3.1.17, however, each C_λ must be either a subset of *A* or a subset of *B.* Since *A* and *B* are disjoint sets we must have $C_\lambda \subseteq A$ for all λ , and so$E \subseteq A$**.** From this we obtain the contradiction that $B = \emptyset$.□

3.1.21 Remark:

1. The structure of the connected subsets of the real line is deceptively simple. For example, if the removal of a single point x from a connected set *C* leaves a disconnected set, then $C/\{x\}$ is the union of two disjoint connected sets.
2. Another geometrically reasonable property of connected sets is given in the following theorem:

3.1.22 Theorem:

If a connected set C has a nonempty intersection with both a set E and the complement of E in a topological space (X, τ),then C has a nonempty intersection with the boundary of E (i.e. $C \cap b(E) \neq \emptyset$).

Proof:

We will show that if we assume that C is disjoint from $b(E)$ we obtain the contradiction that $C = (C \cap E)|(C \cap E^c)$.

From the equation $C = C \cap X = C \cap (E \cup E^c) = (C \cap E) \cup (C \cap E^c)$ we see that C is the union of the two sets. These two sets are nonempty by hypothesis. If we calculate

$$(C \cap E) \cap \overline{(C \cap E^c)} \subseteq (C \cap E) \cap \overline{E^c} = C \cap (E \cap \overline{E^c}) = C \cap b(E),$$

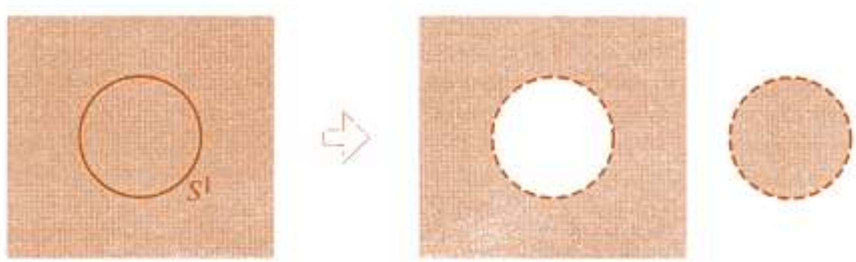
we see that the assumption that $C \cap b(E) = \emptyset$ leads to the conclusion that $(C \cap E) \cap \overline{(C \cap E^c)} = \emptyset$. In the same way we may show that $\overline{(C \cap E)} \cap (C \cap E^c) = \emptyset$, and we have a separation of C.□

3.1.23 Definition:

Let (X, τ) be a connected topological space . A ***cutset*** of X is a subset of X such that X/S is disconnected . A ***cutpoint*** of X is a point $p \in X$ such that $\{p\}$ is a cutset of X. A cutset or cutpoint of X is said to ***separate*** X.

3.1.24 Example:

The plane $\mathbb{R}^2$ is connected . If we remove the circle S^1, we are left with two disjoint nonempty open sets.



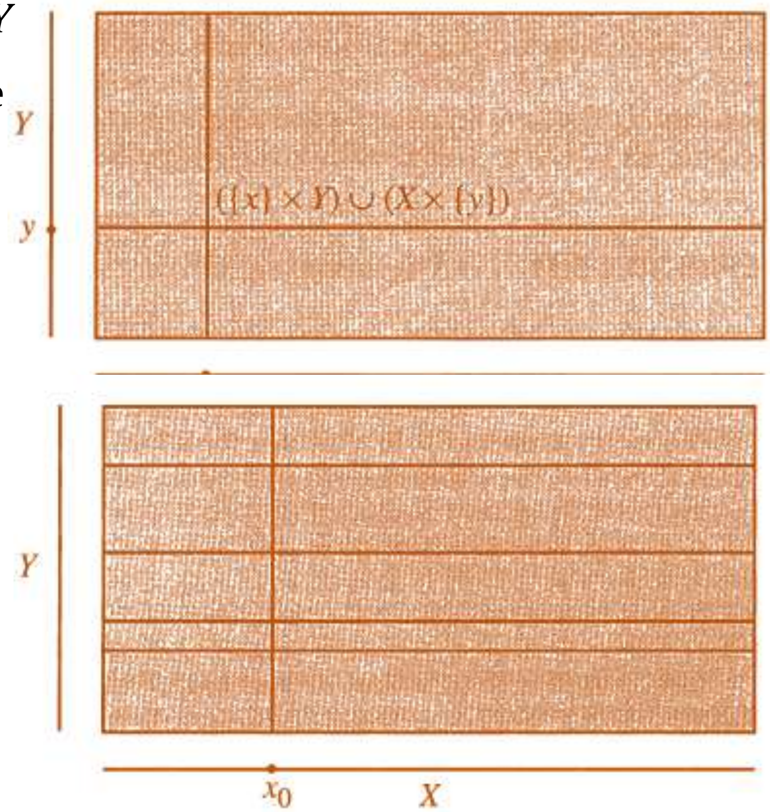
3.1.25 Theorem:

Let $X_1, \cdots, X_n$ be connected spaces. Then the product space $X_1 \times \cdots \times X_n$ is connected.

Proof:

We shall prove the product of two spaces. The general result can then be shown by induction. Assume that X and Y are connected topological spaces. For every

$x \in X$, the subspace $\{x\} \times Y$ of $X \times Y$ is homeomorphic to Y and is therefore connected. Similarly, for every $y \in Y$, the subspace $X \times \{y\}$ of $X \times Y$ is connected. Thus, by Corollary 3.1.20, for every $x \in X$ and $y \in Y$ the set $(\{x\} \times Y) \, U \, (X \times \{y\})$ is connected in $X \times Y$.



Now fix $x_0 \in X$ and let y vary. Each set $(\{x_0\} \times Y) \, U \, (X \times \{y\})$ Contains the set $\{x_0\} \times Y$. It then follows by Corollary 3.1.20 that $\bigcup_{y \in Y}((\{x_0\} \times Y) \cup (X \times \{y\}))$ is connected in $X \times Y$. Furthermore, $\bigcup_{y \in Y}((\{x_0\} \times Y) \cup (X \times \{y\})) = X \times Y$, implying that $X \times Y$ is connected.□

3.2 Components

3.2.1 Definition:

A ***component*** E of a topological space (X, τ) is a maximal connected subset of X i.e. E is connected and E is not A proper subset of any connected subset of X.

3.2.2 Example:

If X is connected then X has one component X itself . Also $(\mathbb{R},\tau)$ the usual topology has one component $\mathbb{R}$ itself.

3.2.3 Example:

Consider the following topology on $X = \{a,\mathrm{b,c,d},e\}$, $\tau = \{X,\emptyset,\{\mathrm{a}\},\{\mathrm{c,d}\},\{\mathrm{a,c,d}\},$ {b,c,d,e}} then the components of X are {a} and {b,c,d,e}.Any other connected subset of X such that $\{b,\mathrm{d},e\}$ is a subset of one of the components.

3.2.4 Theorem:

The components of a topological space(X, τ) are closed subsets of X.

Proof:

If C is a component of X**,** choose a point $x \in C$ and suppose that $y \in \bar{C}$**.** Since $\bar{C}$ is a connected set by Corollary 1,y is in a connected subset of X which contains x**.** Hence$\bar{C} \subseteq C$, and so C must be closed.□

3.2.5 Theorem:

Every connected subset of a topological space (X, τ) is contained in a connected component.

Proof:

Assume A is a connected subset of a topological space (X, τ) . If $\{A_i : i \in I\}$ is a family of connected contained A i.e. $A_i \subseteq A;\ \forall i \in \mathbb{N}$ then $A \neq \emptyset$,so $\bigcap_i A_i \neq \emptyset$ by Corollary (3) we get $C = \bigcup_i A_i$ is a connected contain A .If E is connected contain C then E also contain A , so E=C then C is a component contain A.

3.2.6 Corollary:

Every point in a topological space (X, τ) is contained in a connected component.

Proof:

Since for every $p \in X$ the set $\{p\}$ is connected then by theorem 3.2.5 Every point in a topological space (X, τ) is contained in a connected component.□

3.2.7 Theorem:

The component of a topological space (X, τ) forms a partition of X.

Proof:

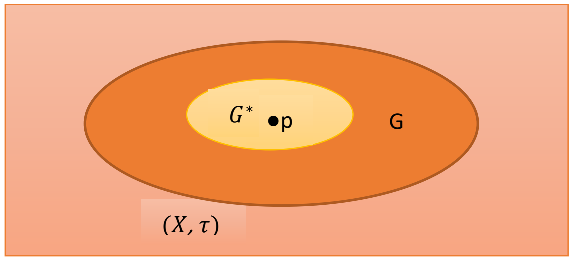
Let $\{C_i\}_{i\in\mathbb{N}}$ be a family of connected component in a topological space (X, τ) then

1. $C_i \cap C_j = \emptyset, \forall i \neq j$ since if $C_i \cap C_j \neq \emptyset$ then by corollary (3) we get $C_i \cup C_j$ is connected contain the sets C_i,C_j and since C_i,C_j are connected component then $C_i = C_i \cap C_j = C_j$ and this is contradiction.
2. It's clear that $X = \bigcup_{i\in\mathbb{N}} C_i$.□

3.3 Locally Connected Spaces

3.3.1 Definition:

A topological space (X, τ) is ***locally connected*** at $p \in X$ iff every open set G containing p, there exists a connected open set G^*containing p and contained in G. Thus a space is ***locally connecte***d iff the family of all open connected sets is a base for the topology for the space.



3.3.2 Remark:

A locally connected set need not be connected. For example, a set consisting of two disjoint open intervals is locally connected but not connected. The connected subsets of the real numbers are locally connected, but this implication need not hold in general i.e. in topological spaces The connected subsets need not be a locally connected set.

locally connected set	$\nRightarrow$	connected sets	in topological spaces
connected sets	$\nRightarrow$	locally connected set	in topological spaces
connected sets	$\Rightarrow$	locally connected set	in real numbers

3.3.3 Example

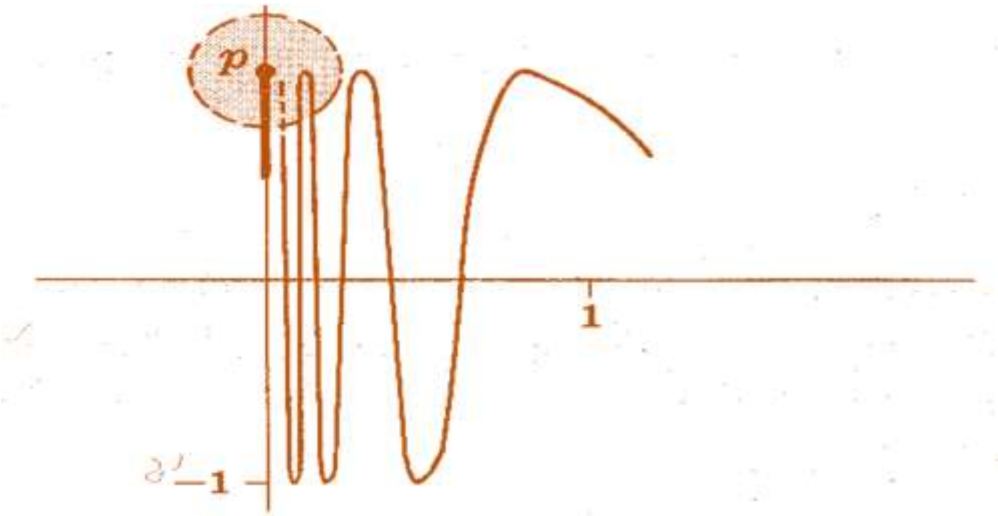
Every discrete topological space (X, τ) is locally connected.

Solution:

If $p \in X$ then $\{p\}$ is an open connected set containing p which is contained in every open set containing p (Note that X is not connected if X contains more than one point).

3.3.4 Example:

Let A and B be subsets of the plane $\mathbb{R}^2$ of example 3.1.9 , $A \cup B$ is a connected set but $A \cup B$ is not locally connected at $p = (0,1)$. For example the open disc with center p and radius $\frac{1}{4}$ does not contain any connected open set contain p.



3.3.5 Theorem:

Let E be a component in locally connected space (X, τ) *then E is open.*

Proof:

Let $p \in E$. Since X is locally connected space then p belongs to at least one connected set G_p but E is the component of p hence $p \in G_p \subset E$ and so $E = \cup \{G_p : p \in E\}$. Therefore, E is open since it is the union of open sets. □

3.3.6 Theorem:

Let (X, τ) be a locally connected space and let Y be an open subset of X then the subspace (Y, τ_Y) is locally connected.

Proof:

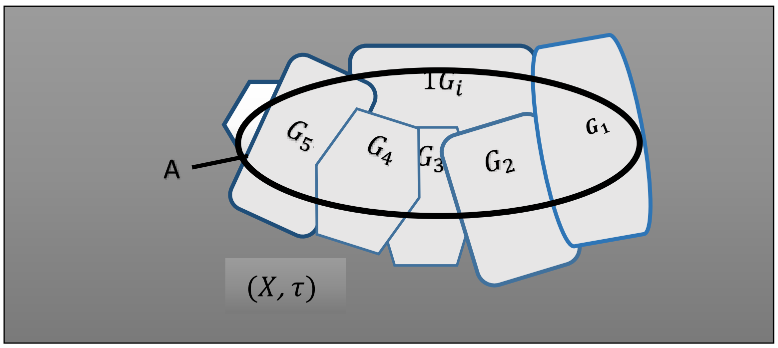
Assume $p \in Y$, N *is an open* set in (Y, τ_Y) contain p so there exist an open set U in X such that $Y \cap U = N$ but Y is an open set in X ,so N is an open set in X contain p and X is locally connected then there exists a connected set W in X such that $p \in W \subseteq U$. Now we have $V = W \cap Y \subseteq Y \cap U = N$ where V is a connected set in Y contain p so (Y, τ_Y) is locally connected.□

3.4 Compact Spaces

3.4.1 Definition:

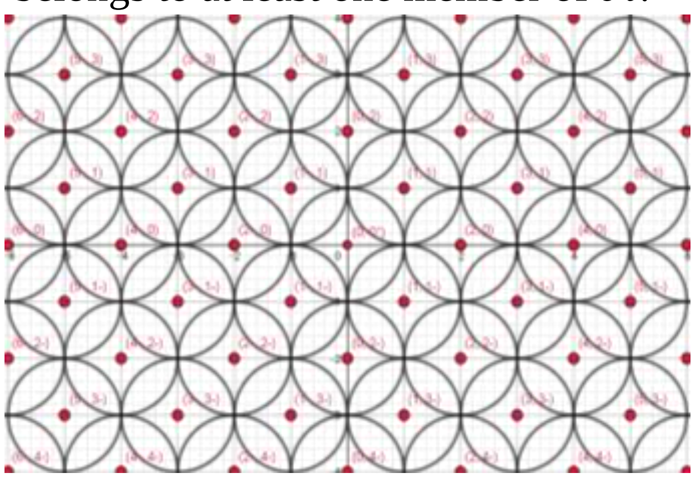
Let A be a subset of a topological space (X, τ) and let $\mathcal{A} = \{G_i\}_i$ be a collection of subsets of X then:

1. The collection $\mathcal{A}$ is said to ***cover*** A or to be a ***cover*** of A is contained in the union of sets in $\mathcal{A}$,(i.e. $A \subseteq \bigcup_i G_i$).
2. If $\mathcal{A}$ covers and each set in $\mathcal{A}$ is open then we call $\mathcal{A}$ an ***open cover*** of A.
3. If $\mathcal{A}$ covers A ,and $\mathcal{A}'$ is a subcollection of $\mathcal{A}$ that also covers A, then $\mathcal{A}'$ is called a ***subcover*** of $\mathcal{A}$.



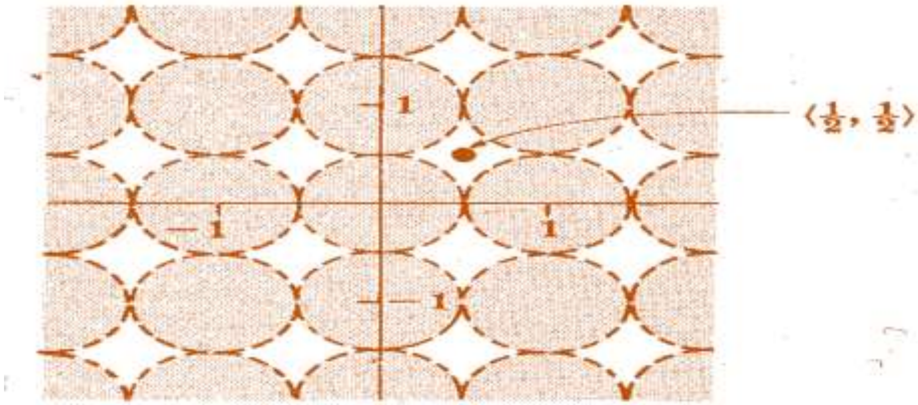
3.4.2 Example:

Consider the class $\mathcal{A} = \{D_p : p \in \mathbb{Z} \times \mathbb{Z}\}$, where D_p is the open disc in the plane $\mathbb{R}^2$with radius 1 and center $p = (m,n)$,m and n integers.Then $\mathcal{A}$ is a cover of $\mathbb{R}^2$, i.e. every point in $\mathbb{R}^2$ belongs to at least one member of $\mathcal{A}$.



3.4.3 Remark:

In example 3.4.2 if we take the collection of open discs $\mathcal{B} = \{D_p^* : p \in \mathbb{Z} \times \mathbb{Z}\}$, where D_p^* has center p and radius $\frac{1}{2}$, is not a cover of $\mathbb{R}^2$.For example the point $(\frac{1}{2},\frac{1}{2}) \in \mathbb{R}^2$ does not belong to any member of $\mathcal{B}$.



3.4.4 Definition:

A topological space (X, τ) is ***compact*** iff every open cover of X has finite subcover,(i.e. if $\mathcal{A} = \{G_i\}_i$ is an open cover for X ($X \subseteq \bigcup_i G_i$) then there exists $\{G_1, G_2, \dots, G_n\}$ finite subcover s.t. $X \subseteq \bigcup_{i=1}^{n} G_i$.

3.4.5 Example:

Let A be any finite subset of a topological space (X, τ) then A is compact.

Solution:

Let $A = \{a_1, a_2, \dots, a_n\}$ be a finite subset of a topological space (X, τ) and let $\mathcal{A} = \{G_i\}_i$ be an open cover for A, i.e. $A \subseteq \bigcup_i G_i$ then

$$\because a_1 \in A \longrightarrow \exists\, G_1 \in \mathcal{A}, \text{ s.t. } a_1 \in G_1$$
$$\because a_2 \in A \longrightarrow \exists\, G_2 \in \mathcal{A}, \text{ s.t. } a_2 \in G_2$$

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$$\because a_n \in A \longrightarrow \exists\, G_n \in \mathcal{A}, \text{ s.t. } a_n \in G_n$$

Then $A = \{a_1, a_2, \dots, a_n\} \subseteq \{G_1, G_2, \dots, G_n\} = \bigcup_{i=1}^{n} G_i$,A is compact.

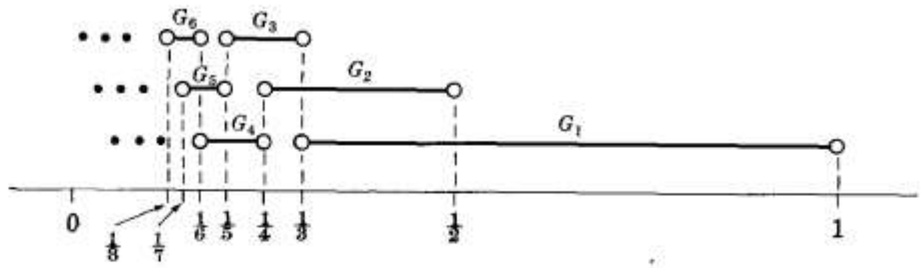
3.4.6 Example:

The open interval $A = (0,1)$ on the real line $\mathbb{R}$ with the usual topology is not compact.

Solution:

Assume A is compact and let $\mathcal{A} = \left\{G_n = \left(\frac{1}{n+2}, \frac{1}{n}\right) : n \in \mathbb{N}\right\} = \left\{\left(\frac{1}{3}, 1\right), \left(\frac{1}{4}, \frac{1}{2}\right), \left(\frac{1}{5}, \frac{1}{3}\right), \dots\right\}$ be an open cover for A such that $A \subseteq \bigcup_{n=1}^{\infty} G_n$ then $\mathcal{A}$ has finite subcover $\mathcal{A}' = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$ for A .

Let $\in = \min\{a_1, a_2, \dots, a_n\}$ then $\in > 0$ and $(a_1, b_1) \cup (a_2, b_2) \cup \dots \cup (a_n, b_n) \subseteq (\in, 1)$.But $(0, \in]$ and $(\in, 1)$ are disjoint hence $\mathcal{A}'$is not a cover of A and A is not compact



3.4.7 Example:

The subset $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is compact in $\mathbb{R}$ with the usual topology.

Solution:

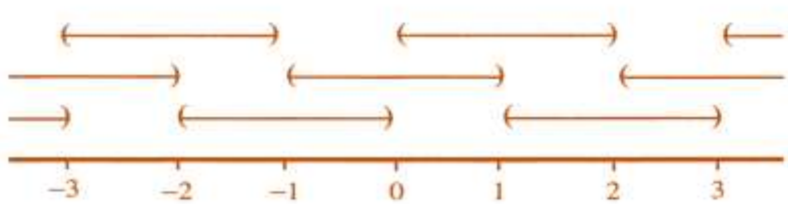
Let $\mathcal{A}$ be an open cover for A. Since $0 \in A$ then there exists at least one open set $U_0 \in \mathcal{A}$, $0 \in U_0$.Let $\varepsilon > 0$, s.t. $0\in(-\varepsilon,\varepsilon)\subseteq U_0$. By Archimedes theorem $\exists k \in \mathbb{N}$, s.t. $\frac{1}{k} < \mathcal{E} \longrightarrow \frac{1}{n}\in(-\varepsilon,\varepsilon)\subseteq U_0$,n>k .Now since $\frac{1}{n} \in A$,$1\leq n\leq k\longrightarrow\exists U_n \in \mathcal{A}$, s.t. $\frac{1}{n} < U_n$,$1\leq n\leq k$, so $\{U_0,U_1,U_2,\ldots,U_k\}$ is a finite subcover of $\mathcal{A}$ for A .Then A is compact.

3.4.8 Example:

Consider (0,1] as a subspace of $\mathbb{R}$ then (0,1] is not compact , since $\mathcal{A} = \left\{\left(\frac{1}{n},2\right): n \in \mathbb{Z}^+\right\}$ is an open cover for (0,1] has no finite subcover of $\mathcal{A}$ that cover (0,1].

3.4.9 Example:

The real line $\mathbb{R}$ with the usual topology is not compact since $\mathcal{A}$=$\{\ldots,(-1,1),(0,2),(1,3),\ldots\}$ is an open cover has no finite subcover for.



3.4.10 Example:

Let (X, τ) be the co-finite topology then X is compact.

Solution:

Let $\mathcal{A} = \{G_i\}$ be an open cover of X .Choose $G_0 \in \mathcal{A}$. Since τ is the co-finite topology, G_0^c is a finite set , i.e. $G_0^c = \{a_1, a_2, \ldots, a_m\}$.Since $\mathcal{A}$be an open cover of X, for each $a_k \in G_0^c$ $\exists G_{i_k} \in \mathcal{A}$ such that $a_k \in G_{i_k}$.Hence $G_0^c \subseteq G_{i_1} \cup G_{i_2} \cup \ldots \cup G_{i_m}$

and $X = G_0 \cup G_0^c = G_0 \cup G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_m}$.Thus X is compact.

3.4.11 Example:

Every infinite subset A of a discrete topological space (X, τ) is not compact.

Solution:

Let $\mathcal{A} = \{\{a\}: a \in A\}$ be a collection of singleton subsets of A,i.e. $A = \cup \{\{a\}: a \in A\}$ then $\mathcal{A}$ is an open cover of A since every subsets of a discrete topology are open. $\mathcal{A}$ is infinite since A is infinite ,so $\mathcal{A}$ has no finite subcover for A.

3.4.12 Remark:

From examples 3.4.5 and 3.4.11we get a subset of a discrete topology is compact iff it is finite.

3.4.13 Example:

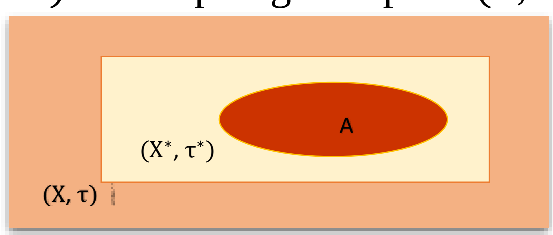
The indiscrete topology (X, τ) is compact.

Solution:

Since $\tau = \{\emptyset, X\}$ then any open cover for X must be of the form $\mathcal{A} = \{X\}$ which is finite cover since it contain X only , X is compact.

3.4.14 Theorem:

If A is a subset of a subspace (X^*, τ^*) of a topological space (X, τ) then A is τ^*-compact iff it is τ -compact.



Proof:

$\Longrightarrow$

Suppose A is τ^*-compact and $\{G_i\}$ is some τ -open covering of A. The family of sets $\{X^* \cap G_i\}$ clearly forms a τ^*-open covering for A since $A = X^* \cap A \subseteq X^* \cap (\cup_i G_i) = \cup_i (X^* \cap G_i)$. Since A is τ^*-compact, there is a finite subcovering $A \subseteq \cup_{i=1}^{n} (X^* \cap G_i) \subseteq \cup_{i=1}^{n} G_i$ of A which yields a finite subcovering of A from $\{G_i\}$.

$\Longleftarrow$

Now suppose that A is τ -compact and $\{G_i^*\}$ is some τ^*-open covering of A. From the definition of the induced topology, each $G_i^* = X^* \cap G_i$ for some τ - open set G_i. The family $\{G_i\}$ is clearly a τ -open covering of A and so there must be some finite subcovering $A \subseteq \cup_{i=1}^{n} G_i$. But then we have $A = X^* \cap A \subseteq X^* \cap (\cup_{i=1}^{n} G_i) = \cup_{i=1}^{n} (X^* \cap G_i) = \cup_{i=1}^{n} G_i^*$.and so a finite subcovering of A from $\{G_i^*\}$.□

3.5 Finite Intersection Property

3.5.1 Definition:

A family $\{A_i\}$ of sets will be said to have the ***Finite Intersection Property*** (denote by F.I.P.) iff every finite subfamily $\{A_i\}_{i=1}^{n}$ of the family has a nonempty intersection $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

3.5.2 Example:

$The\ family\ \mathcal{A} = \left\{(0,\frac{1}{n}) : n \in \mathbb{N}\right\} = \{(0,1),\left(0,\frac{1}{2}\right),\left(0,\frac{1}{3}\right).\left(0,\frac{1}{4}\right), \dots\}\ has\ F.I.P.$

Solution:

Let $\{(0,a_1)\ ,(0,a_2)\ ,(0,a_3)\ , \dots ,(0,a_n)\}$ be a finite subfamily of $\mathcal{A}$ and let $b = \min\{a_1,a_2,a_3,\dots,a_n\} > 0$ then $(0,a_1)\cap(0,a_2)\cap(0,a_3)\cap\dots\cap(0,a_n) = (0,b) \neq \emptyset$,so $\mathcal{A}$ has F.I.P.

3.5.3 Remark:

In example 3.5.2 we have $\bigcap_{n\in\mathbb{N}}(0,\frac{1}{n}) = \emptyset$.

3.5.4 Example:

The family $\mathfrak{B} = \{(-\infty,n] : n \in \mathbb{Z}\} = \{\dots,(-\infty,-2],(-\infty,-1],(-\infty,0],(-\infty,1],(-\infty,2],\dots\}$ has F.I.P.

Solution:

Let $\{(-\infty,a_1]\ ,(-\infty,a_2]\ ,(-\infty,a_3]\ , \dots ,(-\infty,a_n]\}$ be a finite subfamily of $\mathfrak{B}$ and let $b = \min\{a_1,a_2,a_3,\dots,a_n\} > 0$ then $(-\infty,a_1] \cap(-\infty,a_2]\cap (-\infty,a_3] \cap \dots\cap(-\infty,a_n] = (-\infty,b] \neq \emptyset$,so $\mathfrak{B}$ has F.I.P. Note that $\bigcap_{n\in\mathbb{N}}(-\infty,n] = \emptyset$.

3.5.5 Theorem:

A topological space (X, τ) is compact iff any family of closed sets having the finite intersection property has a nonempty intersection.

Proof:

$\Longrightarrow$

Let us suppose that (X,τ) is compact and $\{F_i\}$ is a family of closed sets whose intersection is empty. Since $\bigcap_i F_i = \emptyset$, we may take the complement of each side of the equation and, using DeMorgan's Law, obtain $X = \emptyset^c = (\bigcap_i F_i)^c = \bigcup_i {F_i}^c$. Thus the family $\{{F_i}^c\}$ is an open covering of the compact space X, and so there must exist some finite subcovering. But if $X = \bigcup_{i=1}^{n} {F_i}^c$ then $\emptyset = X^c = (\bigcup_{i=1}^{n} {F_i}^c)^c = \bigcap_{i=1}^{n} F_i$ so that the family $\{F_i\}$ cannot have the finite intersection property.

$\Leftarrow$

Now suppose (X,τ) is not compact. From the definition this means that there must be some open covering $\{G_i\}$ of X which has no finite subcovering. To say that there is no finite subcovering means that the complement of the union of any finite number of members of the cover is nonempty. By DeMorgan's Law, the family $\{G_i{}^c\}$ is then a family of closed sets with the finite intersection property. Since $\{G_i\}$ is a covering of X, however, $\bigcap_i G_i{}^c = \emptyset$ since $\emptyset = X^c = (\bigcup_i G_i)^c = \bigcap_i G_i{}^c$. Thus this family of closed sets with the finite intersection property has an empty intersection. □

3.5.6 Theorem:

Every closed subset of a compact space is compact.

Proof:

Let $\mathcal{A} = \{G_i\}$ be an open cover of F the closed subset of a compact space (X,τ) , i.e. $F = \bigcup_i G_i$.Then $X = F \cup F^c = (\bigcup_i G_i) \cup F^c$, i.e. $\mathcal{A}^* = \{G_i\} \cup \{F^c\}$ is a cover of X . But F^c is open since F is closed , so $\mathcal{A}^*$is an open cover of X. By hypotheses, X is compact ; hence $\mathcal{A}^*$has a finite subcover of X i.e.

$$X = G_1 \cup G_2 \cup \dots \cup G_n \cup F^c,\ G_i \in \mathcal{A}, i=1,2,\dots,n$$

But F andF^c are disjoint ; hence

$$F \subseteq G_1 \cup G_2 \cup \dots \cup G_n\ , G_i \in \mathcal{A}, i=1,2,\dots,n.$$

WE have shown that any open cover $\mathcal{A} = \{G_i\}$ of F contains a finite subcover, i.e. F is compact.□

3.6 Sequentially compact sets

3.6.1 Definition:

A subset A of a topological space (X,τ) is ***sequentially compact*** iff every sequence in A contains a subsequence which converges to a point in A.

3.6.2 Example:

Let A be a finite subset of a topological space (X,τ) then A is sequentially compact.

Solution:

Let $\langle a_1, a_2, a_3, \dots \rangle$ be a sequence in A then at least one of the elements in A say a_0 must appears an infinite number of times in the sequence ,hence $\langle a_0, a_0, a_0, \dots \rangle$ is a subsequence of $\langle a_n \rangle$ it converges to $a_0 \in A$.

3.6.3 Example:

The open interval $A = (0,1)$ in $\mathbb{R}$ with the usual topology is not sequentially compact.

Solution:

Consider the sequence $\langle a_n \rangle = \langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \rangle$ in A which converge to 0 then every subsequence is also converge to 0. But $0 \notin A$, i.e. the sequence $\langle a_n \rangle$ does not contain a subsequence converge to a point in A. So A is not sequentially compact.

3.6.4 Remark:

In general, there exists compact sets which are not sequentially compact and vise versa although in metric spaces they are equivalent.

3.6.5 Example:

Let $\tau = \{\emptyset, U \subseteq X : U^c \text{is countable}\}$ be a topology on a non-empty set X then every infinite subset of X is not sequentially compact.

Solution:

The sequence $\langle a_n \rangle = \langle a_1, a_2, a_3, \dots \rangle$ in X converge to $b \in X$ iff THE sequence of the form $\langle a_1, a_2, a_3, \dots, a_n, \text{b}, \text{b}, \dots \rangle$, i.e.the set A consisting of the terms of $\langle a_n \rangle$

different from b is finite. Now A is countable and so $A^c$is an open set containing b. Hence if $a_n \longrightarrow b$ then A^ccontain all except a finite number of the terms of the sequence and so A is finite .Hence if A is an infinite subset of X , there exists a sequence $\langle b_n \rangle$ in A with distinct terms. Thus $\langle b_n \rangle$ does not contain any convergent subsequence and A is not sequentially compact.

3.6.6 Theorem:

Let A be a sequentially compact subset of a topological space (X,τ) then every countable open cover of A has a finite subcover.

Proof:

Assume A is infinite for otherwise the proof is trivial and assume there exists a countable open cover $\{G_i : i \epsilon \mathbb{N}\}$ with no finite subcover .Let $n_1$be the smallest integer such that $A \cap G_{n_1} \neq \emptyset$. Choose

Let $n_1$be the smallest integer s.t. $A \cap G_{n_1} \neq \emptyset$.Choose $a_1 \in A \cap G_{n_1}$

Let $n_2$be the least positive integer larger than n_1 s.t. $A \cap G_{n_2} \neq \emptyset$. Choose $a_2 \in \left(A \cap G_{n_2}\right) \setminus (A \cap G_{n_1})$.

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We obtain the sequence $\langle a_1, a_2, a_3, \ldots \rangle$with the property that , for every $i \in \mathbb{N}$,

$$a_i \in A \cap G_{n_i} \text{ , } a_i \notin \bigcup_{j=1}^{n-1} \left(A \cap G_{n_j}\right) \text{ and } n_i > n_{i-1}$$

We claim that $\langle a_i \rangle$ has no convergent subsequence in A . Let $p \in A$ then

$$\exists G_{i_0} \in \{G_i\} \text{ s.t. } p \in G_{i_0}.$$

Now $A \cap G_{i_0} \neq \emptyset$ since $p \in A \cap G_{i_0}$, hence $\exists j_0 \in \mathbb{N}$ s.t. $G_{j_{n_0}} = G_{i_0}$.But by the choice of the sequence $\langle a_1, a_2, a_3, \ldots \rangle, i > j_0 \Longrightarrow a_i \notin G_{i_0}$.Accordingly since G_{i_0} is an open set containing p, no subsequence of $\langle a_i \rangle$ converge to p. But p was arbitrary, so A is not sequentially compact and this is contradiction then every countable open cover of A has a finite subcover. □

3.7 Countable Compact Spaces

3.7.1 Definition:

A subset A of a topological space (X,τ) is ***countably compact*** iff every infinite subset B of A has at least one limit point in A.

3.7.2 Theorem (Bolzano-Weierstrass Theorem):

Every bounded infinite set of real numbers has a limit point.

3.7.3 Example:

Every bounded closed interval $A = [a,b]$ is countably compact.

Solution:

Assume B is an infinite subset of A .Since A is bounded and $B \subseteq A$ then by Bolzano-Weierstrass Theorem B has a limit point p .Since A is closed and $d(B) \subseteq d(A)$ then the limit point of B belongs to A, i.e. A is locally compact.

3.7.4 Example:

The open interval $A = (0,1)$ is not countably compact.

Solution:

Consider the infinite subset $B = \{\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$ of A .Observe that B has exactly one limit point which is 0 but $0 \notin A$,hence A is not countably compact.

3.7.5 Remark:

The general relationship between compact, sequentially compact and countably compact sets is given in the following diagram, theorems (3.7.6 , 3.7.7) and example 3.7.8.

Compact	$\Rightarrow$ $\not\Leftarrow$	countably compact	$\Leftarrow$ $\not\Rightarrow$	sequentially compact

3.7.6 Theorem:

A compact subset of a topological space is countably compact.

Proof:

Assume (X,τ) is a compact topological space and let A be infinite subset of ***X*** with *no limit points in X*, i.e. for each point $x \in X$ is not a limit point of A so there must *exist an open* G_x *containing x such that* $G_x \backslash \{x\} \cap A =$

$\emptyset$. Clearly$G_x \cap A$ contains, at most, the one point x itself. Since the family $\{G_x\}_{x\in X}$ forms an open covering of the compact space ***X***, there must be some finite subcovering $X = \bigcup_{i=1}^{n} G_{x_i}$. From this it follows that $A = A \cap X = A \cap \left(\bigcup_{i=1}^{n} G_{x_i}\right) = \bigcup_{i=1}^{n}(A \cap G_{x_i})$ is a finite union of sets, each containing, at most, one element, and so A is finite and this is contradiction. Thus every infinite subset of ***X*** must have at least one limit point. □

3.7.7 Theorem:

A sequentially compact subset of a topological space is countably compact.

Proof:

Let A be any infinite subset of X. Then there exists a sequence $\langle a_1, a_2, a_3, \dots \rangle$ in A with distinct terms. Since X is sequentially compact then the sequence $\langle a_n \rangle$ contains a subsequence $\langle a_{i_1}, a_{i_2}, a_{i_3}, \dots \rangle$ (also with distinct terms) which converges to a point $p \in X$. Hence every open set G_p contain p contains an infinite number of points in A. Since $p \in X$ is a limit point of A, i.e. X is countably compact.□

3.7.8 Example:

Let τ be the topology on $\mathbb{N}$, the set of positive integers generated by sets {{1,2},{3,4},{5,6},...} .Let A be a non – empty infinite subset of $\mathbb{N}$, say $n_0 \in A$. If n_0 is odd then $n_0 + 1$ is a limit point of A , and if n_0 is even then $n_0 - 1$ is a limit point of A. In either case A has a limit point, so $(\mathbb{N}, \tau)$ is countably compact.

On the other hand $(\mathbb{N}, \tau)$ is not compact since $\mathcal{A} = \{\{1,2\},\{3,4\},\{5,6\},\dots\}$ is an open cover of $\mathbb{N}$ with no finite subcove. Also $(\mathbb{N}, \tau)$ is not sequentially compact since the sequence $\langle 1,2,3, \dots \rangle$ contains no convergent subsequence.

3.7.9 Theorem:

A closed subset of countably compact is countably compact.

Proof:

Let F be a closed subset of countably compact space (X, τ) and let A be any infinite subset of F.

Since $A \subseteq F$ then $A \subseteq X$ but X is countably compact, so A has a limit point $p \in X$.Since $A \subseteq F$ and F is closed set then F is countably compact.□

3.8 Locally Compact Spaces

3.8.1 Definition:

A topological(X,τ) is ***locally compact*** iff each point of X is contained in a compact neighborhood.

3.8.2 Remark:

Since a compact space is a compact neighborhood of each of its points, it is clear that $every\ compact\ space\ is\ locally\ compact$, i.e. every compact space is locally compact but the converse is not true as the following example.

3.8.3 Example:

Let $(\mathbb{R},\tau)$ be the usual topology .For each point $p \in \mathbb{R}$ there exists a closed interval $[p-\mathcal{E},p+\mathcal{E}]$ contain p. Since every closed interval is closed and bounded then its compact by Heine-Borel Theorem (A subset of the real line is compact iff it is closed and bounded). Hence $\mathbb{R}$ $is\ a\ locally\ compact\ space$. On the other hand $\mathbb{R}$ is not compact since the class $\mathcal{A}=$ {..,(-3,-1),(-2,0),(-1,1),(0,2),(1,3),...} is an open cover of $\mathbb{R}$ but contains no finite subcover.

3.8.4 Example:

The discrete topology (X,τ) is locally compact since $\forall p \in X\ \exists\{p\}$ a compact neighborhood of p.

3.8.5 Example:

The indiscrete topology (X,τ) is locally compact since X is compact.

3.8.6 Theorem:

A closed subset of a locally compact space is locally compact space.

Proof:

Let A be a closed subset of locally compact space (X,τ) and let $p \in$A then there exists a compact neighborhood H of p.Since A is closed then $F=A\cap H$ is compact (by let (X,τ) is a topological space and $F\subseteq X$ be a closed set. If A is compact then $A\cap F$ is compact) but $p\in H^{\circ}$then $p\in H^{\circ}\cap A\subseteq F$, where $H^{\circ}\cap A\in\tau_A$,so p has compact neighborhood $F=A\cap H$, i.e. A is locally compact. □