Calculus II

Second Semester

Lecturer 1

Dr. Ban Jaffar AL-Taiy
Taghreed Hussein Abed

Contents

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References

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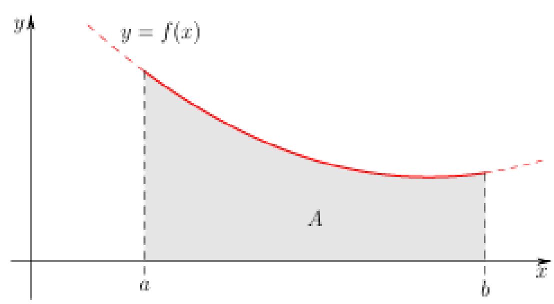
Chapter four: Integrals.

A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method, called integration, to calculate the areas and volumes of more general shapes. The definite integral is the key tool in calculus for defining and calculating areas and volumes. We also use it to compute quantities such as the lengths of curved paths, probabilities, and averages.

We show that the process of computing these definite integrals is closely connected to finding antiderivatives. This is one of the most important relationships in calculus; it gives us an efficient way to compute definite integrals, providing a simple and powerful method that eliminates the difficulty of directly computing limits of approximations. This connection is captured in the Fundamental Theorem of Calculus.

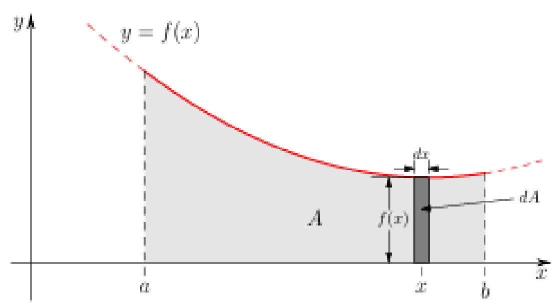
4.1 The Meaning of the Definite Integral:

The definite integral of the function $f(x)$ between $x = a$ and $x = b$ is written: $\int_a^b f(x)dx$.



Geometrically it equals area A between the curve $y = f(x)$ and the x-axis between the vertical lines $x = a$ and $x = b$.

More precisely, assuming $a < b$, the definite integral is the net sum of the ***signed*** areas between the curve $y = f(x)$ and the x-axis where areas below the x-axis (i.e. where $f(x)$ dips below the x-axis) are counted ***negatively***.



The notation used for definite integral, $\int_a^b f(x)dx$, is elegant and intuitive. We are $\int$ umming ($\int dA$) the (infinitesimally) small differential rectangular areas $dA = f(x) \cdot dx$ of height $f(x)$ and width dx at each value x between $x = a$ and $x = b$.

Definition:

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b i.e. $A = \int_a^b f(x)dx$.

Remark:

An easy way to evaluate definite integrals is due to the Fundamental Theorem of Calculus which relates the calculation of a definite integral with the evaluation of the antiderivative $F(x)$ of $f(x)$:

Theorem (The Fundamental Theorem of Calculus):

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$ for any F an antiderivative of f, i.e. $F'(x) = f(x)$.

Notational, we write $F(b) - F(a)$ with shorthand $F(x)|_a^b$, i.e. $F(x)|_a^b = F(b) - F(a)$, where, unlike the integral sign, the bar is placed on the right.

Table 1. Rules satisfied by definite integrals

1. ***Order of Integration:*** $\int_b^a f(x)\,dx = -\int_a^b f(x)dx$ **A definition, $a < b$**
2. ***Zero Width Interval:*** $\int_a^b f(x)dx = 0$ **A definition when $f(x)$ exists $(a = b)$**
3. ***Constant Multiple:*** $\int_a^b kf(x)\,dx = k\int_a^b f(x)dx$ **Any constant k**
   $\int_a^b k\,dx = k(b - a)$
4. ***Sum and Difference:*** $\int_a^b (f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. ***Additivity:*** $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ $[a, b] = [a, c] \cup c, b]$
6. ***Max-Min Inequality:*** If f has maximum value max f and minimum value min f on $[a, b]$, then
   $$(min\ f) \cdot (b - a) \leq \int_a^b f(x)dx \leq (maxf\ ) \cdot (b - a).$$
7. ***Domination:*** If $f(x) \geq g(x)$ on $[a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
   If $f(x) \geq 0$ on $[a, b]$ then $\int_a^b f(x)dx \geq 0$. **Special case**

4.2 indefinite integrals and the Substitution method:

Because of the intimate relationship between the antiderivative and the definite integral, we define the ***indefinite integral*** of $f(x)$ (with no limits a or b) to just be the antiderivative, i.e.

$$\int f(x)dx = F(x) + C,$$

where $F(x)$ is an antiderivate of $f(x)$ (so $F'(x) = f(x)$) and C is an arbitrary constant. The latter is required since the antiderivative of a function is not unique as $\frac{d}{dx}C = 0$ implies we can always add a constant to an antiderivative to get another antiderivative of the same function.

If u is a differentiable function of x and n is any number different from -1, then $\int u^n du = \frac{u^{n+1}}{n+1} + C$.

Example:

Find the integral $\int (x^3 + x)^5 (3x^2 + 1)dx$

Solution:

We set $u = x^3 + x$. Then $du = \frac{du}{dx}dx = (3x^2 + 1)\,dx$, so that by substitution we have

$$\int (x^3 + x)^5 (3x^2 + 1)dx = \int u^5 du \qquad \textbf{Let } u = x^3 + x, du = (3x^2 + 1)\,dx.$$

$$= \frac{u^6}{6} + C \qquad \textbf{Integrate with respect to u.}$$

$$= \frac{(x^3+x)^6}{6} + C \qquad \textbf{Substitute } x^3 + x \textbf{ for } u. \blacksquare$$

Example:

Find the integral $\int \sqrt{2x + 1}dx$

Solution:

The integral does not fit the formula $\int u^n du$ with $u = 2x + 1$ and $n = 1/2$, because $du = \frac{du}{dx}dx = 2\,dx.$

which is not precisely dx. The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by introducing a factor of $1/2$ in front of the integral sign. So, we write

$$\int \sqrt{2x+1}dx = \frac{1}{2}\int \sqrt{\underbrace{2x+1}_{u}} \cdot \underbrace{2dx}_{du}$$

$$= \frac{1}{2}\int u^{1/2}du$$ **Let $u = 2x+1, du = 2\,dx$.**

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$ **Integrate with respect to u.**

$$= \frac{1}{2}\frac{(2x+1)^{3/2}}{3/2} + C$$ **Substitute $2x + 1$ for u.** ■

Remark:

The substitutions in previous Examples are instances of the following general rule.

Theorem (The Substitution rule):

If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du.$$

Remark (The Substitution method to evaluate $\int f(g(x)) \cdot g'(x)dx$):

1. Substitute $u = g(x)$ and $du = \frac{du}{dx}dx = g'(x)dx$ to obtain $\int f(u)du$.

2. Integrate with respect to u.

3. Replace u by $g(x)$.

Example:

Evaluate $\int x\sqrt{2x+1}dx$

Solution:

The substitution $u = 2x+1$ with $du = 2\,dx$. Then

$$\sqrt{2x+1}dx = \frac{1}{2}\sqrt{u}du.$$

However, in this example the integrand contains an extra factor of x that multiplies the term $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ for x to obtain $x = u - 1)/2$, and find that

$$x\sqrt{2x+1}dx = \frac{1}{2}(u-1).\frac{1}{2}\sqrt{u}\,du.$$

The integration now becomes

$$\int x\sqrt{2x+1}dx = \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u-1)\,u^{\frac{1}{2}}\,du \qquad \textbf{Substitute.}$$

$$= \frac{1}{4}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\,du \qquad \textbf{Multiply terms.}$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) + C \qquad \textbf{Integrate.}$$

$$= \frac{1}{4}\left(\frac{2}{5}(2x+1)^{\frac{5}{2}} - \frac{2}{3}(2x+1)^{\frac{3}{2}}\right) + C \quad \textbf{Replace } \boldsymbol{u} \textbf{ by } \boldsymbol{2x+1}. \blacksquare$$

Example:

Evaluate $\int \frac{2\,z\,dz}{\sqrt[3]{z^2+1}}$

Solution:

We will use the substitution method of integration as an exploratory tool: We substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. In this example both substitutions turn out to be successful, but that is not always the case. If one substitution does not help, a different substitution may work instead.

Method 1: Substitute $u = z^2 + 1$.

$$\int \frac{2\,z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{\frac{1}{3}}} \qquad \textbf{Let } \boldsymbol{u = z^2+1, du = 2z\,dz}.$$

$$= \int u^{-\frac{1}{3}}\,du \qquad \textbf{In the form } \boldsymbol{\int u^n\,du}$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \textbf{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \textbf{Replace } \boldsymbol{u} \textbf{ by } \boldsymbol{z^2+1}.$$

Method 2: Substitute $u = \sqrt[3]{z^2+1}$

$$\int \frac{2\,z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3\,u^2du}{u} \qquad \textbf{Let } \boldsymbol{u = \sqrt[3]{z^2+1}, u^3 = z^2+1, 3u^2\,du = 2z\,dz}.$$

$$= 3\int u\,du$$

$$= 3\frac{u^2}{2} + C \qquad \textbf{Integrate.}$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \textbf{Replace } \boldsymbol{u} \textbf{ by } \boldsymbol{(z^2+1)^{1/3}}. \blacksquare$$

Using our notation for indefinite integrals and our knowledge of derivatives gives the following.

Table of Indefinite Integrals for trigonometric functions

1. $\int sin\ ax = \frac{-1}{a} cos\ ax + C$
2. $\int cos\ ax = \frac{1}{a} sin\ ax + C$
3. $\int sec^2\ ax = \frac{1}{a} tan\ ax + C$
4. $\int csc^2\ ax = \frac{-1}{a} cot\ ax + C$
5. $\int sec\ ax \tan ax = \frac{1}{a} sec\ ax + C$
6. $\int csc\ ax \cot ax = \frac{-1}{a} csc\ ax + C$

Example:

Find $\int sec^2(5x+1) \cdot 5dx$

Solution:

We substitute $u = 5x + 1$ and $du = 5\ dx$. Then,

$\int sec^2(5x+1) \cdot 5dx = \int sec^2 u\ du$ **Let $u = 5x + 1, du = 5\ dx$.**

$= \tan u + C$ **$\frac{d}{du} \tan u = sec^2 u$**

$= \tan(5x + 1) + C$ **Substitute $5x + 1$ for u.** ■

Example:

Find $\int cos(7\theta + 3) d\theta$

Solution:

We let $u = 7\theta + 3$ so that $du = 7\ d\theta$. The constant factor 7 is missing from the du term in the integral. We can compensate for it by multiplying and dividing by 7,

$\int cos(7\theta+3)d\theta = \frac{1}{7}\int cos(7\theta+3) \cdot 7d\theta$ **Place factor $\frac{1}{7}$ in front of integral.**

$= \frac{1}{7}\int \cos u\ du$ **Let $u = 7u + 3, du = 7\ d\theta$.**

$= \frac{1}{7}\sin u + C$ **Integrate.**

$= \frac{1}{7}\sin(7\theta + 3) + C$ **Substitute $7\theta + 3$ for u.**

There is another approach to this problem. With $u = 7\theta + 3$ and $du = 7\ d\theta$ as before, we solve for du to obtain $d\theta = (1/7)\ du$. Then the integral becomes

$\int cos(7\theta + 3)d\theta = \int cosu \cdot \frac{1}{7} du$ **Let $u = 7\theta + 3, du = 7\, d\theta$, and $d\theta = \frac{1}{7}\, du$.**

$= \frac{1}{7} \sin u + C$ **Integrate.**

$= \frac{1}{7} \sin(7\theta + 3) + C$ **Substitute $7\theta + 3$ for u.**

We can verify this solution by differentiating and checking that we obtain the original function $cos(7\theta + 3)$. ■

Remark:

Sometimes we observe that a power of *x* appears in the integrand that is one less than the power of *x* appearing in the argument of a function we want to integrate. This observation immediately suggests we try a substitution for the higher power of *x*. This situation occurs in the following integration.

$\int x^2 cos\, x^3 dx = \int cos\, x^3 \cdot x^2\, dx$

$= \int cos\, u \cdot \frac{1}{3} du$ **Let $u = x^3, du = 3x^2\, dx$, $(1/3)\, du = x^2\, dx$.**

$= \frac{1}{3} \int cos\, u\, du$

$= \frac{1}{3} \sin u + C$ **Integrate with respect to u.**

$= \frac{1}{3} \sin x^3 + C$ **Replace u by x^3.** ■

Example:

Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can evaluate using the Substitution Rule.

a) $\int sin^2 x\, dx = \int \frac{1-\cos 2x}{2} dx$ $\quad sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{1}{2} \int (1 - \cos 2x)\, dx$

$= \frac{1}{2} x - \frac{1}{2} \frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C.$

b) $\int cos^2 x\, dx = \int \frac{1+\cos 2x}{2} dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$ $\quad cos^2 x = \frac{1+\cos 2x}{2}$

c) $\int (1 - 2\sin^2 x)\sin 2x\, dx = \int (\cos^2 x - \sin^2 x)\sin 2x\, dx$

$= \int \cos 2x \sin 2x\, dx$ $\quad$ $\cos 2x = \cos^2 x - \sin^2 x$

$= \int \frac{1}{2}\sin 4x\, dx = \int \frac{1}{8}\sin u\, du$ $\quad$ $u = 4x, du = 4x\, dx$

$= -\cos 4x + C.$ ■

Exercises:

1. Evaluate the indefinite integrals in following by using the given substitutions to reduce the integrals to standard form.

a) $\int 2(2x+4)^5 dx, u = 2x+4$

b) $\int 7(7x-1)^5 dx, u = 7x-1$

c) $\int 2x(x^2+5)^{-4} dx, u = x^2+5$

d) $\int \frac{4x^3}{(x^4+1)^2} dx, u = x^4+1$

e) $\int (3x+2)(3x^2+4x)^4 dx, u = 3x^2+4x$

f) $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx, u = 1+\sqrt{x}$

g) $\int \sin 3x\, dx, u = 3x$

h) $\int x \sin(2x^2)\, dx, u = 2x^2$

i) $\int \sec 2t\, dt, u = 2t$

j) $\int \left(1-\cos\frac{t}{2}\right)\sin\frac{t}{2} dt, u = 1-\cos\frac{t}{2}$

k) $\int \frac{9r^2}{\sqrt{1-r^3}} dx, u = 1-r^3$

l) $\int 12(y^4+4y^2+1)^2(y^3+2y) dx, u = y^4+4y^2+1$

m) $\int \sqrt{x}\sin^2(x^{3/2}-1)\, dx, u = x^{3/2}-1$

n) $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right) dx, u = -\frac{1}{x}$

o) $\int \csc^2 2\theta \cot 2\theta\, d\theta$

I. Using $u = \cot 2\theta$

II. Using $u = \csc 2\theta$

p) $\int \frac{1}{\sqrt{5x+8}} dx$

I. Using $u = 5x+8$

II. Using $u = \sqrt{5x+8}$

2. Evaluate the integrals in following Exercises

1. $\int \sqrt{3-2s}\, ds$

2. $\int \frac{1}{\sqrt{5s+4}} ds$

3. $\int \theta\sqrt[4]{1-\theta^2}\, d\theta$

4. $\int 3y\sqrt{7-3y^2}\, dy$

5. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2} dx$

6. $\int \sqrt{\sin x}\cos^3 x\, dx$

7. $\int \sec^2(3x+2)\, dx$

8. $\int \tan^2 x \sec^2 x\, dx$

9. $\int \sin^5\frac{x}{3}\cos\frac{x}{3} dx$

10. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2} dx$

11. $r^2\left(\frac{r^3}{18}-1\right)^5 dr$

12. $r^4\left(7-\frac{r^5}{10}\right)^3 dr$

13. $\int x^{1/2}\sin(x^{3/2}+1)\, dx$

14. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right) dv$

15. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)} dt$

16. $\int \frac{\sec z \tan z}{\sqrt{\sec z}} dz$

17. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right) dt$

18. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\, dt$

19. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta} d\theta$

20. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}} d\theta$

21. $\int \frac{x}{\sqrt{1+x}}\,dx$ **22.** $\int \sqrt{\frac{x-1}{x^5}}\,dx$ **23.** $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

24. $\int \frac{1}{x^3}\sqrt{\frac{x^2-1}{x^2}}\,dx$ **25.** $\int \sqrt{\frac{x^3-3}{x^{11}}}\,dx$ **26.** $\int \sqrt{\frac{x^4}{x^3-1}}\,dx$

27. $\int x(x-1)^{10}\,dx$ **28.** $\int x\sqrt{4-x}\,dx$ **29.** $\int (x+1)^2(1-x)^5 dx$

30. $\int (x+5)(x-5)^{1/3} dx$ **31.** $\int x^3\sqrt{x^2+1}\,dx$ **32.** $\int 3x^5\sqrt{x^3+1}\,dx$

33. $\int \frac{x}{(x^2-4)^3}\,dx$ **34.** $\int \frac{x}{(2x-1)^{2/3}}\,dx$

3. If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in the following Exercises

a) $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

I. $u = tan\ x$, followed by $v = u^3$, then by $w = 2 + v$

II. $u = \tan^3 x$, followed by $v = 2 + u$

III. $u = 2 + \tan^3 x$

b) $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

I. $u = x - 1$, followed by $v = sin\,u$, then by $w = 1 + v^2$

II. $u = sin\,(x-1)$, followed by $v = 1 + u^2$

III. $u = 1 + \sin^2(x-1)$

4. Evaluate the integrals in following Exercises

a) $\int \frac{(2r-1)\cdot \cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

b) $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}\,d\theta$

4.3 Definite integral Substitutions and the Area Between Curves:

There are two methods for evaluating a definite integral by substitution. One method is to find an antiderivative using substitution and then to evaluate the definite integral by applying the Evaluation Theorem. The other method extends the process of substitution directly to definite integrals by changing the limits of integration. We will use the new formula that we introduce here to compute the area between two curves.

4.3.1 The Substitution Formula:

The following formula shows how the limits of integration change when we apply a substitution to an integral.

Theorem (Substitution in Definite integrals):

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Remark:

To use Theorem, make the same u-substitution $u = g(x)$ and $du = g'(x)\,dx$ that you would use to evaluate the corresponding indefinite integral. Then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

Example (*):

Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\,dx$.

Solution:

We will show how to evaluate the integral using Theorem (Substitution in Definite integrals), and how to evaluate it using the original limits of integration.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem.

$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$ **Let $u = x^3 + 1, du = 3x^2\, dx$.**

When $x = -1, u = (-1)^3 + 1 = 0$.

When $x = 1, u = (1)^3 + 1 = 2$.

$= \int_0^2 \sqrt{u}du$

$= \frac{2}{3}u^{3/2}\Big]_0^2$ **Evaluate the new definite integral**

$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}[2\sqrt{2}] = \frac{4\sqrt{2}}{3}.$

<u>Method 2:</u> Transform the integral as an indefinite integral, integrate, change back to *x*, and use the original *x*-limits.

$\int 3x^2\sqrt{x^3+1} = \int \sqrt{u}du$ **Let $u = x^3 + 1, du = 3x^2\, dx$.**

$= \frac{2}{3}u^{3/2} + C$ **Integrate with respect to u.**

$= \frac{2}{3}(x^3+1)^{3/2} + C$ **Replace u by $x^3 + 1$.**

$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$ **Use the integral just found, with limits of integration for x.**

$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$

$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}[2\sqrt{2}] = \frac{4\sqrt{2}}{3}$. ■

<u>Remark:</u>

Which method is better - evaluating the transformed definite integral with transformed limits using Theorem, or transforming the integral, integrating, and transforming back to use the original limits of integration? In Example (*), the first method seems easier, but that is not always the case. Generally, it is best to know both methods and to use Whichever one seems better at the time.

<u>Example:</u>

We use the method of transforming the limits of integration

a) $\int_{\pi/4}^{\pi/2} \cot\theta\, csc^2\,\theta\, d\theta = \int_1^0 u\cdot(-du)$ **Let $u = cot\,\theta, du = -csc^2\,\theta\, d\theta$, $-du = csc^2\, u\, du$.**

When $\theta = \frac{\pi}{4}, u = cot\left(\frac{\pi}{4}\right) = 1$.

When $\theta = \frac{\pi}{2}, u = cot\left(\frac{\pi}{2}\right) = 0$.

$$= -\int_1^0 u\,du$$
$$= -[\tfrac{u^2}{2}]_1^0$$
$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

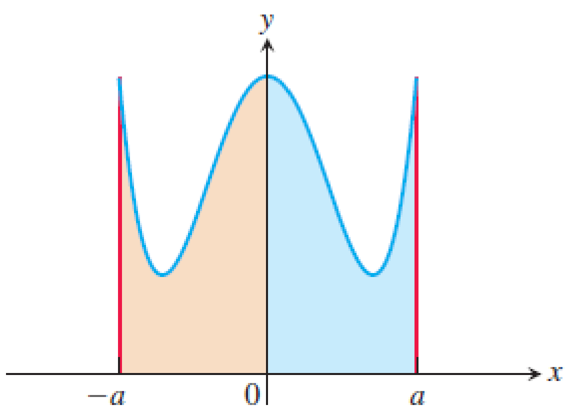
b) $\int_0^{\pi/2} \frac{2\sin x \cos x}{(1+\sin^2 x)^3}\,dx = \int_0^1 \frac{1}{u^3}\,du$

Let $u = 1 + \sin^2 x$, $du = 2\,sin\,x\,cos\,x\,dx$.
When $x = 0, u = 1$.
When $x = \pi/2, u = 2$.

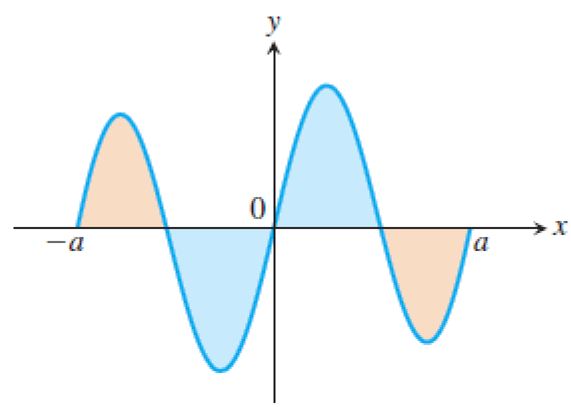
$$= \frac{1}{2u^2}\Big]_1^2 = -\frac{1}{8} - \left(-\frac{1}{2}\right) = \frac{3}{8}\,. \blacksquare$$

4.3.2 Definite Integrals of Symmetric Functions:

The Substitution Formula in Theorem (Substitution in Definite integrals) simplifies the calculation of definite integrals of even and odd functions over a symmetric interval $[-a, a]$.



For f an even function, the integral from $-a$ to a is twice the integral from 0 to a.



For f an odd function, the integral from $-a$ to a equal 0.

Theorem:

Let f be continuous on the symmetric interval $[-a, a]$.

a) If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.

b) If f is odd, then $\int_{-a}^{a} f(x)dx = 0$.

Example:

Evaluate $\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx$.

Solution:

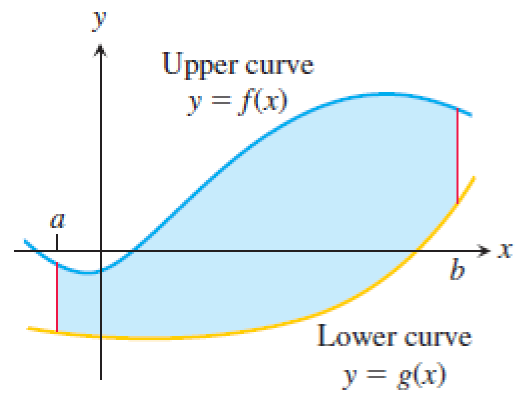
Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx = 2\int_0^2(x^4 - 4x^2 + 6)\,dx$$
$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$
$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right)$$
$$= \frac{232}{15}. \blacksquare$$

4.3.3 Areas Between Curves:

Definition:

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the ***area of the region between the curves*** $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b:



The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

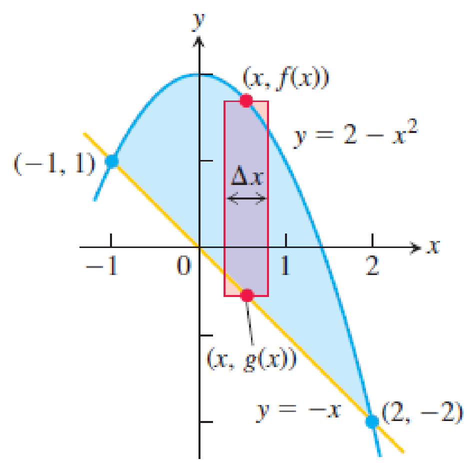
Remark:

When applying this definition, it is usually helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g. It also helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x. Then you can integrate the function $f - g$ for the area between the intersections.

Example:

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution:



First, we sketch the two curves. The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$2 - x^2 = -x \qquad \textbf{Equate } f(x) \textbf{ and } g(x).$$
$$x^2 - x - 2 = 0 \qquad \textbf{Rewrite.}$$
$$(x+1)(x-2) = 0 \qquad \textbf{Factor.}$$
$$x = -1, x = 2 \qquad \textbf{Solve.}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1, b = 2$. The area between the curves is

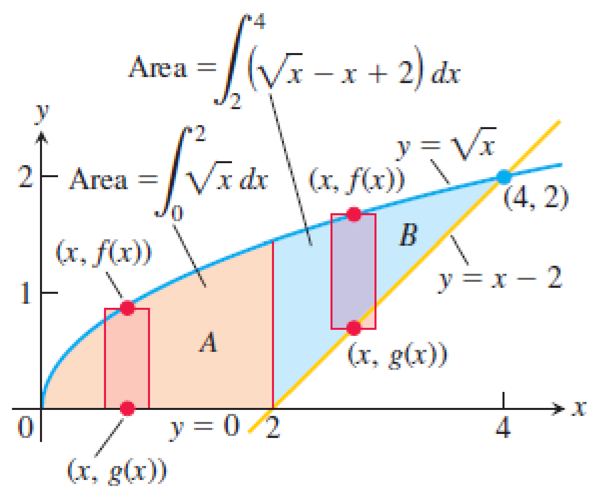
$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^2 [(2 - x^2) - (-x)]\, dx$$
$$= \int_{-1}^2 (2 + x - x^2) dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}. \blacksquare$$

Remark:

If the formula for a bounding curve change at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

Example (**):

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x - axis and the line $y = x - 2$.



Solution:

The sketch shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\sqrt{x} = x - 2 \qquad \textbf{Equate } f(x) \textbf{ and } g(x).$$
$$x = (x-2)^2 = x^2 - 4x + 4 \qquad \textbf{Square both sides.}$$

$x^2 - 5x + 4 = 0$ **Rewrite.**

$(x-1)(x-4) = 0$ **Factor.**

$x = 1, x = 4$ **Solve.**

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

For $0 \leq x \leq 2$: $f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \leq x \leq 4$: $f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

We add the areas of subregions A and B to find the total area:

$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\, dx}_{\text{area of } B}$$

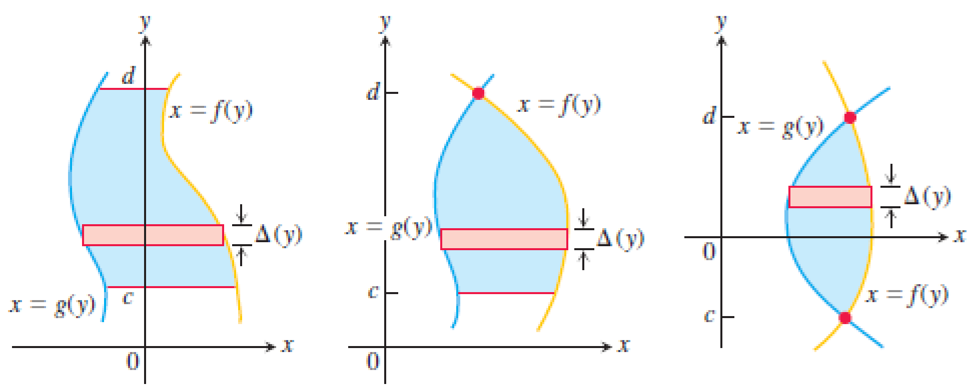
$$= \left[\tfrac{2}{3}x^{3/2}\right]_0^2 + \left[\tfrac{2}{3}x^{3/2} - \tfrac{x^2}{2} + 2x\right]_2^4$$

$$= \tfrac{2}{3}(2)^{3/2} - 0 + \left(\tfrac{2}{3}(4)^{\frac{3}{2}} - 8 + 8\right) - \left(\tfrac{2}{3}(2)^{\frac{3}{2}} - 2 + 4\right)$$

$$= \tfrac{2}{3}(8) - 2 = \tfrac{10}{3}. \blacksquare$$

4.3.4 integration with respect to *y*:

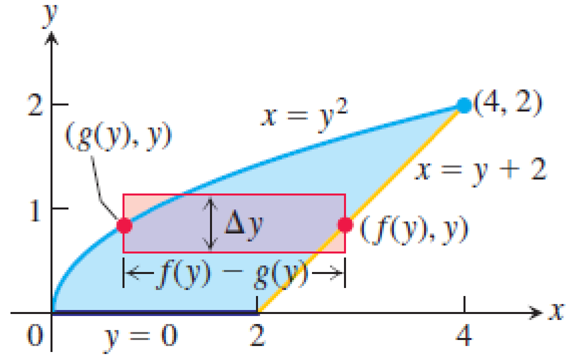
If a region's bounding curves are described by functions of *y*, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of *x*. For regions like these:



use the formula $A = \int_a^b [f(x) - g(x)]\, dx$. In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

Example(***):

Find the area of the region in Example (**) by integrating with respect to y.



Solution:

We first sketch the region and a typical horizontal rectangle based on a partition of an interval of y-values. The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$y + 2 = y^2$ **Equate $f(x) = y + 2$ and $g(x) = y^2$.**

$y^2 - y - 2 = 0$ **Rewrite.**

$(y + 1)(y - 2) = 0$ **Factor.**

$y = -1, y = 2$ **Solve.**

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection below the x-axis.). The area of the region is

$$A = \int_c^d [f(y) - g(y)]\,dx = \int_0^2 [y + 2 - y^2]\,dy$$

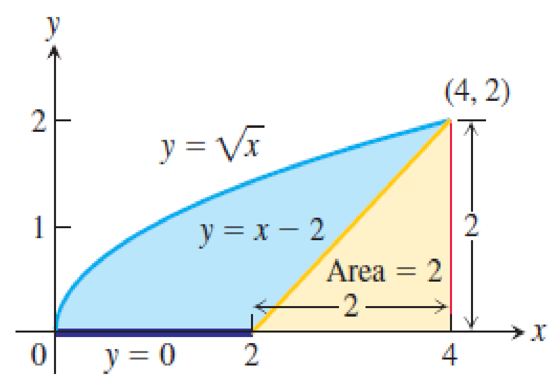
$$= \int_0^2 [2 + y - y^2]\,dy = [2y + \frac{y^2}{2} - \frac{y^3}{3}]_0^2$$

$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) = \frac{10}{3}.$$

This is the result of Example (*), found with less work. ■

Remark:

Although it was easier to find the area in Example (**) by integrating with respect to y rather than x (just as we did in Example (***)), there is an easier way yet. The area we want is the area between the curve $y = \sqrt{x}$ and the x-axis for $0 \le x \le 4$, minus the area of



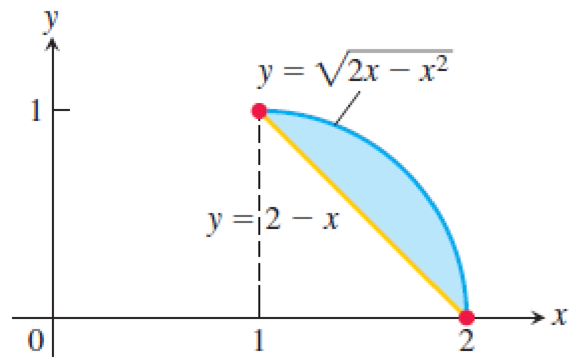
The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle.

an isosceles triangle of base and height equal to 2. So by combining calculus with some geometry, we find

$$\text{Area} = \int_0^4 \sqrt{x}\,dx - \tfrac{1}{2}(2)(2) = \tfrac{2}{3}x^{\frac{3}{2}}\Big]_0^4 - 4 = \tfrac{2}{3}(8) - 0 - 2 = \tfrac{10}{3}.$$

Example:

Find the area of the region bounded below by the line $y = 2 - x$ and above by the curve $y = \sqrt{2x - x^2}$.



Solution:

A sketch of the region is displayed in Figure, and we see that the line and curve intersect at the points (1, 1) and (2, 0). Using vertical rectangles, the area of the region is given by

$$A = \int_1^2 (\sqrt{2x - x^2} + x - 2)\,dx.$$

However, we don't know how to find an antiderivative for the term involving the radical, and no simple substitution is apparent.

To use horizontal rectangles, we first need to express each bounding curve as a function of the variable y. The line on the left is easily found to be $x = 2 - y$. For the curve $y = \sqrt{2x - x^2}$ on the right-hand side we have

$$y^2 = 2x - x^2$$
$$= -(x^2 - 2x + 1) + 1 \quad \text{Complete the square.}$$
$$= -(x - 1)^2 + 1$$

Solving for x,

$$(x - 1)^2 = 1 - y^2$$
$$x = 1 + \sqrt{1 - y^2} \qquad x \geq 1, 0 \leq y \leq 1$$

The area of the region is then given by

$$A = \int_0^1 [(1 + \sqrt{1 - y^2}) - (2 - y)]\,dx = \int_0^1 (\sqrt{1 - y^2} + y - 1)\,dx$$

Again, we don't know yet how to integrate the radical term. We conclude that neither vertical nor horizontal rectangles lead to an integral we currently can evaluate.

Nevertheless, as we found with Example (***), sometimes a little observation proves to be helpful. If we look again at the algebra for expressing the right-hand side curve $y = \sqrt{2x - x^2}$ as a function of y, we see that $(x-1)^2 + y^2 = 1$, which is the equation of the unit circle with center shifted to the point (1, 0). From Figure, we can then see that the area of the region we want is the area of the upper right quarter of the unit circle minus the area of the triangle with vertices (1, 1), (1, 0), and (2, 0). That is, the area is given by

$$A = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4} \approx 0.285. \blacksquare$$

Exercises:

1. Use the Substitution Formula in Theorem (Substitution in Definite integrals) to evaluate the integrals in following Exercises

1) **a)** $\int_0^3 \sqrt{y+1}\,dy$ **b)** $\int_{-1}^0 \sqrt{y+1}\,dy$

2) **a)** $\int_0^1 r\sqrt{1-r^2}\,dr$ **b)** $\int_{-1}^1 r\sqrt{1-r^2}\,dr$

3) **a)** $\int_0^{\pi/4} \tan x \ \sec^2 x\, dx$ **b)** $\int_{-\pi/4}^0 \tan x \ \sec^2 x\, dx$

4) **a)** $\int_0^{\pi} 3\ \cos^2 x \sin x\, dx$ **b)** $\int_{2\pi}^{3\pi} 3\ \cos^2 x \sin x\, dx$

5) **a)** $\int_0^1 t^3(1+t^4)^3\, dt$ **b)** $\int_{-1}^1 t^3(1+t^4)^3\, dt$

6) **a)** $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\, dt$ **b)** $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\, dt$

7) **a)** $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\, dr$ **b)** $\int_0^1 \frac{5r}{(4+r^2)^2}\, dr$

8) **a)** $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$ **b)** $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$

9) **a)** $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$ **b)** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$

10) **a)** $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\, dx$ **b)** $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\, dx$

11) **a)** $\int_0^1 t\sqrt{4+5t}\,dt$ **b)** $\int_1^9 t\sqrt{4+5t}\,dt$

12) **a)** $\int_0^{\pi/6} (1-\cos 3t)\sin 3t dt$ **b)** $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t dt$

13) **a)** $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$

14) **a)** $\int_{-\pi/2}^0 (2+\tan\frac{t}{2})\sec^2\frac{t}{2}\, dt$

b) $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$

b) $\int_{-\pi/2}^{\pi/2} (2 + \tan\frac{t}{2}) \sec^2\frac{t}{2}\, dt$

15) $\int_0^1 \sqrt{t^5 + 2t}\,(4t^4 + 2)dt$

16) $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

17) $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$

18) $\int_{\pi}^{3\pi/2} \cot^5(\frac{\theta}{6}) \sec^2(\frac{\theta}{6})\, d\theta$

19) $\int_0^{\pi} 5(5 - 4\cot t)^{1/4} \sin t\, dt$

20) $\int_0^{\pi/4} (1 - \sin 2t)^{3/2} \cos 2t\, dt$

21) $\int_0^1 (4y - y^2 + 4y^3 + 1)^{-2/3}(12y^2 - 2y + 4)dy$

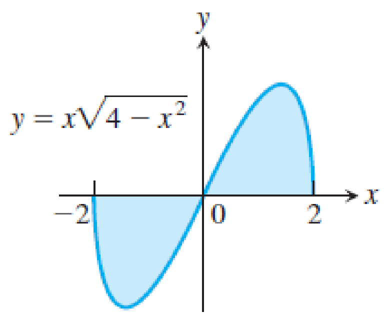
23) $\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2}(y^2 + 4y - 4)dy$

23) $\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta} \cos^2(\theta^{3/2})\, d\theta$

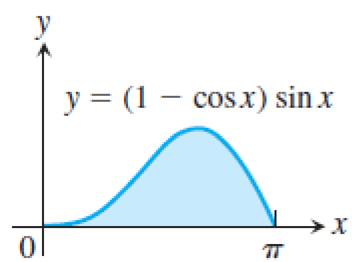
24) $\int_{-1}^{-1/2} t^{-2} \sin^2(1 + \frac{1}{t})\, dt$

2. Find the total areas of the shaded regions in the following Exercises:

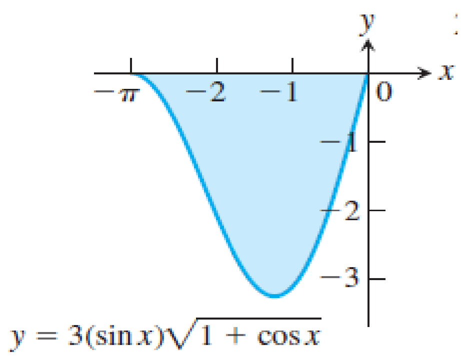
a)



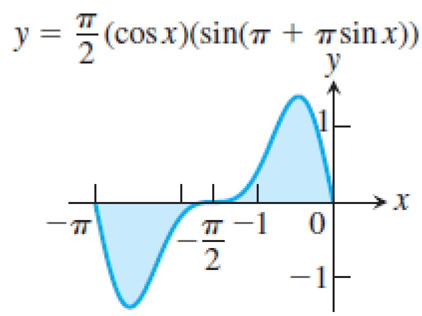
b)



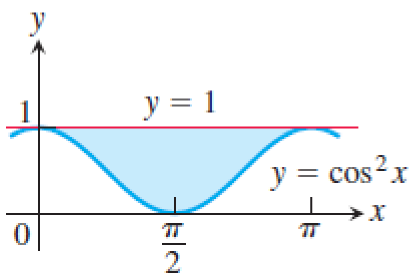
c)



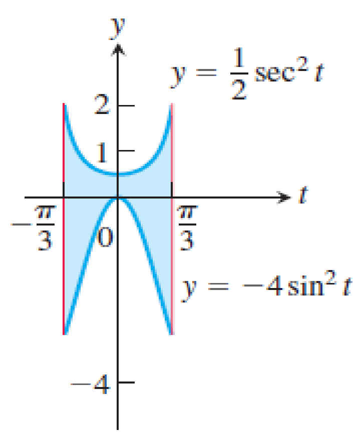
d)



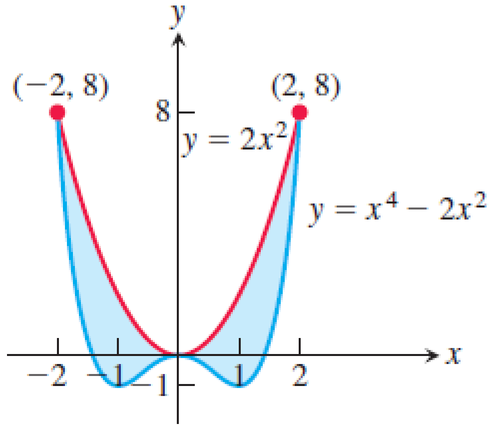
e)



f)



g)



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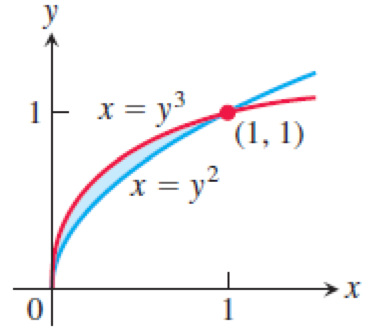
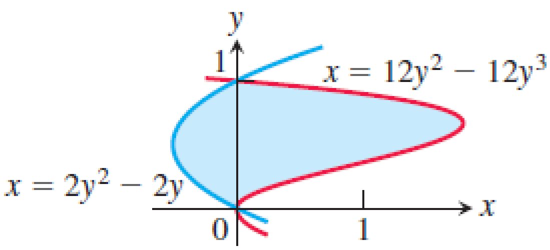
h)



i)



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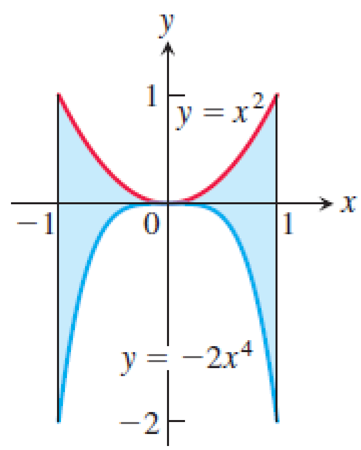

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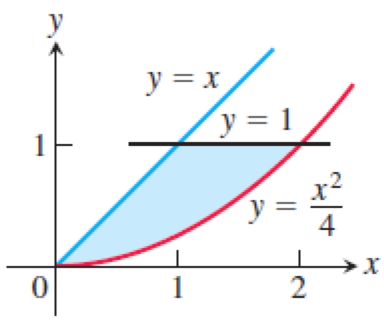

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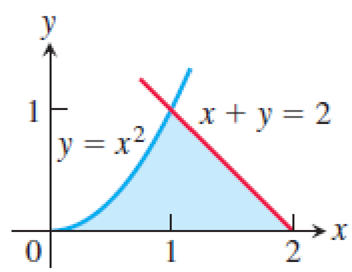

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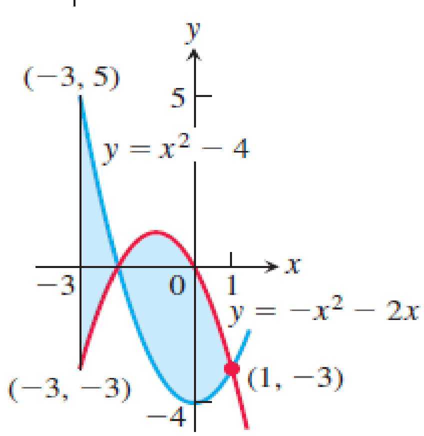
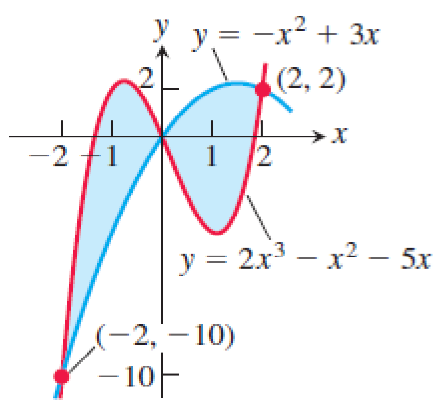
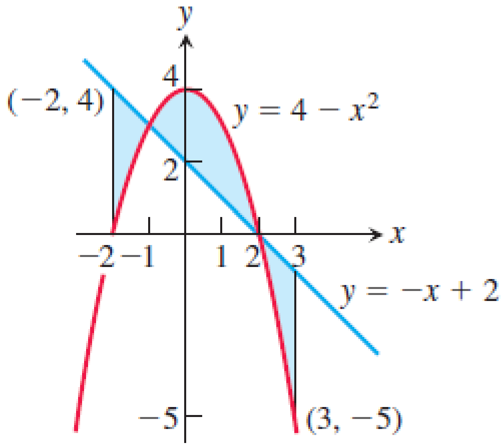


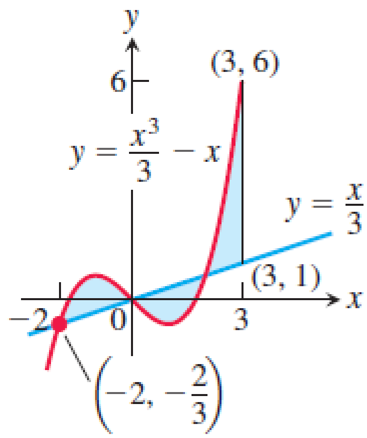
n)



o)



p)



3. Find the areas of the regions enclosed by the lines and curves in the following Exercises:

1) $y = x^2 - 2$ and $y = 2$.
2) $y = 2x - x^2$ and $y = -3$.
3) $y = x^4$ and $y = 8x$.
4) $y = x^2 - 2x$ and $y = x$.
5) $y = x^2$ and $y = -x^2 + 4x$.
6) $y = 7 - 2x^2$ and $y = x^2 + 4$.
7) $y = x^4 - 4x^2 + 4$ and $y = x^2$.
8) $y = x\sqrt{a^2 - x^2}, a > 0$ and $y = 0$.
9) $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)
10) $y = |x^2 - 4|$ and $y = (x^2/2) + 4$
11) $x = 2y^2, x = 0$ and $y = 3$.
12) $x = y^2$ and $x = y + 2$.
13) $y^2 - 4x = 4$ and $4x - y = 16$.
14) $x - y^2 = 0$ and $x + 2y^2 = 3$.
15) $x + y^2 = 0$ and $x + 3y^2 = 2$.
16) $x - y^{2/3} = 0$ and $x + y^4 = 2$.
17) $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$.
18) $x = y^3 - y^2$ and $x = 2y$.
19) $4x^2 + y = 4$ and $x^4 - y = 1$.
20) $x^3 - y = 0$ and $3x^2 - y = 4$.
21) $x + 4y^2 = 4$ and $x + y^4 = 1$ for $x \geq 0$.
22) $x + y^2 = 3$ and $4x + y^2 = 0$.
23) $y = 2\sin x$ and $y = \sin 2x, 0 \leq x \leq \pi$.
24) $y = 8\cos x$ and $y = \sec^2 x, -\pi/3 \leq x \leq \pi/3$.
25) $y = \cos(\pi x/2)$ and $y = 1 - x^2$
26) $y = \sin(\pi x/2)$ and $y = x$
27) $y = \sec^2 x, y = \tan^2 x, x = -\pi/4$, and $x = \pi/4$
28) $x = \tan^2 y, x = -\tan^2 y, -\pi/4 \leq y \leq \pi/4$.
29) $x = 3\sin y\sqrt{\cos y}$ and $x = 0, 0 \leq y \leq \pi/2$
30) $y = \sec^2(\pi x/3)$ and $y = x^{1/3}, -1 \leq x \leq 1$.

4. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

5. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

6. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

7. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = sin\ x$ and $y = cos\ x$.

8. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a) Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

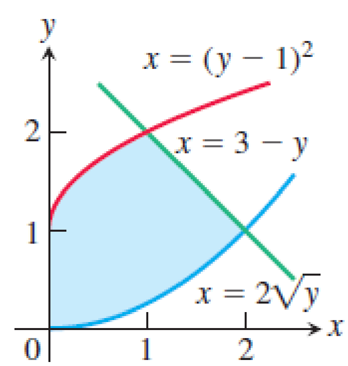
b) Find c by integrating with respect to y. (This puts c in the limits of integration.)

c) Find c by integrating with respect to x. (This puts c into the integrand as well.)

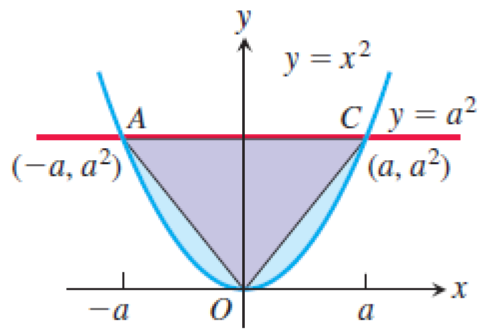
9. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a)** x, **b)** y.

10. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

11. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



12. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

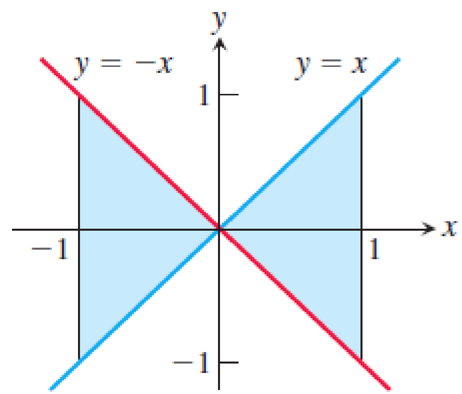


13. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

14. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a) $\int_{-1}^{1}\left(x - (-x)\right)dx = \int_{-1}^{1} 2x\,dx.$

b) $\int_{-1}^{1}\left(-x - (x)\right)dx = \int_{-1}^{1} -2x\,dx.$



15. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_{a}^{b}[f(x) - g(x)]\,dx.$$

Give reasons for your answer.

Calculus II

Second Semester

Lecturer 2

Dr. Ban Jaffar AL-Taiy

Taghreed Hussein Abed

4.4 Integrals and Transcendental Functions:

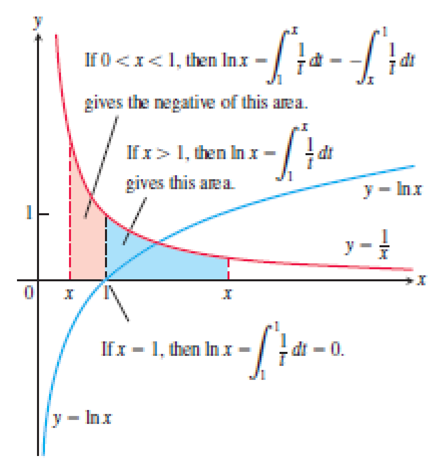
Our treatment of the logarithmic and exponential functions has been rather informal. In this chapter, we give a rigorous analytic approach to the definitions and properties of these functions. We also introduce hyperbolic functions and their inverses. Like trigonometric functions, these functions belong to the class of transcendental functions.

4.4.1 The Logarithm Defined as an Integral:

In section 1.4, we introduced the natural logarithm function $ln\,x$ as the inverse of the exponential function e^x. The function e^x was chosen as that function in the family of general exponential functions a^x , $a > 0$, whose graph has slope 1 as it crosses the *y*-axis. The function a^x was presented intuitively, however, based on its graph at rational values of *x*.

In this section we recreate the theory of logarithmic and exponential functions from an entirely different point of view. Here we define these functions analytically and derive their behaviors. To begin, we use the Fundamental Theorem of Calculus to define the natural logarithm function $ln\,x$ as an integral. We quickly develop its properties, including the algebraic, geometric, and analytic properties with which we are already familiar. Next, we introduce the function e^x as the inverse function of $ln\,x$, and establish its properties.

Defining $ln\,x$ as an integral and e^x as its inverse is an indirect approach that gives an elegant and powerful way to obtain and validate the key properties of logarithmic and exponential functions.



Definition:

The natural logarithm is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt\ , \quad x > 0.$$

From the Fundamental Theorem of Calculus, $ln\,x$ is a continuous function. Geometrically, if $x > 1$, then $ln\,x$ is the area under the curve

$y = 1/t$ from $t = 1$ to $t = x$. For $0 < x < 1$, $ln\, x$ gives the negative of the area under the curve from x to 1, and the function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have $\ln 1 = \int_1^1 \frac{1}{t} dt = 0$.

Notice that the graph shows $y = 1/x$ but use $y = 1/t$ in the integral. Using x for everything would have us writing $\ln x = \int_1^x \frac{1}{x} dx$, with x meaning two different things. So, we change the variable of integration to t.

Definition:

The number e is the number in the domain of the natural logarithm that satisfies $\ln(e) = \int_1^e \frac{1}{t} dt = 1$.

Remark:

If u is a differentiable function that is never zero, then

$$\int \frac{1}{u} du = \ln|u| + C.$$

This equation applies anywhere on the domain of $\frac{1}{u}$, which is the set of points where $u \neq 0$. It says that integrals that have the form $\int \frac{du}{u}$ lead to logarithms. Whenever $u = f(x)$ is a differentiable function that is never zero, we have that $du = f'(x)\, dx$ and

$$\int \frac{f'(x)}{f(x)} du = \ln|f(x)| + C.$$

Example:

We rewrite an integral in the form $\int \frac{du}{u}$.

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta} = \int_1^5 \frac{2}{u} du \qquad \boldsymbol{u = 3 + 2\sin\theta, du = 2\cos\theta\, d\theta,\ u(-\pi/2) = 1, u(\pi/2) = 5}$$

$$= 2\, ln|u|]_1^5 = 2\, ln|5| - 2\, ln|5| = 2\, ln\, 5$$

Note that $u = 3 + 2\sin\theta$ is always positive on $[-\pi/2, \pi/2]$. ■

Remark:

If $f(x) = e^x$, then $\int e^x dx = e^x + C$.

Example:

a) $\int_0^{\ln 2} e^{3x}dx = \int_0^{\ln 8} e^u \frac{1}{3}\,du$ $\qquad u = 3x, \frac{1}{3}du = dx, u(0) = 0,$ $u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8.$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du = \frac{1}{3}e^u\Big]_0^{\ln 8} = \frac{1}{3}(8-1) = \frac{7}{3}$$

b) $\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2} = e^1 - e^0 = e - 1.$ ■

Remark:

If $f(x) = a^x$, then $\int a^x dx = \frac{a^x}{\ln a} + C.$

Example:

a) $\int 2^x dx = \frac{2^x}{\ln 2} + C.$

b) $\int 2^{\sin x}\cos x\,dx = \int 2^u du = \frac{2^u}{\ln 2} + C$ $\qquad \boldsymbol{u = \sin x, du = \cos x\,dx},$

$$= \frac{2^{\sin x}}{\ln 2} + C\ .\ ■$$

4.4.2 Derivatives and Integrals Involving $\boldsymbol{\log_a x}$:

To find derivatives or integrals involving base a logarithm, we convert them to natural logarithms. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$$

$$\boldsymbol{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}}.$$

Example:

a) $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\cdot\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$

b) $\int \frac{\log_2 x}{x}dx = \frac{1}{\ln 2}\int\frac{\ln x}{x}dx$ $\qquad \boldsymbol{\log_2 x = \frac{\ln x}{\ln 2}}$

$= \frac{1}{\ln 2}\int u\,du$ $\qquad \boldsymbol{u = \ln x,\ du = \frac{1}{x}dx}$

$= \frac{1}{\ln 2}\cdot\frac{u^2}{2} + C = \frac{1}{\ln 2}\cdot\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C.$ ■

4.4.3 The Integrals of $\boldsymbol{\tan x, \cot x, \sec x}$, and $\boldsymbol{\csc x}$:

The integration of trigonometric functions is

$\boldsymbol{\int \tan x\,dx} = \int\frac{\sin x}{\cos x}dx = \int\frac{-du}{u}$ $\qquad \boldsymbol{u = \cos x > 0 \text{ on } (-\pi/2, \pi/2),}$ $\boldsymbol{du = -\sin x\,dx}$

$= -\ln|u| + C = -\ln|\cos x| + C$

$= \ln\left|\frac{1}{\cos x}\right| + C = \boldsymbol{\ln|\sec x| + C}$ $\qquad$ **Reciprocal Rule**

For the cotangent,

$$\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} \qquad u = \sin x,\, du = \cos x\, dx$$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

To integrate $sec\ x$, we multiply and divide by $(sec\ x + tan\ x)$ as an algebraic form of 1.

$$\int \sec x\, dx = \int \sec x \frac{(\sec x+\tan x)}{(\sec x+\tan x)} dx = \int \frac{\sec^2 x+\sec x \tan x}{\sec x+\tan x} dx$$

$$= \int \frac{du}{u} = \ln|u| + C \qquad u = sec\ x + tan\ x,$$

$$= ln\, |\sec x + \tan x| + C \qquad du = (sec\ x\ tan\ x + sec2\ x)dx$$

For $csc\ x$, we multiply and divide by $(csc\ x + cot\ x)$.

$$\int \csc x\, dx = \int \csc x \frac{(\csc x+\cot x)}{(\csc x+\cot x)} dx = \int \frac{\csc^2 x+\csc x \cot x}{\csc x+\cot x} dx$$

$$= \int -\frac{du}{u} = -\ln|u| + C \qquad u = csc\ x + cot\ x,$$

$$= -ln\, |\csc x + \cot x| + C \qquad du = (-csc\ x\ cot\ x - csc^2\ x)dx$$

In summary, we have the following results.

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\, du = \ln|\sec u| + C \qquad \int \sec u\, du = ln\, |\sec u + \tan u| + C$$

$$\int \cot u\, du = -\ln|\csc u| + C \qquad \int \csc u\, du = -ln\, |\csc u + \cot u| + C$$

Example:

$$\int_0^{\pi/6} \tan 2x\, dx = \int_0^{\pi/3} \tan u \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\, du$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi > 3$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2. \ \blacksquare$$

4.5.4 The Integrals of inverse trigonometric functions:

The derivative formulas in section 3.8.5 yield three useful integration formulas in Table 1. The formulas are readily verified by differentiating the functions on the right-hand sides. Since two notations are regularly used to represent the inverse sine function, we state these formulas using both notations, $sin^{-1}\ x$ as well as $arcsin\ x$, and similarly for the other inverse trigonometric functions.

TABLE 1. Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a > 0$.

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

The derivative formulas in section 3.8.5 have $a = 1$, but in most integrations $a \neq 1$, and the formulas in Table 1. are more useful.

Example:

a) $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2}$ **$a = 1, u = x$ in Table 1., Formula 1**

$= \sin^{-1}(\sqrt{3}/2) - \sin^{-1}(\sqrt{2}/2) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

b) $\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}$ **$a = \sqrt{3}, u = 2x$, and $du/2 = dx$**

$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + \mathrm{C}$ **Table 1., Formula 1**

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + \mathrm{C}$

c) $\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{du}{u\sqrt{u^2-a^2}}$ **$u = e^x, du = e^x,$ $dx = du/e^x = du/u, a = \sqrt{6}$**

$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ **Table 1., Formula 3**

$= \frac{1}{\sqrt{6}}\sec^{-1}\left|\frac{e^x}{\sqrt{6}}\right| + C.$ ■

Example:

Evaluate **a)** $\int \frac{dx}{\sqrt{4x-x^2}}$, **b)** $\int \frac{dx}{4x^2+4x+2}$.

Solution:

a) The expression $\sqrt{4x-x^2}$ does not match any of the formulas in Table 1., so we first rewrite $4x - x^2$ by completing the square:

$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x-2)^2.$

$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$

$= \int \frac{du}{\sqrt{a^2-u^2}}$ **$a = 2, u = x - 2$, and $du = dx$**

$= \sin^{-1}\left(\frac{u}{a}\right) + C$ **Table 1., Formula 1**

$= \sin^{-1}\left(\frac{x-2}{2}\right) + C.$

b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \tfrac{1}{4}\right) + 2 - \tfrac{4}{4}$$
$$= 4\left(x + \tfrac{1}{2}\right)^2 + 1 = (2x+1)^2 + 1.$$

Then,

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1} = \frac{1}{2}\int \frac{du}{u^2+a^2} \qquad a = 1, u = 2x+1, \text{ and } du/2 = dx$$
$$= \frac{1}{2}\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Table 1., Formula 2}$$
$$= \frac{1}{2}\tan^{-1}(2x+1) + C. \qquad a = 1, u = 2x+1. \blacksquare$$

Exercises:

1. Evaluate the integrals in following Exercises

a) $\int_{-3}^{-2} \frac{dx}{x}$ **b)** $\int_{-1}^{0} \frac{3\,dx}{3x-2}$ **c)** $\int \frac{2y\,dy}{y^2-25}$

d) $\int \frac{8r\,dr}{4r^2-5}$ **e)** $\int_0^{\pi} \frac{\sin t}{2-\cos t}\,dt$ **f)** $\int_0^{\pi/3} \frac{4\sin\theta}{1-4\cos\theta}\,d\theta$

g) $\int_1^2 \frac{2\ln x}{x}\,dx$ **h)** $\int_2^4 \frac{dx}{x\ln x}$ **i)** $\int_2^4 \frac{dx}{x(\ln x)^2}$

j) $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$ **k)** $\int \frac{3\sec^2 t}{6+3\tan t}\,dt$ **l)** $\int \frac{\sec y \cdot \tan y}{2+\sec y}\,dy$

m) $\int_0^{\pi/2} \tan\frac{x}{2}\,dx$ **n)** $\int_{\pi/4}^{\pi/2} \cot t\,dt$ **o)** $\int_{\pi/2}^{\pi} 2\cot\frac{\theta}{3}\,d\theta$

p) $\int_0^{\pi/12} 6\tan 3x\,dx$ **q)** $\int \frac{dx}{2\sqrt{x}+2x}$ **r)** $\int \frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

2. In the following Exercises, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

a) $y = \sqrt{x(x+1)}$ **b)** $y = \sqrt{(x^2+1)(x-1)^2}$ **c)** $y = \sqrt{\frac{t}{t+1}}$

d) $y = \sqrt{\frac{1}{t(t+1)}}$ **e)** $y = \sqrt{\theta+3}\sin\theta$ **f)** $y = (\tan\theta)\sqrt{2\theta+1}$

g) $y = t(t+1)(t+2)$ **h)** $y = \frac{1}{t(t+1)(t+2)}$ **i)** $y = \frac{\theta+5}{\theta\cos\theta}$

j) $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}$ **k)** $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$ **l)** $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$

m) $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$ **n)** $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

3. Evaluate the integrals in the following Exercises:

1) $\int (e^{3x} + 5e^{-x})\,dx$ **2)** $\int (2e^{x} - 3e^{-2x})\,dx$ **3)** $\int_{\ln 2}^{\ln 3} e^x\,dx$

4) $\int_{-\ln 2}^{0} e^{-x} dx$

5) $\int 8e^{(x+1)} dx$

6) $\int 2e^{(2x-1)} dx$

7) $\int_{\ln 4}^{\ln 9} e^{x/2} dx$

8) $\int_{0}^{\ln 16} e^{x/4} dx$

9) $\int \frac{e^{\sqrt{r}}}{\sqrt{r}} dr$

10) $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}} dr$

11) $\int 2te^{-t^2} dt$

12) $\int t^3 e^{(t^4)} dt$

13) $\int \frac{e^{1/x}}{x^2} dx$

14) $\int \frac{e^{-1/x^2}}{x^3} dx$

15) $\int \frac{e^r}{1+e^r} dr$

16) $\int_{0}^{\pi/4} \left(1 + e^{\tan\theta}\right) \sec^2\theta \, d\theta$

17) $\int_{\pi/4}^{\pi/2} \left(1 + e^{\cot\theta}\right) \csc^2\theta \, d\theta$

18) $\int e^{\sec \pi t} \sec \pi t \cdot \tan \pi t \; dt$

19) $\int e^{\csc(\pi+t)} \csc(\pi+t) \cdot \cot(\pi+t) \; dt$

20) $\int_{\ln(\frac{\pi}{6})}^{\ln(\frac{\pi}{2})} 2e^v \cos e^v \, dv$

21) $\int_{0}^{\sqrt{\ln \pi}} 2xe^{x^2} \cos(e^{x^2}) \, dx$

22) $\int \frac{dx}{1+e^x}$

23) $\int 5^x dx$

24) $\int \frac{3^x}{3-3^x} dx$

25) $\int_{0}^{1} 2^{-\theta} \, d\theta$

26) $\int_{-2}^{0} 5^{-\theta} \, d\theta$

27) $\int_{1}^{\sqrt{2}} x2^{(x^2)} \, dx$

28) $\int_{1}^{4} \frac{2^{\sqrt{x}}}{\sqrt{x}} \, dx$

29) $\int_{0}^{\pi/2} 7^{\cos t} \sin t \; dt$

30) $\int_{0}^{\pi/4} (\frac{1}{3})^{\tan t} \sec^2 t \; dt$

31) $\int_{2}^{4} x^{2x} (1 + \ln x) dx$

32) $\int \frac{x2^{x^2}}{1+2^{x^2}} dx$

33) $\int 3x^{\sqrt{3}} \, dx$

34) $\int x^{\sqrt{2}-1} \, dx$

35) $\int_{0}^{3} (\sqrt{2}+1)x^{\sqrt{2}} \, dx$

36) $\int_{1}^{e} x^{(\ln 2)-1} \, dx$

37) $\int \frac{log_{10}x}{x} dx$

38) $\int_{1}^{4} \frac{log_2 x}{x} dx$

39) $\int_{1}^{4} \frac{\ln 2 \cdot log_2 x}{x} dx$

40) $\int_{1}^{e} \frac{2 \ln 10 \cdot log_{10} x}{x} dx$

41) $\int_{0}^{2} \frac{log_2(x+2)}{x+2} dx$

42) $\int_{1/10}^{10} \frac{log_{10}(10x)}{x} dx$

43) $\int_{0}^{9} \frac{2\, log_{10}(x+1)}{x+1} dx$

44) $\int_{2}^{3} \frac{2\, log_2(x-1)}{x-1} dx$

45) $\int \frac{dx}{x\, log_{10} x}$

46) $\int \frac{dx}{x\,(log_8 x)^2}$

47) $\int_{1}^{\ln x} \frac{1}{t} \, dt, x > 1$

48) $\int_{1}^{e^x} \frac{1}{t} \, dt$

49) $\int_{1}^{1/x} \frac{1}{t} \, dt, x > 0$

50) $\frac{1}{\ln a} \int_{1}^{x} \frac{1}{t} \, dt, x > 0$

51) $\int \frac{dx}{\sqrt{9-x^2}}$

52) $\int \frac{dx}{\sqrt{1-4x^2}}$

53) $\int \frac{dx}{17+x^2}$

54) $\int \frac{dx}{9+3x^2}$

55) $\int \frac{dx}{x\sqrt{25x^2-2}}$

56) $\int \frac{dx}{x\sqrt{5x^2-4}}$

57) $\int_{0}^{1} \frac{4\, ds}{\sqrt{4-s^2}}$

58) $\int_{0}^{3\sqrt{2}/4} \frac{ds}{\sqrt{9-4s^2}}$

59) $\int_{0}^{2} \frac{dt}{8+2t^2}$

60) $\int_{-2}^{2} \frac{dt}{4+3t^2}$

61) $\int_{-1}^{-\sqrt{2}/2} \frac{dy}{y\sqrt{4y^2-1}}$

62) $\int_{-2/3}^{-\sqrt{2}/3} \frac{dy}{y\sqrt{9y^2-1}}$

63) $\int \frac{3\, dr}{\sqrt{1-4(r-1)^2}}$

64) $\int \frac{6\, dr}{\sqrt{4-(r+1)^2}}$

65) $\int \frac{dx}{2+(x-1)^2}$

66) $\int \frac{dx}{1+(3x+1)^2}$

67) $\int \frac{dx}{(2x-1)\sqrt{(2x-1)^2-4}}$

68) $\int \frac{dx}{(x+3)\sqrt{(x+3)^2-25}}$

69) $\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta d\theta}{1+(\sin\theta)^2}$

70) $\int_{\pi/6}^{\pi/4} \frac{\csc^2 x dx}{1+(\cot x)^2}$

71) $\int_0^{\ln\sqrt{3}} \frac{e^x dx}{1+e^{2x}}$

72) $\int_1^{e^{\pi/4}} \frac{4\,dt}{t(1+\ln^2 t)}$

73) $\int \frac{y\,dy}{\sqrt{1-y^4}}$

74) $\int \frac{\sec^2 y\,dy}{\sqrt{1-\tan^2 y}}$

75) $\int \frac{dx}{\sqrt{-x^2+4x-3}}$

76) $\int \frac{dx}{\sqrt{2x-x^2}}$

77) $\int_{-1}^{0} \frac{6\,dt}{\sqrt{3-2t-t^2}}$

78) $\int_{1/2}^{1} \frac{6\,dt}{\sqrt{3+4t-4t^2}}$

79) $\int \frac{dy}{y^2-2y+5}$

80) $\int \frac{dy}{y^2+6y+10}$

81) $\int_1^2 \frac{8\,dx}{x^2-2x+2}$

82) $\int_2^4 \frac{2\,dx}{x^2-6x+10}$

83) $\int \frac{x+4}{x^2+4}dx$

84) $\int \frac{t-2}{t^2-6t+10}dt$

85) $\int \frac{x^2+2x-1}{x^2+9}dx$

86) $\int \frac{t^3-2t^2+3t-4}{t^2+1}dt$

87) $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

88) $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

89) $\int \frac{e^{\sin^{-1}x}dx}{\sqrt{1-x^2}}$

90) $\int \frac{e^{\cos^{-1}x}dx}{\sqrt{1-x^2}}$

91) $\int \frac{(\sin^{-1}x)^2 dx}{\sqrt{1-x^2}}$

92) $\int \frac{\sqrt{\tan^{-1}x}\,dx}{1+x^2}$

93) $\int \frac{dy}{(\tan^{-1}y)(1+y^2)}$

94) $\int \frac{dy}{(\sin^{-1}y)\sqrt{1-y^2}}$

95) $\int_{\sqrt{2}}^{2} \frac{\sec^2(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$

96) $\int_{2/\sqrt{3}}^{2} \frac{\cos(\sec^{-1}x)\,dx}{x\sqrt{x^2-1}}$

97) $\int \frac{e^x \sin^{-1}e^x}{\sqrt{1-e^{2x}}}dx$

98) $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

99) $\int_{-\sqrt{3}}^{1/\sqrt{3}} \frac{\cos(\tan^{-1}3x)}{1+9x^2}dx$

100) $\int \frac{1}{\sqrt{x}(x+1)\left((\tan^{-1}\sqrt{x})^2+9\right)}dx$

4.5 Hyperbolic Functions:

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}. The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical and engineering applications.

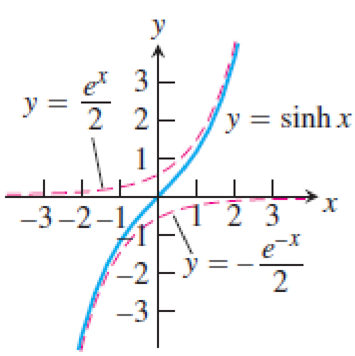
Definition:

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

We pronounce $sinh\ x$ as "cinch x," rhyming with "pinch x," and $cosh\ x$ as "kosh x," rhyming with "gosh x." From this basic pair, we define the hyperbolic tangent, cotangent, secant, and cosecant functions. The defining equations and graphs of these functions are shown in Table 2. We will see that the hyperbolic functions bear many similarities to the trigonometric functions after which they are named.

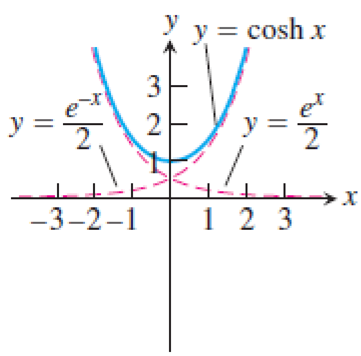
TABLE 2. The six basic hyperbolic functions



(a)

Hyperbolic sine:

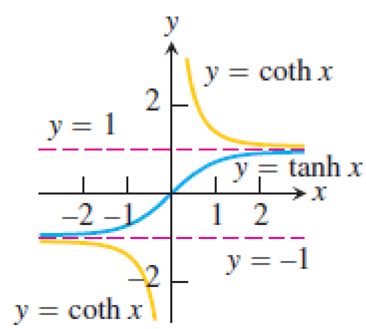
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



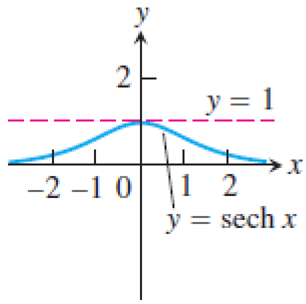
(c)

Hyperbolic tangent:

$$tanh\ x = \frac{sinh\ x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

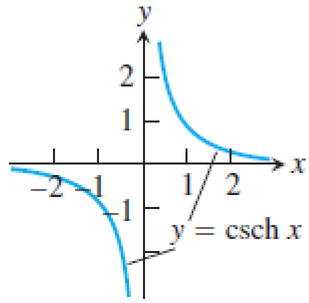
$$coth\ x = \frac{cosh\ x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$sech\, x = \frac{1}{cosh\, x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$csch\, x = \frac{1}{sinh\, x} = \frac{2}{e^x - e^{-x}}$$

Remark:

Hyperbolic functions satisfy the following identities in Table 3.

TABLE 3. Identities for hyperbolic functions

1. $cosh^2 x - sinh^2 x = 1$
2. $sinh\, 2x = 2\, sinh\, x\, cosh\, x$
3. $cosh\, 2x = cosh^2 x + sinh^2 x$
4. $cosh^2 x = \frac{cosh\, 2x + 1}{2}$
5. $sinh^2 x = \frac{cosh\, 2x - 1}{2}$
6. $tanh^2 x = 1 - sech^2 x$
7. $coth^2 x = 1 + csch^2 x$

These identities are proved directly from the definitions, as we show here for the second one:

$$\mathbf{2\, sinh\, x\, cosh\, x} = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = 2\left(\frac{e^{2x} - e^{-2x}}{4}\right) = \frac{e^{2x} - e^{-2x}}{2} = \mathbf{sinh\, 2x}.$$

The other identities are obtained similarly, by substituting in the definitions of the hyperbolic functions and using algebra.

Remark:

For any real number u, we know the point with coordinates $(cos\, u, sin\, u)$ lies on the unit circle $x^2 + y^2 = 1$. So, the trigonometric functions are sometimes called *circular functions*. Because of the first identity $cosh^2 x - sinh^2 x = 1$, with *u* substituted for *x* in Table 3, the point having coordinates $(cosh\, u, sinh\, u)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$. This is where hyperbolic functions get their names.

4.5.1 Derivatives and Integrals of Hyperbolic Functions:

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table 4.). Again, there are similarities with trigonometric functions.

TABLE 4. Derivatives of hyperbolic functions

1. $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$	**2.** $\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$
3. $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$	**4.** $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$
5. $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$	**6.** $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$

The derivative formulas are derived from the derivative of e^u:

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) \qquad \textbf{Definition of } \sinh u$$

$$= \frac{e^u \frac{du}{dx} - e^{-u}\frac{du}{dx}}{2} \qquad \textbf{Derivative of } e^u$$

$$= \cosh u \frac{du}{dx} \qquad \textbf{Definition of } \cosh u$$

This gives the first derivative formula. From the definition, we can calculate the derivative of the hyperbolic cosecant function, as follows:

$$\frac{d}{dx}(\operatorname{csch} u) = \frac{d}{dx}\left(\frac{1}{\sinh u}\right) \qquad \textbf{Definition of } \operatorname{csch} u$$

$$= -\frac{\cosh u du}{\sinh^2 u dx} \qquad \textbf{Quotient Rule for derivatives}$$

$$= -\frac{1}{\sinh u} \cdot -\frac{\cosh u du}{\sinh u dx} \qquad \textbf{Rearrange terms.}$$

$$= -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \textbf{Definitions of } \operatorname{csch} u \textbf{ and } \coth u$$

The other formulas in Table 4. are obtained similarly.

The derivative formulas lead to the integral formulas in Table 5.

TABLE 5. Integral formulas for hyperbolic functions

1. $\int \sinh u \, du = \cosh u + C$	**2.** $\int \cosh u \, du = \sinh u + C$
3. $\int \operatorname{sech}^2 u \, du = \tanh u + C$	**4.** $\int \operatorname{csch}^2 u \, du = -\coth u + C$
5. $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$	**6.** $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

Example:

We illustrate the derivative and integral formulas

a) $\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2}\,\frac{d}{dx}\sqrt{1+t^2} = \frac{\operatorname{sech}^2\sqrt{1+t^2}}{\sqrt{1+t^2}}$

b) $\int \coth 5x\,dx = \int \frac{\cosh 5x}{\sinh 5x}dx = \frac{1}{5}\int\frac{du}{u}$ **$u = \sinh 5x$, $du = 5\cosh 5x\,dx$**

$= \frac{1}{5}ln|u| + C = \frac{1}{5}ln|\sinh 5x| + C$

c) $\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}dx$ **Table 3.**

$= \frac{1}{2}\int_0^1(\cosh 2x - 1)dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$

$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$ **Evaluate with a calculator**

d) $\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x\frac{e^x - e^{-x}}{2}dx = \int_0^{\ln 2}(2e^{2x} - 2)dx$

$= [e^{2x} - 2x]_0^{\ln 2} = \left(e^{2\ln 2} - 2\ln 2\right) - (1 - 0)$

$= 4 - 2\ln 2 - 1 \approx 1.6137.$ ■

4.5.2 Inverse Hyperbolic Functions:

The inverses of the six basic hyperbolic functions are very useful in integration. Since $\frac{d(sinh\ x)}{dx} = cosh\ x > 0$, the hyperbolic sine is an increasing function of x. We denote its inverse by $y = \sinh^{-1} x$. For every value of x in the interval $-\infty < x < \infty$, the value of $y = \sinh^{-1} x$ is the number whose hyperbolic sine is x.

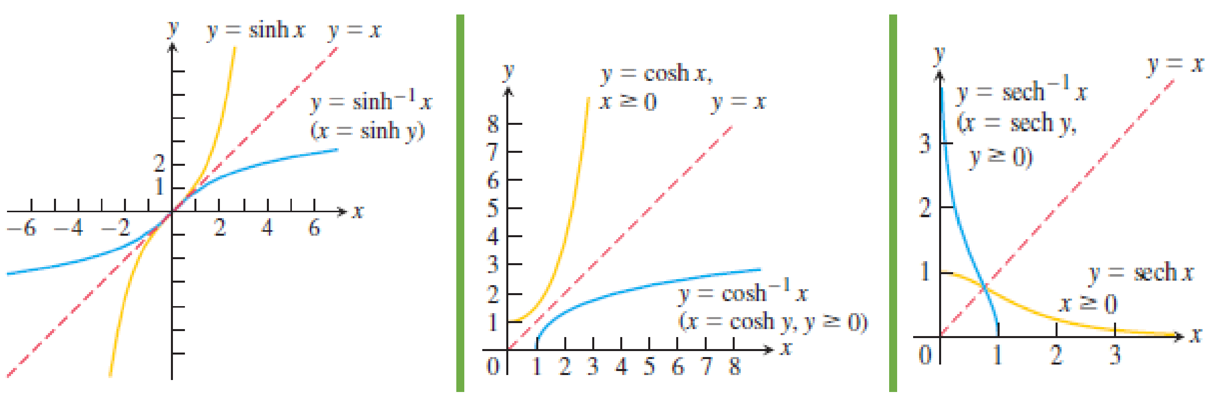


FIGURE. The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetry about the line $y = x$.

Remark:

The function $y = cosh\, x$ is not one-to-one because of its graph in Table 2. does not pass the horizontal line test. The restricted function $y = cosh\, x, x \geq 0$, however, is one to - one and therefore has an inverse, denoted by $y = cosh^{-1} x$. For every value of $x \geq 1, y = cosh^{-1} x$ is the number in the interval $0 \leq y < \infty$ whose hyperbolic cosine is x.

Like $y = cosh\, x$, the function $y = sech\, x = 1/cosh\, x$ fails to be one-to-one, but its restriction to nonnegative values of x does have an inverse, denoted by $y = sech^{-1} x$. For every value of x in the interval $(0, 1], y = sech^{-1}x$ is the nonnegative number whose hyperbolic secant is x.

The hyperbolic tangent, cotangent, and cosecant are one-to-one on their domains and therefore have inverses, denoted by

$$y = tanh^{-1} x, \quad y = coth^{-1} x, \quad y = csch^{-1} x.$$

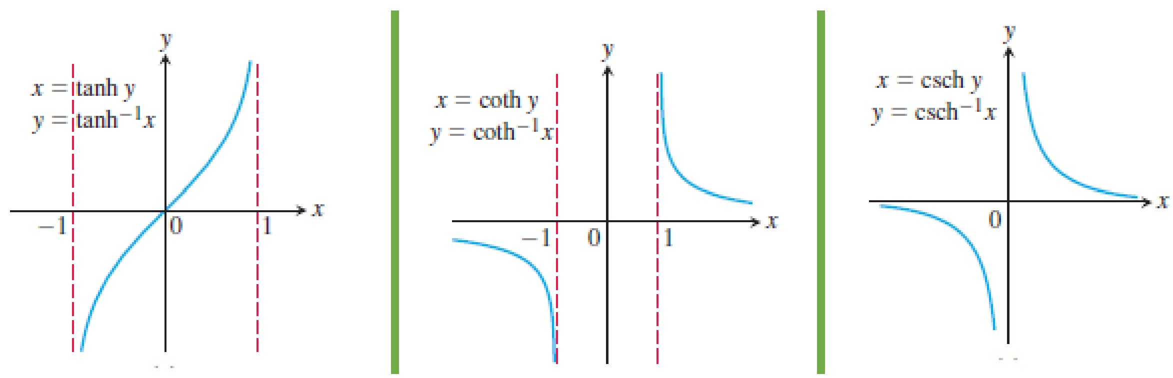


FIGURE. The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

TABLE 6. Identities for inverse hyperbolic functions		
1. $sech^{-1}x = cosh^{-1}\frac{1}{x}$	**2.** $csch^{-1}x = sinh^{-1}\frac{1}{x}$	**3.** $coth^{-1}x = tanh^{-1}\frac{1}{x}$

Remark:

We use the identities in Table 6. to calculate the values of $sech^{-1}x$, $csch^{-1} x$, and $coth^{-1}x$ on calculators that give only $cosh^{-1}x$, $sinh^{-1}x$, and $tanh^{-1}x$. These identities are direct consequences of the definitions. For example, if $0 < x \leq 1$, then

$$\text{sech}(cosh^{-1}(\tfrac{1}{x})) = \frac{1}{\cosh(cosh^{-1}(\frac{1}{x}))} = \frac{1}{(\frac{1}{x})} = x.$$

We also know that $sech\,(sech^{-1}\,x) = x$, so because the hyperbolic secant is one-to-one on $(0, 1]$, we have $cosh^{-1}\frac{1}{x} = sech^{-1}x$.

4.5.3 Derivatives of Inverse Hyperbolic Functions:

An important use of inverse hyperbolic functions lies in antiderivatives that reverse the derivative formulas in Table 7.

TABLE7. Derivatives of inverse hyperbolic functions

1. $\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$	**2.** $\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}\,, \quad u > 1$
3. $\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}\,, \quad \lvert u\rvert < 1$	**4.** $\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}\,, \quad \lvert u\rvert > 1$
5. $\frac{d(\text{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, 0 < u < 1$	**6.** $\frac{d(\text{csch}^{-1} u)}{dx} = -\frac{1}{\lvert u\rvert\sqrt{1+u^2}}\frac{du}{dx}, u \neq 0$

The restrictions $|u| < 1$ and $|u| > 1$ on the derivative formulas for $\tanh^{-1} u$ and $\coth^{-1} u$ come from the natural restrictions on the values of these functions. The distinction between $|u| < 1$ and $|u| > 1$ becomes important when we convert the derivative formulas into integral formulas.

We illustrate how the derivatives of the inverse hyperbolic functions are found in following Example, where we calculate $\frac{d(\cosh^{-1} u)}{dx}$. The other derivatives are obtained by similar calculations.

Example:

Show that if *u* is a differentiable function of *x* whose values are greater than 1, then $\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

Solution:

First, we find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem (The Derivative Rule for Inverses) of Section 3.7.1 with $f(x) = cosh\,x$ and $f^{-1}(x) = cosh^{-1}\,x$. Theorem (The Derivative Rule for Inverses) can be applied because the derivative of $cosh\,x$ is positive when $x > 0$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{\sinh(\cosh^{-1}(x))}$$

$$= \frac{1}{\sqrt{\cosh^2(\cosh^{-1}x)-1}}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

Theorem (The Derivative Rule for Inverses)

$f'(u) = \sinh u$

$\cosh^2 u - \sinh^2 u = 1$

$\sinh u = \sqrt{\cosh^2 u - 1}$

$\cosh(\cosh^{-1} x) = x$

The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}\,. \blacksquare$$

Remark:

With appropriate substitutions, the derivative formulas in Table 7. lead to the integration formulas in Table 8. Each of the formulas in Table 8. can be verified by differentiating the expression on the right-hand side.

TABLE 8. Integrals leading to inverse hyperbolic functions

1. $\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \quad a > 0$
2. $\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \quad u > a > 0$
3. $\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$
4. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \ 0 < u < a$
5. $\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \ u \neq 0$ and $a > 0$

Example:

Evaluate $\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$.

Solution:

The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

$u = 2x, du = 2\,dx, a = \sqrt{3}$

Formula from Table 8.

Therefore, $\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$

$$= sinh^{-1}(\frac{2}{\sqrt{3}}) - 0 \approx 0.98665. \blacksquare$$

Exercises:

1. Each of following Exercises gives a value of $sinh\ x$ or $cosh\ x$. Use the definitions and the identity $cosh^2\ x - sinh^2\ x = 1$ to find the values of the remaining five hyperbolic functions.

a) $\sinh x = -\frac{3}{4}$. **b)** $\sinh x = \frac{4}{3}$ **c)** $\cosh x = \frac{17}{15}, x > 0$ **d)** $\cosh x = \frac{13}{5}, x > 0$

2. Rewrite the expressions in following Exercises in terms of exponentials and simplify the results as much as you can.

a) $2\cosh(\ln x)$ **b)** $\sinh(2\ln x)$ **c)** $\cosh 5x + \sinh 5x$

d) $\cosh 3x + \sinh 3x$ **e)** $(\sinh x + \cosh x)^4$

f) $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x)$

3. Prove the identities

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y\,,$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y\,.$$

Then use them to show that **a)** $\sinh 2x = 2\sinh x \cosh x$

b) $\cosh 2x = \cosh 2x \sinh 2x$

4. Use the definitions of $cosh\ x$ and $sinh\ x$ to show that

$$\cosh^2 x - \sinh^2 x = 1$$

5. In following Exercises, find the derivative of y with respect to the appropriate variable.

a) $y = 6\sinh\frac{x}{3}$ **b)** $y = \frac{1}{2}\sinh(2x + 1)$ **c)** $y = 2\sqrt{t}\tanh\sqrt{t}$

d) $y = t^2\tanh\frac{1}{t}$ **e)** $y = \ln(\sinh z)$ **f)** $y = \ln(\cosh z)$

g) $y = (\operatorname{sech}\theta)(1 - \ln\operatorname{sech}\theta)$ **h)** $y = (\operatorname{csch}\theta)(1 - \ln\operatorname{csch}\theta)$

i) $y = \ln\cosh v - \frac{1}{2}\tanh^2 v$ **j)** $y = \ln\sinh v - \frac{1}{2}\coth^2 v$

k) $y = (x^2 + 1)\operatorname{sech}(\ln x)$ **l)** $y = (4x^2 - 1)\operatorname{csch}(\ln 2x)$

m) $\sinh^{-1}\sqrt{x}$ **n)** $y = \cosh^{-1} 2\sqrt{x+1}$

o) $y = (1 - \theta)\tanh^{-1}\theta$ **p)** $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta + 1)$

q) $y = (1 - t)\coth^{-1}\sqrt{t}$ **r)** $y = (1 - t^2)\coth^{-1} t$

s) $y = \cos^{-1} x - x\operatorname{sech}^{-1} x$ **t)** $y = \ln x + \sqrt{1 - x^2}\operatorname{sech}^{-1} x$

u) $y = \operatorname{csch}^{-1}(\frac{1}{2})^{\theta}$ **v)** $y = \operatorname{csch}^{-1} 2^{\theta}$ **w)** $y = \sinh^{-1}(\tan x)$

x) $y = cosh^{-1}(sec\ x), 0 < x < \pi/2$

6. Verify the integration formulas in following Exercises.

a) **I)** $\int \operatorname{sech} x \; dx = \tan^{-1}(\sinh x) + C$

II) $\int \operatorname{sech} x \; dx = sin^{-1}(\tanh x) + C$

b) $\int x \operatorname{sech}^{-1} x \; dx = \frac{x^2}{2}\operatorname{sech}^{-1} x - \frac{1}{2}\sqrt{1-x^2} + C$

c) $\int x \coth^{-1} x \; dx = \frac{x^2-1}{2}\coth^{-1} x + \frac{x}{2} + C$

d) $\int \tanh^{-1} x \; dx = x \operatorname{tahh}^{-1} x + \frac{1}{2}\ln(1-x^2) + C$

7. Evaluate the integrals in following Exercises

a) $\int \sinh 2x\, dx$ **b)** $\int \sinh \frac{x}{5}\, dx$ **c)** $\int 6\cosh(\frac{x}{2} - \ln 3)\, dx$

d) $\int 4\cosh(3x - \ln 2)\, dx$ **e)** $\int \tanh \frac{x}{7}\, dx$ **f)** $\int \coth \frac{\theta}{\sqrt{3}}\, d\theta$

g) $\int \operatorname{sech}^2(x - \frac{1}{2})\, dx$ **h)** $\int \operatorname{csch}^2(5-x)\, dx$ **i)** $\int \frac{\operatorname{sech}\sqrt{t}\tanh\sqrt{t}\, dt}{\sqrt{t}}$

j) $\int \frac{\operatorname{csch}(\ln t)\coth(\ln t)\, dt}{t}$ **k)** $\int_{\ln 2}^{\ln 4} \coth x \; dx$ **l)** $\int_0^{\ln 2} \tanh 2x \; dx$

m) $\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta \; d\theta$ **n)** $\int_0^{\ln 2} 4e^{-\theta}\sinh\theta \; d\theta$ **o)** $\int_{-\pi/4}^{\pi/4} \cosh(\tan\theta)\sec^2\theta\, d\theta$

p) $\int_0^{\pi/2} 2\sinh(\sin\theta)\cos\theta\, d\theta$ **q)** $\int_1^2 \frac{\cosh(\ln t)}{t}\, dt$ **r)** $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}}\, dx$

s) $\int_{-\ln 2}^{0} \cosh^2(\frac{x}{2})\, dx$ **t)** $\int_0^{\ln 10} 4\sinh^2(\frac{x}{2})\, dx$

8. Since the hyperbolic functions can be written in terms of exponential functions, it is possible to express the inverse hyperbolic functions in terms of logarithms, as shown in the following table

1. $sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right), -\infty < x < \infty$

2. $cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right), \;\; x \geq 1$

3. $tanh^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x}\;, \;\; |x| < 1$

4. $sech^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \;\; 0 < x \leq 1$

5. $csch^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), \;\; x \neq 0$

6. $coth^{-1}x = \frac{1}{2}\ln\frac{x+1}{x-1}\;, \;\; |x| > 1$

Use these formulas to express the numbers in following Exercises in terms of natural logarithms.

a) $\sinh^{-1}(-5/12)$ **b)** $\cosh^{-1}(5/3)$ **c)** $\tanh^{-1}(-1/2)$

d) $\coth^{-1}(5/4)$ **e)** $\operatorname{sech}^{-1}(3/5)$ **f)** $\operatorname{csch}^{-1}(-1/\sqrt{3})$

9. Evaluate the integrals in following Exercises in terms of

a) inverse hyperbolic functions.

b) natural logarithms.

1) $\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}}$ **2)** $\int_0^{1/3} \frac{6dx}{\sqrt{1+9x^2}}$ **3)** $\int_{5/4}^{2} \frac{dx}{1-x^2}$

4) $\int_0^{1/2} \frac{dx}{1-x^2}$ **5)** $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}}$ **6)** $\int_1^{2} \frac{dx}{x\sqrt{4+x^2}}$

7) $\int_0^{\pi} \frac{\cos x\, dx}{\sqrt{1+\sin^2 x}}$ **8)** $\int_1^{e} \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Calculus II
Second Semester
Lecturer 3
Dr. Ban Jaffar AL-Taiy
Taghreed Hussein Abed

Chapter five: Techniques of Integration.

The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. However, finding anti-derivatives (or indefinite integrals) is not as straightforward as finding derivatives. In this chapter we study several important techniques that apply to finding integrals for specialized classes of functions such as trigonometric functions, products of certain functions, and rational functions. Since we cannot always find an antiderivative, we develop numerical methods for calculating definite integrals. We also study integrals whose domain or range are infinite, called improper integrals.

5.1 Using Basic Integration Formulas:

Table 1. summarizes the indefinite integrals of many of the functions we have studied so far are summarized in the following Table and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this section we combine the Substitution Rules with algebraic methods and trigonometric identities to help us use Table

TABLE (1) Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)	12. $\int \tan x\,dx = ln\lvert\sec x\rvert + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$	13. $\int \cot x\,dx = ln\lvert\sin\ x\rvert + C$
3. $\int \frac{dx}{x} = ln\lvert x\rvert + C$	14. $\int \sec x\,dx = ln\lvert\sec x + \tan x\rvert + C$
4. $\int e^x\,dx = e^x + C$	15. $\int \csc x\,dx = -ln\lvert\csc x + \cot x\rvert + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C(a > 0, a \neq 1)$	16. $\int \sinh x\ \ dx = \cosh x + C$
6. $\int \sin x\ \ dx = -\cos x + C$	17. $\int \cosh x\ \ dx = \sinh x + C$
7. $\int \cos x\ \ dx = \sin x + C$	18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
8. $\int \sec^2 x\ \ dx = \tan x + C$	19. $\int \frac{dx}{\sqrt{a^2+x^2}} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
9. $\int \csc^2 x\ \ dx = -\cot x + C$	20. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\text{arcsec}\left\lvert\frac{x}{a}\right\rvert + C$
10. $\int \sec x \cdot \tan x\ \ dx = \sec x + C$	21. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C(a > 0)$
11. $\int \csc x \cdot \cot x\ \ dx = -\csc x + C$	22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C(\text{x} > a > 0)$

Sometimes we must rewrite an integral to match it to a standard form of the type displayed in Table (1).

Example:

Complete the square to evaluate $\int \frac{dx}{\sqrt{8x-x^2}}$.

Solution:

We completed the square to simplify the denominator:

$$8x - x^2 = -x^2 + 8x + 16 - 16 = (-x^2 - 8x - 16) + 16$$
$$= -(x^2 - 8x + 16) + 16) = -(x-4)^2 + 16$$

Then

$$\int \frac{dx}{\sqrt{8x-x^2}} = \int \frac{dx}{\sqrt{16-(x-4)^2}}$$

$$= \int \frac{dx}{\sqrt{a^2-u^2}} \qquad \boldsymbol{a = 4, u = (x-4), du = dx}$$

$$= \arcsin(\tfrac{u}{a}) + C \qquad \textbf{Table 1, Formula 18}$$

$$= \arcsin(\tfrac{x-4}{4}) + C. \blacksquare$$

Example:

Evaluate the integral $\int (\cos x \cdot \sin 2x + \sin x \cdot \cos 2x)\, dx$.

Solution:

We can replace the integrand with an equivalent trigonometric expression using the Sine Addition Formula to obtain a simple substitution:

$$\int (\cos x \cdot \sin 2x + \sin x \cdot \cos 2x)\, dx = \int (\sin(x + 2x))\, dx$$

$$= \int \sin 3x \, dx$$

$$= \frac{1}{3}\int \sin u \, du \qquad \boldsymbol{u = 3x, du = 3dx}$$

$$= -\frac{1}{3}\cos 3x + C. \qquad \textbf{Table 1, Formula 6.} \blacksquare$$

Example:

Find $\int_0^{\pi/4} \frac{dx}{1-\sin x}$.

Solution:

We multiply the numerator and denominator of the integrand by $1 + \sin x$.This procedure transforms the integral into one we can evaluate:

$$\int_0^{\pi/4} \frac{dx}{1-\sin x} = \int_0^{\pi/4} \frac{1}{1-\sin x} \cdot \frac{1+\sin x}{1+\sin x} dx$$

Multiply and divide by conjugate.

$$= \int_0^{\pi/4} \frac{1+\sin x}{1-\sin^2 x} dx$$

Simplify

$$= \int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x} dx$$

$\mathbf{1 - \sin^2 x = \cos^2 x}$

$$= \int_0^{\pi/4} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \cdot \tan x) dx$$

Use Table 1, Formulas 8 and 10

$$= [\tan x + \sec x]_0^{\pi/4} = \left(1 + \sqrt{2} - (0 + 1)\right) = \sqrt{2}. \blacksquare$$

Example:

Evaluate the integral $\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}} dx$.

Solution:

We rewrite the integral and apply the Substitution Rule for Definite Integrals, to find

$$\int_3^5 \frac{2x-3}{\sqrt{x^2-3x+1}} dx = \int_1^{11} \frac{du}{\sqrt{u}}$$

$\mathbf{u = x^2 - 3x + 1, du = (2x - 3)\, dx;}$
$\mathbf{u = 1 \text{ when } x = 3, u = 11 \text{ when } x = 5}$

$$= \int_1^{11} u^{-1/2} du$$

$$= 2\sqrt{u}\Big]_1^{11} = 2(\sqrt{u} - 1) \approx 4.63.$$

Table 1, Formula 2. $\blacksquare$

Example:

Evaluate $\int \frac{3x^2-7x}{3x+2} dx$.

Solution:

The integrand is an improper fraction since the degree of the numerator is greater than the degree of the denominator. To integrate it, we perform long division to obtain a quotient plus a remainder that is a proper fraction: $\frac{3x^2-7x}{3x+2} = (x-3) + \frac{6}{3x+2}$.

$$\begin{array}{r|l} & x - 3 \\ \hline 3x + 2 & 3x^2 - 7x \\ & 3x^2 + 2 \\ \hline & \quad -9x \\ & \quad -9x - 6 \\ \hline & \qquad\quad +6 \end{array}$$

Therefore,

$$\int \frac{3x^2-7x}{3x+2}\,dx = \int \left((x-3)+\frac{6}{3x+2}\right)dx = \frac{x^2}{2} - 3x + 2\ln|3x+2| + C. \blacksquare$$

Remark:

Reducing an improper fraction by long division does not always lead to an expression we can integrate directly.

Example:

Evaluate $\int \frac{3x+2}{\sqrt{1-x^2}}\,dx$.

Solution:

We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = 3\int \frac{x}{\sqrt{1-x^2}}\,dx + 2\int \frac{1}{\sqrt{1-x^2}}\,dx.$$

In the first of these new integrals, we substitute $u = 1 - x^2, du = -2xdx,$ so $xdx = -\frac{1}{2}\,du$. Then we obtain

$$3\int \frac{x\ dx}{\sqrt{1-x^2}} = 3\int \frac{(-1/2)\ du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}du = -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1-x^2} + C_1.$$

The second of the new integrals is a standard form,

$$2\int \frac{1}{\sqrt{1-x^2}}\,dx = 2\sin^{-1}x + C_2.$$ **Table 1, Formula 18**

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x+2}{\sqrt{1-x^2}}\,dx = -3\sqrt{1-x^2} + 2\sin^{-1}x + C. \blacksquare$$

Remark:

The question of what to substitute for in an integrand is not always quite so clear. Sometimes we simply proceed by trial-and-error, and if nothing works out, we then try another method altogether. The next several sections of the text present some of these new methods, but substitution works in the following example.

Example:

Evaluate $\int \frac{dx}{(1+\sqrt{x})^3}\,dx$.

Solution:

We might try substituting for the term $\sqrt{x}$, but the derivative factor $1/\sqrt{x}$ is missing from the integrand, so this substitution will not help. The other possibility is to substitute for $(1+\sqrt{x})$, and it turns out this works:

$$\int \frac{dx}{(1+\sqrt{x})^3} dx = \int \frac{2(u-1)}{u^3} du$$

$\boldsymbol{u = 1 + \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx;}$

$\boldsymbol{dx = 2\sqrt{x}. du = 2(u-1) du}$

$$= \int \left(\frac{2}{u^2} - \frac{2}{u^3}\right) du$$
$$= -\frac{2}{u} + \frac{1}{u^2} + C$$
$$= \frac{1-2u}{u^2} + C$$
$$= \frac{1-2(1+\sqrt{x})}{(1+\sqrt{x})^2} + C$$
$$= C - \frac{1+2\sqrt{x}}{(1+\sqrt{x})^2}. \blacksquare$$

Remark:

When evaluating definite integrals, a property of the integrand may help us in calculating the result.

Example:

Evaluate $\int_{-\pi/2}^{\pi/2} x^3 cos\ x\, dx.$

Solution:

No substitution or algebraic manipulation is clearly helpful here. But we observe that the interval of integration is the symmetric interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, the factor x^3 is an odd function, and $cos\ x$ is an even function, so their product is odd. Therefore,

$$\int_{-\pi/2}^{\pi/2} x^3 cos\ x\, dx = 0. \blacksquare$$

Exercises:

The integrals in following are in no particular order. Evaluate each integral using any algebraic method or trigonometric identity you think is appropriate. When necessary, use a substitution to reduce it to a standard form.

1. $\int_0^1 \frac{16x}{8x^2+2} dx$
2. $\int \frac{x^2}{x^2+1} dx$
3. $\int (\sec x - \tan x)^2 dx$
4. $\int_{\pi/4}^{\pi/3} \frac{1}{\cos^2 x \cdot \tan x} dx$
5. $\int \frac{1-x}{\sqrt{1-x}} dx$
6. $\int \frac{1}{x-\sqrt{x}} dx$
7. $\int \frac{e^{-\cot z}}{\sin^2 z} dz$
8. $\int \frac{2^{\ln z^3}}{16 z} dz$
9. $\int \frac{1}{e^z+e^{-z}} dz$
10. $\int_1^2 \frac{8}{x^2-2x+2} dx$
11. $\int_{-1}^0 \frac{4}{1+(2x+1)^2} dx$
12. $\int_{-1}^3 \frac{4x^2-7}{2x+3} dx$

13. $\int \frac{1}{1-\sec t}\,dt$

14. $\int \csc t \cdot \sin 3t\,dt$

15. $\int_0^{\pi/4} \frac{1+\sin\theta}{\cos^2\theta}\,d\theta$

16. $\int \frac{1}{\sqrt{2\theta-\theta^2}}\,d\theta$

17. $\int \frac{\ln y}{y+4yln^2 y}\,dy$

18. $\int \frac{2^{\sqrt{y}}}{2\sqrt{y}}\,dy$

19. $\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

20. $\int \frac{1}{t\sqrt{3+t^2}}\,dt$

21. $\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$

22. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

23. $\int_0^{\pi/2} \sqrt{1-\cos\theta}\,d\theta$

24. $\int (\sec t+\cot t)^2\,dt$

25. $\int \frac{1}{\sqrt{e^{2y}-1}}\,dy$

26. $\int \frac{6}{\sqrt{y}(1+y)}\,dy$

27. $\int \frac{2}{x\sqrt{1-4\ln^2 x}}\,dx$

28. $\int \frac{1}{(x-2)\sqrt{x^2-4x+3}}\,dx$

29. $\int \frac{\tan\theta+3}{\sin\theta}\,d\theta$

30. $\int 3\sinh(\frac{x}{2}+\ln 5)\,dx$

31. $\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx$

32. $\int_{-1}^{1} \sqrt{1+x^2}\sin x\,dx$

33. $\int_{-1}^{0} \sqrt{\frac{1+y}{1-y}}\,dy$

34. $\int e^{z+e^z}\,dz$

35. $\int \frac{7}{(x-1)\sqrt{x^2-2x-48}}\,dx$

36. $\int \frac{dx}{(2x+1)\sqrt{4x+4x^2}}$

37. $\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

38. $\int \frac{d\theta}{\cos\theta-1}$

39. $\int \frac{1}{1+e^x}\,dx$

40. $\int \frac{\sqrt{x}}{1+x^3}\,dx$

41. $\int \frac{e^{3x}}{e^x+1}\,dx$

42. $\int \frac{2^x-1}{3^x}\,dx$

43. $\int \frac{1}{\sqrt{x}(1+x)}\,dx$

44. $\int (\csc x-\sec x)(\sin x+\cos x)\,dx$

5.2 Integration by Parts:

Integration by parts is a technique for simplifying integrals of the form $\int u(x)v'(x)dx$. It is useful when u can be differentiated repeatedly and v' can be integrated repeatedly without difficulty.

The integrals $\int x\cos x\,dx$ and $\int x^2e^x dx$ are such integrals because $u(x) = x$ or $u(x) = x^2$ can be differentiated repeatedly to become zero, and $v'(x) = cos\,x$ or $v'(x) = e^x$ can be integrated repeatedly without difficulty.

Integration by parts also applies to integrals like $\int \ln x\,dx$ and $\int e^x \cos x\ dx$. In the first case, the integrand ln x can be rewritten as $(ln\,x)(1)$, and $u(x) = ln\,x$ is easy to differentiate while $v'(x) = 1$ easily integrates to x. In the second case, each part of the integrand appears again after repeated differentiation or integration.

5.2.1 Product Rule in Integral Form:

If u and v are differentiable functions of x, the Product Rule says that $\frac{d}{dx}[u(x)v(x) = u'(x)v(x) + u(x)v'(x)$. In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[u(x)v(x) = \int[u'(x)v(x) + u(x)v'(x)]dx$$

or $$\int \frac{d}{dx}[u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int u(x)v'(x)\,dx = \int \frac{d}{dx}[u(x)v(x) - \int v(x)u'(x)dx,$$

leading to the integration by parts formula.

Integration by Parts Formula:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)dx \qquad \textbf{(1)}$$

This formula allows us to exchange the problem of computing the integral $\int u(x)v'(x)\,dx$ with the problem of computing a different integral, $\int v(x)u'(x)dx$. In many cases, we can choose the functions u and v so that the second integral is easier to compute than the first. There can be many choices for u and v, and it is not always clear which choice works best, so sometimes we need to try several.

The formula is often given in differential form. With $v'(x)dx = dv$ and $u'(x)\,dx = du$, the integration by parts formula becomes

Integration by Parts Formula - Differential Version:

$$\int u dv = uv - \int v du \qquad (2)$$

Example:

Find $\int x \cos x\, dx$.

Solution:

There is no obvious antiderivative of $x\, cos\, x$, so we use the integration by parts formula $\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)dx$ to change this expression to one that is easier to integrate. We first decide how to choose the functions $u(x)$ and $v(x)$. In this case we factor the expression $x\, cos\, x$ into

$$u(x) = x \quad \text{and} \quad v'(x) = \cos x.$$

Next, we differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 1 \quad \text{and} \quad v(x) = sinx.$$

When finding an antiderivative for $v'(x)$ we have a choice of how to pick a constant of integration C. We choose the constant $C = 0$, since that makes this antiderivative as simple as possible. We now apply the integration by parts formula:

$$\int \underbrace{x}_{u(x)} \underbrace{\cos x}_{v'(x)}\, dx = \underbrace{x}_{u(x)} \underbrace{\sin x}_{v(x)} - \int \underbrace{\sin x}_{v(x)}\, \underbrace{(1)}_{u'(x)} dx \qquad \text{Integration by parts formula}$$

$$= x \sin x + \cos x + C \qquad \text{Integrate and simplify.} \ ■$$

Remark:

There are four apparent choices available for $u(x)$ and $v'(x)$ in previous Example:

1. Let $u(x) = 1$ and $v'(x) = x\cos x$
2. Let $u(x) = x$ and $v'(x) = \cos x$
3. Let $u(x) = x cos\, x$ and $v'(x) = 1$
4. Let $u(x) = \cos x$ and $v'(x) = x$

Choice 2 was used in previous Example. The other three choices lead to integrals we don't know how to integrate. For instance, Choice 3, with $u'^{(x)} = -x sin\, x + \cos x$, leads to the integral $\int (x\cos x - x^2 sin\, x)dx$. The goal of integration by parts is to go from an integral $\int u(x)v'(x)\,dx$ that we don't see how to evaluate to an integral $\int v(x)u'(x)dx$ that we can evaluate. Generally, you choose $v'(x)$ first to be as much of the integrand as we can readily integrate; $u(x)$ is the leftover part. When finding $v(x)$ from $v'(x)$, any antiderivative will work, and we usually pick the simplest one; no arbitrary constant of integration is needed in $v(x)$ because it would simply cancel out of the right-hand side of Equation (2).

Example:

Find $\int \ln x\, dx$.

Solution:

We have not yet seen how to find an antiderivative for $ln\, x$. If we set $u(x) = ln\, x$, then $u'(x)$ is the simpler function $1/x$. It may not appear that a second function $v'(x)$ is multiplying ln x, but we can choose $v'(x)$ to be the constant function $v'(x) = 1$. We use the integration by parts formula Equation (1) with

$$u(x) = \ln x \quad \text{and} \quad v'(x) = 1.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = \frac{1}{x} \quad \text{and} \quad v(x) = x.$$

Then

$$\int \underbrace{\ln x}_{u(x)} \cdot \underbrace{1}_{v'(x)} dx = \underbrace{(\ln x)}_{u(x)}\underbrace{x}_{v(x)} - \int \underbrace{x}_{v(x)}\underbrace{\frac{1}{x}}_{u'(x)} dx \qquad \textbf{Integration by parts formula}$$

$$= x ln\, x - x + C \qquad \textbf{Simplify and integrate.} \ \blacksquare$$

Remark:

In the following examples we use the differential form to indicate the process of integration by parts. The computations are the same, with du and dv providing shorter expressions for $u'(x)\,dx$ and $v'(x)\,dx$. Sometimes we have to use integration by parts more than once, as in the next example.

Example:

Evaluate $\int x^2 e^x dx$.

Solution:

We use the integration by parts formula Equation (1) with

$$u(x) = x^2 \quad \text{and} \quad v'(x) = e^x.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 2x \quad \text{and} \quad v(x) = e^x.$$

We summarize this choice by setting $du = u'(x)\,dx$ and $dv = v'(x)\,dx$, so $du = 2x\,dx$ and $dv = e^x dx.$ We then have

$$\int \underset{u}{x^2}\,\underbrace{e^x dx}_{dv} = \underset{u}{x^2}\,\underset{v}{e^x} - \int \underset{v}{e^x}\,\underbrace{2x dx}_{du}$$ **Integration by parts formula**

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x, dv = e^x dx$. Then $du = dx, v = e^x$, and

$$\int \underset{u}{x}\,\underbrace{e^x dx}_{dv} = \underset{u}{x}\,\underset{v}{e^x} - \int \underset{v}{e^x}\,\underbrace{dx}_{du} = xe^x - e^x + C$$ **Integration by parts Equation (2)**

Using this last evaluation, we then obtain

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2xe^x + 2e^x + C.$$

where the constant of integration is renamed after substituting for the integral on the right. ■

Remark:

The technique of previous Example works for any integral $\int x^n e^x dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy.

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

Example:

Evaluate $\int e^x \cos x \, dx$.

Solution:

Let $u = e^x$ and $dv = cos\ x\ dx$. Then $du = e^x dx, v = sin\ x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx. \qquad \boldsymbol{u(x) = e^x, v(x) = sin\ x}$$

The second integral is like the first except that it has $sin\ x$ in place of $cos\ x$. To evaluate it, we use integration by parts with

$$u = e^x, \ \ dv = sin\ x\ dx, \ \ v = -cos\ x, \ \ du = e^x\ dx.$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x dx)\right)$$

$$\boldsymbol{u(x) = e^x, v(x) = -cos\ x}$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

The unknown integral now appears on both sides of the equation, but with opposite signs. Adding the integral to both sides and adding the constant of integration give

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \ \blacksquare$$

Example:

Obtain a formula that expresses the integral $\int cos^n x dx$ in terms of an integral of a lower power of $cos\, x$.

Solution:

We may think of $cos^n x$ as $cos^{n-1}x \cdot\ cos\, x$. Then we let

$$u = cos^{n-1}x \text{ and } dv = cos\, x\, dx,$$

so that $du = (n\ -\ 1)\, cos^{n-2}x(-sin\, x\, dx)$ and $v = sin\, x.$ Integration by parts then gives

$$\int cos^n x dx = cos^{n-1}x\ \sin x + (n-1)\int \sin^2 x\, cos^{n-2}x dx$$

$$= cos^{n-1}x\ \sin x + (n-1)\int (1-\cos^2 x) cos^{n-2}x dx$$

$$= cos^{n-1}x\ \sin x + (n-1)\int cos^{n-2}x dx - (n-1)\int cos^n x dx.$$

If we add $(n-1)\int cos^n x dx$ to both sides of this equation, we obtain

$$n\int cos^n x dx = cos^{n-1}x\ \sin x + (n-1)\int cos^{n-2}x dx.$$

We then divide through by n, and the final result is

$$\int cos^n x dx = \frac{cos^{n-1}x\ \sin x}{n} + \frac{(n-1)}{n}\int cos^{n-2}x dx. \blacksquare$$

Remark:

The formula found in previous Example is called a ***reduction formula*** because it replaces an integral containing some power of a function with an integral of the same form having the power reduced. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is easy to evaluate. For example, the result in previous Example tells us that

$$\int cos^3 x dx = \frac{cos^2 x\ \sin x}{3} + \frac{2}{3}\int \cos x\, dx = \frac{cos^2 x\ \sin x}{3} + \frac{2}{3}\sin x + C.$$

5.2.2 Evaluating Definite Integrals by Parts:

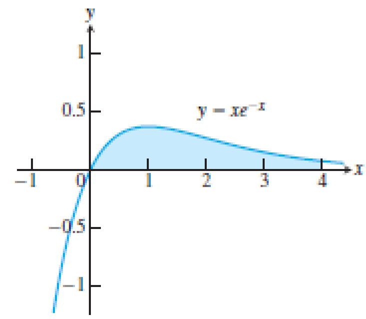
The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both u' and v' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)]_a^b - \int_a^b v(x)u'(x)dx \qquad (3)$$

Example:

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.



Solution:

The region is shaded in Figure. Its area is

$$\int_0^4 xe^{-x}dx.$$

Let $u = x, dv = e^{-x}dx, v = -e^{-x}$, and $du = dx$.

Then,

$$\int_0^4 xe^{-x}dx = -xe^{-x}]_0^4 - \int_0^4 (-e^{-x})dx \quad \text{Integration by parts Formula (3)}$$

$$= [-4e^{-4} - (0e^{-0})] + \int_0^4 e^{-x}dx$$

$$= -4e^{-4} - e^{-x}]_0^4$$

$$= -4e^{-4} - (e^{-4} - e^0) = 1 - 5e^{-4} \approx 0.91. \blacksquare$$

Exercises:

1. Evaluate the integrals in following using integration by parts.

- **a)** $\int x \sin \frac{x}{2} dx$ **b)** $\int \theta \cos \pi\theta \, d\theta$ **c)** $\int t^2 \cos t \, dt$
- **d)** $\int x^2 \sin x dx$ **e)** $\int_1^2 x \ln x \, dx$ **f)** $\int_1^e x^3 \ln x \, dx$
- **g)** $\int xe^x dx$ **h)** $\int xe^{3x} dx$ **i)** $\int x^2 e^{-x} dx$
- **j)** $\int (x^2 - 2x + 1)e^{2x} dx$ **k)** $\int \tan^{-1} y \, dy$ **l)** $\int \sin^{-1} y \, dy$
- **m)** $\int x sec^2 x \, dx$ **n)** $\int 4x \cdot sec^2 2x \, dx$ **o)** $\int x^3 e^x dx$
- **p)** $\int p^4 e^{-q} dp$ **q)** $\int (x^2 - 5x)e^x dx$ **r)** $\int (r^2 + r + 1)e^r dr$
- **s)** $\int x^5 e^x dx$ **t)** $\int t^2 e^{4t} dt$ **u)** $\int e^\theta \sin \theta \, d\theta$
- **v)** $\int e^{-y} \cos y \, dy$ **w)** $\int e^{2x} \cos 3x \, dx$ **x)** $\int e^{-2x} \sin 2x \, dx$

2. Evaluate the integrals in following by using a substitution prior to integration by parts.

- **a)** $\int e^{\sqrt{3s+9}} ds$ **b)** $\int_0^1 x\sqrt{1 - x} dx$ **c)** $\int_0^{\pi/3} x \tan^2 x \, dx$
- **d)** $\int \ln(x + x^2) \, dx$ **e)** $\int \sin(\ln x) \, dx$ **f)** $\int z(\ln z)^2 dz$

3. Evaluate the integrals in the following. Some integrals do not require integration by parts.

a) $\int x \sec x^2 \, dx$
b) $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$
c) $\int x(\ln x)^2 dx$
d) $\int \frac{1}{x(\ln x)^2} dx$
e) $\int \frac{\ln x}{x^2} dx$
f) $\int \frac{(\ln x)^3}{x} dx$
g) $\int x^3 e^{x^4} \, dx$
h) $\int x^5 e^{x^3} \, dx$
i) $\int x^3\sqrt{x^2+1} \, dx$
j) $\int x^2 \sin x^3 \, dx$
k) $\int \sin 3x \cdot \cos 2x \, dx$
l) $\int \sin 2x \cdot \cos 4x \, dx$
m) $\int \sqrt{x} \ln x \, dx$
n) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$
o) $\int \cos\sqrt{x} \, dx$
p) $\int \sqrt{x} e^{\sqrt{x}} \, dx$
q) $\int_0^{\pi/2} \theta^2 \sin 2\theta \, d\theta$
r) $\int_0^{\pi/2} x^3 \cos 2x \, dx$
s) $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t \, dt$
t) $\int_0^{1/\sqrt{2}} 2x \tan^{-1}(x^2) \, dx$
u) $\int x \tan^{-1} x \, dx$
v) $\int x^2 \tan^{-1} \frac{x}{2} dx$
w) $\int (1+2x^2)e^{x^2} \, dx$
x) $\int \frac{xe^x}{(x+1)^2} dx$
y) $\int \sqrt{x}(sin^{-1}\sqrt{x}) \, dx$
z) $\int \frac{(sin^{-1}x)^2}{\sqrt{1-x^2}} dx$