Calculus I

First Semester

Lecturer 10

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3.7 Derivatives of Inverse Functions and Logarithms

In Section 1.5 we saw how the inverse of a function undoes, or inverts, the effect of that function. We defined there the natural logarithm function $f^{-1}(x) = ln\, x$ as the inverse of the natural exponential function $f(x) = e^x$. This is one of the most important function inverse pairs in mathematics and science. We learned how to differentiate the exponential function in Section 3.3. Here we develop a rule for differentiating the inverse of a differentiable function, and we apply the rule to find the derivative of the natural logarithm function.

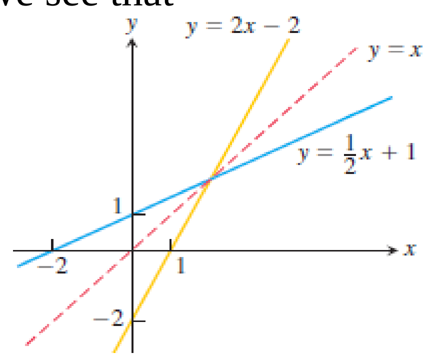
3.7.1 Derivatives of Inverse Functions and Logarithms

Remark:

Take the inverse of the function $f(x) = (1/2)x + 1$ to be $f^{-1}(x) = 2x - 2$ then we calculate their derivatives, we see that

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left(\left(\frac{1}{2}\right)x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx}f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2\ .$$

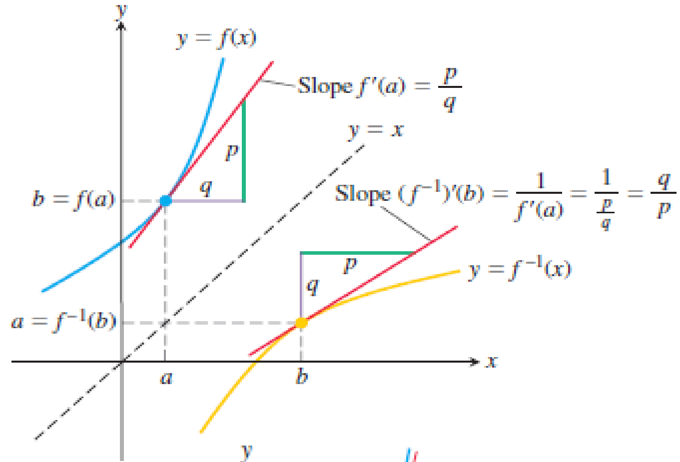


The derivatives are reciprocals of one another, so the slope of one line is the reciprocal of the slope of its inverse line.

This is not a special case. Reflecting any nonhorizontal or nonvertical line across the line $y = x$ always inverts the line's slope. If the original line has slope $m \neq 0$, the reflected line has slope $1/m$.

Theorem(The Derivative Rule for Inverses):

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f at the point $a = f^{-1}(b)$:

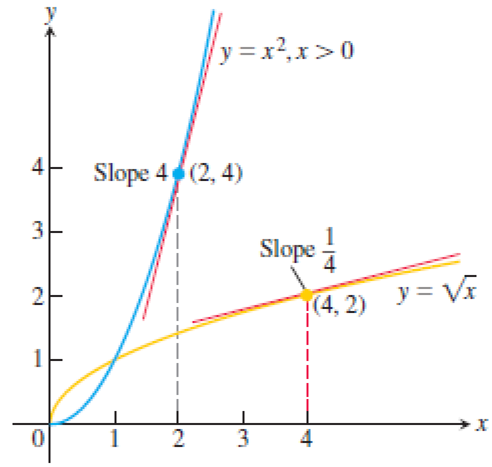


$$(\boldsymbol{f^{-1}})'(\boldsymbol{b}) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Example:



The function $f(x) = x^2, x > 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Let's verify that Theorem (The Derivative Rule for Inverses) gives the same formula for the derivative of $f^{-1}(x)$:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{2(f^{-1}(x))} \qquad \boldsymbol{f'(x) = 2x \text{ with } x \text{ replaced by } f^{-1}(x)}$$

$$= \frac{1}{2(\sqrt{x})} \qquad \boldsymbol{f^{-1}(x) = \sqrt{x}.}$$

Theorem (The Derivative Rule for Inverses) gives a derivative that agrees with the known derivative of the square root function.

Let's examine Theorem (The Derivative Rule for Inverses) at *a* specific point. We pick $x = 2$ (the number a) and $f(2) = 4$ (the value b). Theorem (The Derivative Rule for Inverses) says that the derivative of f at 2, which is $f'(2) = 4$, and the derivative of f^{-1} at $f(2)$, which is $f^{-1}(4)$, are reciprocals. It states that

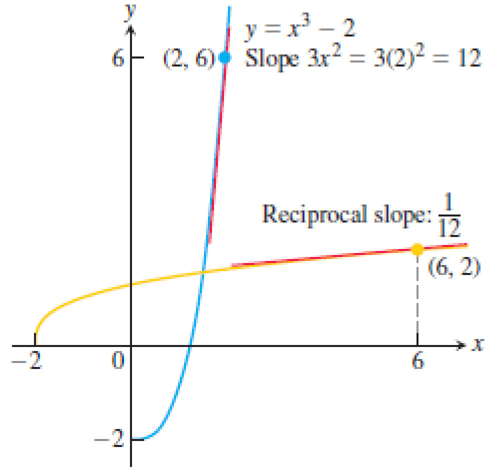
$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(2)} = \left.\frac{1}{2x}\right|_{x=2} = \frac{1}{4}.$$

Remark:

We will use the procedure illustrated in previous Example to calculate formulas for the derivatives of many inverse functions throughout this semester. Equation $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$ sometimes enables us to find specific values of df^{-1}/dx without knowing a formula for f^{-1}.

Example:

Let $f(x) = x^3 - 2, x > 0$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.



Solution:

We apply Theorem (The Derivative Rule for Inverses) to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\left.\frac{df}{dx}\right|_{x=2} = 3x^2\big|_{x=2} = 12.$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=2}} = \frac{1}{12}.$$

3.7.2 Derivative of the Natural Logarithm Function

Since we know that the exponential function $f(x) = e^x$ is differentiable everywhere, we can apply Theorem (The Derivative Rule for Inverses) to find the derivative of its inverse $f^{-1}(x) = ln\, x$:

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$	**Theorem (The Derivative Rule for Inverses)**
$= \frac{1}{e^{f^{-1}(x)}}$	$f'(u) = e^u$
$= \frac{1}{e^{\ln x}}$	$x > 0$
$= \frac{1}{x}$.	**Inverse function relationship**

Instead of applying Theorem (The Derivative Rule for Inverses) directly, we can find the derivative of $y = ln\, x$ using implicit differentiation, as follows:

$y = \ln x$	$x > 0$
$e^y = x$	**Inverse function relationship**
$\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$	**Differentiate implicitly.**
$e^y \frac{dy}{dx} = 1$	**Chain Rule**
$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.	$e^y = x$

No matter which derivation we use, the derivative of y = ln x with respect to x is

$$\frac{d}{dx}\ln x = \frac{1}{x}\,,\ \ x > 0.$$

The Chain Rule extends this formula to positive differentiable functions $u(x)$:

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx},\ \ u > 0.$$

Example:

a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}\cdot 2 = \frac{1}{x}, x > 0.$

b) $\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$

c) $\frac{d}{dx}(\ln x)^4 = 4(\ln x)^3\frac{d}{dx}(\ln x) = 4(\ln x)^3\frac{1}{x} = \frac{4(\ln x)^3}{x}, x > 0.$

d) $\frac{d}{dx}\ln|x| = \frac{d}{du}\ln u\cdot\frac{du}{dx}$ $u = |x|, x \neq 0$

$= \frac{1}{u}\cdot\frac{x}{|x|}$ $\frac{d}{dx}(|x|) = \frac{x}{|x|}$

$= \frac{1}{|x|}\cdot\frac{x}{|x|}$ Substitute for u.

$= \frac{x}{x^2}$

$= \frac{1}{x}.$

So, $1/x$ is the derivative of $ln\,x$ on the domain $x > 0$, and the derivative of $ln(-x)$ on the domain $x < 0$.

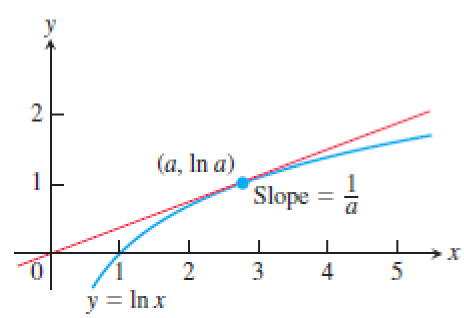
Remark:

Notice from previse Example that the function $y = ln\,2x$ has the same derivative as the function $y = ln\,x$. This is true of $y = ln\,bx$ for any constant b, provided that $bx > 0$:

$$\frac{d}{dx}\ln bx = \frac{1}{bx}\cdot\frac{d}{dx}(bx) = \frac{1}{bx}(b) = \frac{1}{x}.$$

Example:

A line with slope m passes through the origin and is tangent to the graph of $y = ln\,x$. What is the value of m?



Solution:

Suppose the point of tangency occurs at the unknown point $x = a > 0$. Then we know that the point $(a, ln\ a)$ lies on the graph and that the tangent line at that point has slope $m = 1/\ a$. Since the tangent line passes through the origin, its slope is

$$m = \frac{\ln a - 0}{a - 0} = \frac{\ln a}{a}.$$

Setting these two formulas for m equal to each other, we have

$$\frac{\ln a}{a} = \frac{1}{a} \Rightarrow \ln a = 1 \Rightarrow e^{\ln a} = e^1 \Rightarrow a = e \Rightarrow m = \frac{1}{e}.$$

3.7.3 The Derivatives of a^x and $\log_a x$:

We start with the equation $a^x = e^{\ln (a^x)} = e^{x ln\ a}, a > 0$ which was seen in Section 1.5, where it was used to define the function a^x :

$$\begin{aligned} \frac{d}{dx}a^x &= \frac{d}{dx}e^{x ln\ a} \\ &= e^{x ln\ a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx}e^u = e^u \frac{du}{dx} \\ &= a^x \ln a \qquad ln\ a \text{ is a constant.} \end{aligned}$$

That is, if $a > 0$, then a^x is differentiable and

$$\frac{d}{dx}a^x = a^x \ln a.$$

This equation shows why e^x is the preferred exponential function in calculus. If $a = e$, then $ln\ a = 1$ and the derivative of a^x simplifies to

$$\frac{d}{dx}e^x = e^x \ln e = e^x. \qquad (\ln e = 1)$$

If $a > 0$ and *u* is a differentiable function of *x*, then by the Chain Rule, a^u is a differentiable function of *x* and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}.$$

Example:

a) $\frac{d}{dx}3^x = 3^x \ln 3.$

b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3.$

c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

d) $\frac{d}{dx}\sin(3^x) = \cos(3^x)\frac{d}{dx}(3^x) = \cos(3^x) \cdot 3^x \ln 3.$

Remark:

To find the derivative of $\log_a x$ for an arbitrary base ($a > 0, a \neq 1$), we use the change-of-base formula for logarithms to express $\log_a x$ in terms of natural logarithms: $\log_a x = \frac{\ln x}{\ln a}$. Then we take derivatives

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a} = \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x = \frac{1}{\ln a}\cdot\frac{1}{x}.$$

which yields

$$\frac{d}{dx}\log_a x = \frac{1}{\ln a}\cdot\frac{1}{x}, a > 0, a \neq 1.$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives a more general formula:

$$\boldsymbol{\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot\frac{du}{dx}, a > 0, a \neq 1.}$$

Remark:

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called logarithmic differentiation, is illustrated in the next example

Example:

Find dy/dx if $y = \frac{(x^2+1)(x+3)^{1/3}}{x-1}, x > 1$.

Solution:

We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms from Theorem (Algebraic Properties of the Natural Logarithm):

$$\begin{aligned} lny &= ln\frac{(x^2+1)(x+3)^{1/2}}{x-1} \\ &= ln(x^2+1)(x+3)^{1/2} - \ln(x-1) && \textbf{Rule 2} \\ &= ln(x^2+1) + ln(x+3)^{1/2} - \ln(x-1) && \textbf{Rule 1} \\ &= ln(x^2+1) + \tfrac{1}{2}ln(x^2+1) - \ln(x-1) && \textbf{Rule 4} \end{aligned}$$

We then take derivatives of both sides with respect to x

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next, we solve for $dy/\,dx$:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2+1)(x+3)^{1/3}}{x-1}\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right).$$

Remark:

The computation in previous Example would be much longer if we used the product, quotient, and power rules.

Exercises:

1. In following:
- **a)** Find $f^{-1}(x)$.
- **b)** Graph f and f^{-1} together.
- **c)** Evaluate df/dx at $x = a$ and df^{-1}/dx at $x = f(a)$ to show that $(df^{-1}/dx)|_{x=f(a)} = 1/(df/dx)|_{x=a}$.

I) $f(x) = 2x + 3,\ a = -1$ **II)** $f(x) = \frac{x+2}{1-x},\ a = \frac{1}{2}$

III) $f(x) = 5 - 4x,\ a = \frac{1}{2}$ **IV)** $f(x) = 2x^2, x \geq 0, a = 5$

2.
- **a)** Show that $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses of one another.
- **a)** Graph f and g over an x-interval large enough to show the graphs intersecting at $(1,1)$ and $(-1,-1)$. Be sure the picture shows the required symmetry about the line $y = x$.
- **a)** Find the slopes of the tangent lines to the graphs of f and g at $(1,1)$ and $(-1,-1)$ (four tangent lines in all).
- **b)** What lines are tangent to the curves at the origin?

3.
- **a)** Show that $h(x) = x^3/4$ and $k(x) = (4x)^{1/3}$ are inverses of one another.
- **b)** Graph h and k over an x-interval large enough to show the graphs intersecting at $(2, 2)$ and $(-2, -2)$. Be sure the picture shows the required symmetry about the line $y = x$.
- **c)** Find the slopes of the tangent lines to the graphs at h and k at $(2, 2)$ and $(-2, -2)$.
- **d)** What lines are tangent to the curves at the origin?

4. Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of df^{-1}/dx at the point $x = -1 = f(3)$.

5. Let $f(x) = x^2 - 4x - 5, x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

6. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(2, 4)$ and has a slope of $1/3$ there. Find the value of df^{-1}/dx at $x = 4$.

7. Suppose that the differentiable function $y = g(x)$ has an inverse and that the graph of g passes through the origin with slope 2. Find the slope of the graph of g^{-1} at the origin.

8. In following, find the derivative of y with respect to x, t, or θ ,as appropriate.

a) $y = ln(t^2)$ **b)** $y = ln\left(t^{3/2}\right) + \sqrt{t}$ **c)** $y = \ln(sinx)$

d) $y = (\cos\theta)\ln(2\theta + 2)$ **e)** $y = (\ln x)^3$ **f)** $y = t\sqrt{t}$

g) $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$ **h)** $y = (x^2 \ln x)^4$ **i)** $y =$

j) $y = \frac{t}{\sqrt{\ln t}}$ **k)** $y = \frac{x\ln x}{1+\ln x}$ **l)** $y =$

m) $y = \ln(\ln(\ln x))$ **n)** $y = \theta(\sin(\ln\theta) + \cos(\ln\theta))$ **o)** $y = \ln(\sec\theta + \tan\theta)$

p) $y = \ln\frac{1}{x\sqrt{x+1}}$ **q)** $y = \frac{1}{2}\ln\frac{1+x}{1-x}$ **r)** $y = \frac{1+\ln t}{1-\ln t}$

s) $y = \sqrt{\ln\sqrt{t}}$ **t)** $y = \ln(\sec(\ln\theta))$ **u)** $y = \ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$

v) $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right)$ **w)** $y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}}$

9. In following, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

a) $y = \sqrt{x(x+1)}$ **b)** $y = \sqrt{(x^2+1)(x-1)^2}$ **c)** $y = \sqrt{\frac{t}{t+1}}$

d) $y = \sqrt{\frac{1}{t(t+1)}}$ **e)** $y = (\tan\theta)\sqrt{2\theta+1}$ **f)** $y = \frac{1}{t(t+1)(t+2)}$

g) $y = \frac{\theta+5}{\theta\cos\theta}$ **h)** $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}$ **i)** $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

j) $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$ **k)** $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}}$ **l)** $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$

10. In following, find dy/dx.

a) $ln\, y = e^y sin\, x$ **b)** $ln\, xy = e^{x+y}$

c) $x^y = y^x$ **d)** $tan\, y = e^x + ln\, x$

11. In following, find the derivative of y with respect to the given independent variable.

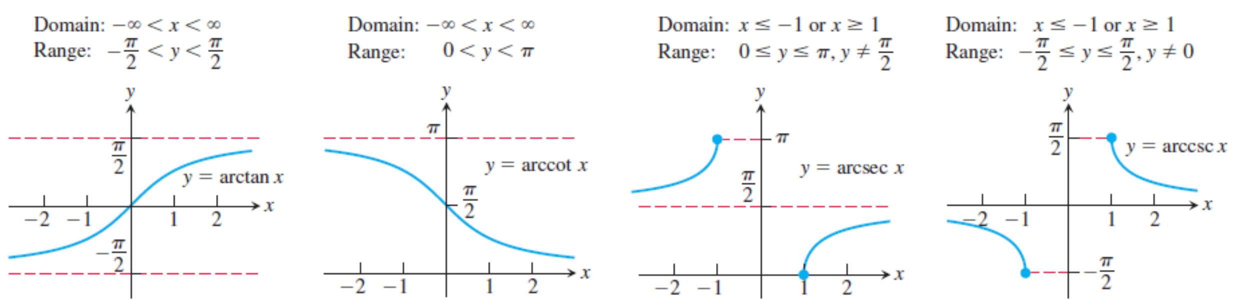
a) $y = 2^x$
b) $y = 2^{(s^2)}$
c) $y = t^{1-e}$
d) $y = \log_3(1 + \theta \ln 3)$
e) $y = \log_4 x + \log_4 x^2$
f) $y = \log_{25} e^x - \log_5 \sqrt{x}$
g) $y = \log_3 r \cdot \log_9 r$
h) $y = \log_3((\frac{x+1}{x-1})^{\ln 3})$
i) $y = \log_5 \sqrt{(\frac{7x}{3x+2})^{\ln 5}}$
j) $y = \theta \sin(\log_7 \theta)$
k) $y = \log_7(\frac{\sin\theta \cos\theta}{e^{\theta} 2^{\theta}})$
l) $y = \log_5 e^x$
m) $y = \log_2(\frac{x^2 e^2}{2\sqrt{x+1}})$
n) $y = 3^{\log_2 t}$
o) $y = 3\log_8(\log_2 t)$
p) $y = t \log_3(e^{(\sin t)(\ln 3)})$

3.8 Inverse Trigonometric Functions

We introduced the six basic inverse trigonometric functions in Section 1.5 but focused there on the arcsine and arccosine functions. Here we complete the study of how all six basic inverse trigonometric functions are defined, graphed, and evaluated, and how their derivatives are computed

3.8.1 Inverses of tan *x*, cot *x*, sec *x*, and csc *x*

The graphs of these four basic inverse trigonometric functions are shown in following Figure.



We obtain these graphs by reflecting the graphs of the restricted trigonometric functions (as discussed in Section 1.5) through the line $y = x$. Let's take a closer look at the arctangent, arccotangent, arcsecant, and arccosecant functions.

The arctangent of *x* is a radian angle whose tangent is *x*. The arccotangent of *x* is an angle whose cotangent is *x*, and so forth. The angles belong to the restricted domains of the tangent, cotangent, secant, and cosecant functions.

Definition:

$y = \arctan x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \text{arccot } x$ is the number in $(0, \pi)$ for which $\cot y = x$.

y = arcsec x is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

y = arccsc x is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

Remark:

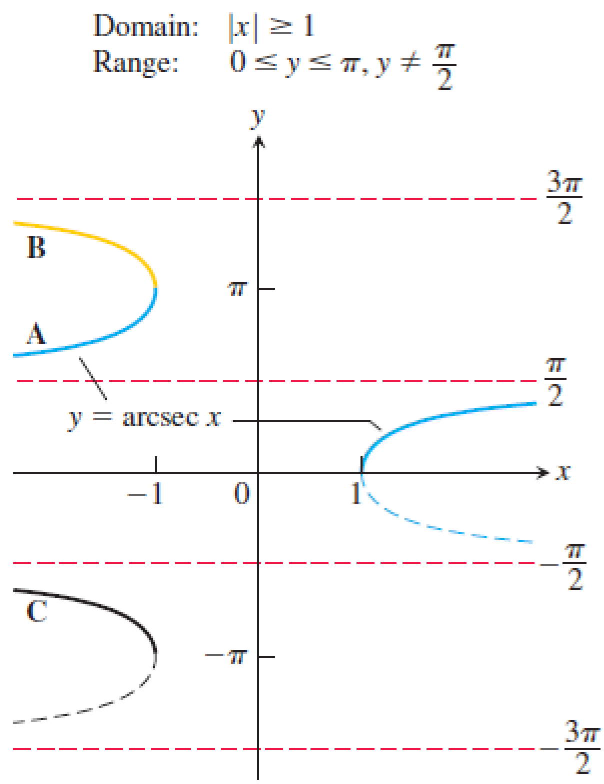
We use open or half-open intervals to avoid values for which the tangent, cotangent, secant, and cosecant functions are undefined.

As we discussed in Section 1.5, the arcsine and arccosine functions are often written as $sin^{-1}x$ and $cos^{-1}x$ instead of arcsin x and arccos x. Likewise, we often denote the other inverse trigonometric functions by $tan^{-1}x$, $cot^{-1}x$, $sec^{-1}x$, and $csc^{-1}x$.

The graph of $y = arctan\, x$ is symmetric about the origin because it is a branch of the graph $x = tan\, y$ that is symmetric about the origin. Algebraically this means that $\boldsymbol{arctan(-x) = -arctan\, x}$; the arctangent is an odd function. The graph of $y = arccot\, x$ has no such symmetry. Notice that the graph of the arctangent function has two horizontal asymptotes: one at $y = \pi/2$ and the other at $y = -\pi/2$.

Remark:

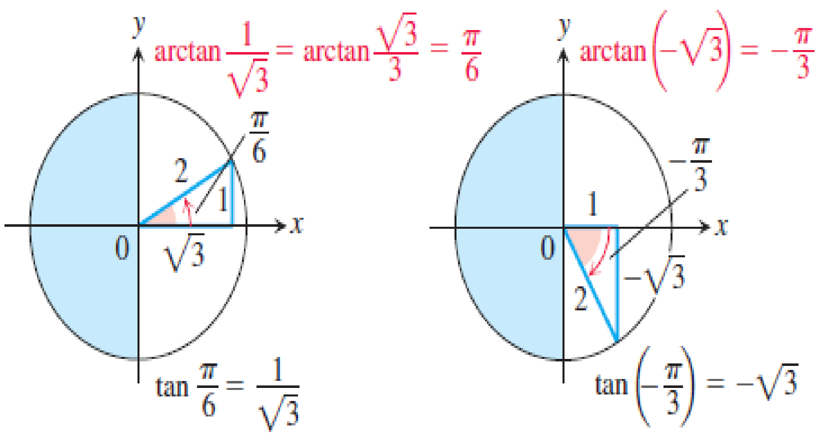
There is no general agreement about how to define $arcsec\, x$ for negative values of *x*. We chose angles in the second quadrant between $\pi/2$ and π. This choice makes $arcsec\, x = arccos(1/x)$. It also makes $arcsec\, x$ an increasing function on each interval of its domain. Some tables choose arcsec x to lie in $[-\pi, -\pi/2)$ for $x < 0$, and some texts choose it to lie in $[\pi, 3\pi/2)$. These choices simplify the formula for the derivative (our formula needs absolute value signs) but fail to satisfy the computational equation $arcsec\, x = arccos(1/x)$. From this, we can derive the identity



$$arcsec\, x = arccos\left(\frac{1}{x}\right) = \frac{\pi}{2} - \arcsin\left(\frac{1}{x}\right).$$

Example:

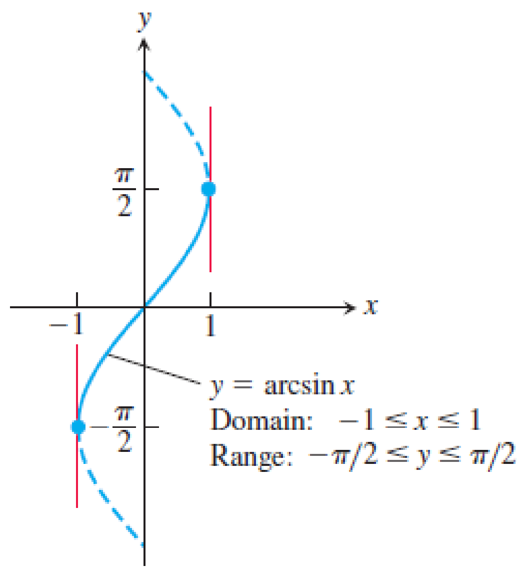
The accompanying figures show two values of $arctan\ x$.



x	$\arctan x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
0	0
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

The angles come from the first and fourth quadrants because the range of arctan x is $(-\pi/2, \pi/2)$.

3.8.2 The Derivative of $y = arcsin\ u$



We know that the function $x = sin\ y$ is differentiable in the interval $-\pi/2 < y < \pi/2$ and that its derivative, the cosine, is positive there. Therefore assures us that the inverse function $y = arcsin\ x$ is differentiable throughout the interval $-1 < x < 1$. We cannot expect it to be differentiable at $x = 1$ or $x = -1$ because the tangent lines to the graph are vertical at these points.

We find the derivative of $y = arcsin\ x$ by applying Theorem (The Derivative Rule for Inverses) with $f(x) = sin\ x$ and $f^{-1}(x) = arcsin\ x$:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \textbf{Theorem (The Derivative Rule for Inverses)}$$

$$= \frac{1}{\cos(\arcsin x)} \qquad f'(y) = \cos y$$

$$= \frac{1}{\sqrt{1 - sin^2(\arcsin x)}} \qquad \cos y = \sqrt{1 - sin^2 y}$$

$$= \frac{1}{\sqrt{1 - x^2}} \qquad sin(\arcsin x) = x$$

For x < 1, $\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$.

If u is a differentiable function of x with $u < 1$, we apply the Chain Rule to get the general formula

$$\frac{d}{dx}(\arcsin u)=\frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx},\ |u| < 1 .$$

Example:

Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\arcsin x^2) = \frac{1}{\sqrt{1-(x^2)^2}}\cdot\frac{d}{dx}(x^2)=\frac{2x}{\sqrt{1-x^4}} .$$

3.8.3 The Derivative of $y = arctan\, u$

We find the derivative of $y = arctan\, x$ by applying Theorem (The Derivative Rule for Inverses) with $f(x) = tan\, x$ and $f^{-1}(x) = arctan\, x$. Theorem (The Derivative Rule for Inverses) can be applied because the derivative of $tan\, x$ is positive for $-\pi/2 < y < \pi/2$:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \text{Theorem (The Derivative Rule for Inverses)}$$

$$= \frac{1}{\sec^2(\arctan x)} \quad f'(u) = \sec^2 u$$

$$= \frac{1}{\sqrt{1+tan^2(\arctan x)}} \quad \sec^2 u = 1 + tan^2 u$$

$$= \frac{1}{1+x^2} \quad tan(\arctan x) = x$$

The derivative is defined for all real numbers:

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\cdot\frac{du}{dx} .$$

The Chain Rule can also be combined with the arctangent function in other ways, as illustrated by the following example

Example:

$$\frac{d}{dx}\left(\frac{1}{\arctan x}\right) = \frac{d}{dx}(\arctan x)^{-1}$$

Derivative of the reciprocal (not the inverse) of arctangent
Apply the Chain Rule.

$$= (-1)(\arctan x)^{-2}\frac{d}{dx}(\arctan x)$$

$$= \frac{-1}{(\arctan x)^2}\cdot\frac{1}{1+x^2}$$

3.8.4 The Derivative of $y = arcsec\ u$

Theorem (The Derivative Rule for Inverses) does not apply to the function $sec\ x$ directly, since its domain is not connected. However, we can apply Theorem to each of the two intervals in its domain to see that the inverse of the one-to-one function $sec\ x$ is indeed differentiable. The formula for the derivative of $arcsec\ x$ on its domain $|x| > 1$ can then be found by using implicit differentiation and the Chain Rule as follows:

$$y = arcsec\ x$$

$$sec\ y = x \qquad \textbf{Inverse function relationship}$$

$$\frac{d}{dx}(sec\ y) = \frac{d}{dx}x \qquad \textbf{Differentiate both sides.}$$

$$(sec\ y \cdot \tan y)\frac{dy}{dx} = 1 \qquad \textbf{Chain Rule}$$

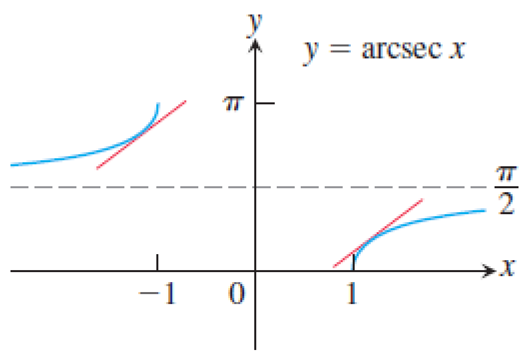
$$\frac{dy}{dx} = \frac{1}{sec\ y \cdot \tan y}$$

Since $|x| > 1$ و y lies in $(0, \pi/2) \cup (\pi/2, \pi)$and $sec\ y \cdot tany \neq 0$.

To express the result in terms of x, we use the relationships $sec\ y = x$ and $tan\ y = \pm\sqrt{sec^2 y - 1} = \pm\sqrt{x^2 - 1}$ to get

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Can we do anything about the ± sign? A glance at the slope of the graph $y = arcsec\ x$ is always positive. Thus,



$$\frac{d}{dx}arcsec\ x = \begin{cases} +\frac{1}{x\sqrt{x^2-1}} & if\ x > 1 \\ -\frac{1}{x\sqrt{x^2-1}} & if\ x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the "±" ambiguity:

$$\frac{d}{dx}(arcsec\ x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1.$$

If u is a differentiable function of x with $u > 1$, we have the formula

$$\frac{d}{dx}(arcsec\ u) = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}, \quad |u| > 1.$$

Example:

$$\frac{d}{dx}(arcsec\ 5x^4) = \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}} \cdot \frac{d}{dx}(5x^4)$$

$$= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \qquad \mathbf{5x^4 > 1}$$

$$= \frac{4}{x\sqrt{25x^8 - 1}}.$$

3.8.5 Derivatives of the Other Three Inverse Trigonometric Functions:

We could use the same techniques to find the derivatives of the other three inverse trigonometric functions—arccosine, arccotangent, and arccosecant—but there is an easier way, thanks to the following identities.

Remark(Inverse Function–Inverse Cofunction Identities):

$$\arccos x = \pi/2 - \arcsin x$$
$$\text{arccot}\, x = \pi/2 - \arctan x$$
$$\text{arccsc}\, x = \pi/2 - \text{arcsec}\, x$$

We saw the first of these identities in Section 1.5. The others are derived in a similar way. It follows easily that the derivatives of the inverse cofunctions are the negatives of the derivatives of the corresponding inverse functions. For example, the derivative of $arccos\ x$ is calculated as follows:

$$\frac{d}{dx}(\arccos x) = \frac{d}{dx}(\pi/2 - \arcsin x) \qquad \textbf{Identity}$$

$$= -\frac{d}{dx}(\arcsin x)$$

$$= -\frac{1}{\sqrt{1-x^2}}. \qquad \textbf{Derivative of arcsine}$$

The derivatives of the inverse trigonometric functions are summarized in following Table.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad (|x| < 1) \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \quad (|x| < 1)$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\text{arccot}\, x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(arcsec\ x) = \frac{1}{|x|\sqrt{x^2 - 1}} \quad (|x| > 1) \qquad \frac{d}{dx}(arccsc\ x) = -\frac{1}{|x|\sqrt{x^2 - 1}} \quad (|x| > 1)$$

Exercises:

1. Use reference triangles in an appropriate quadrant, to find the angles in following:

1. **a)** $\tan^{-1} 1$ **b)** $\arctan(-\sqrt{3})$ **c)** $\tan^{-1}(\frac{1}{\sqrt{3}})$

2. **a)** $\arctan(-1)$ **b)** $\tan^{-1}\sqrt{3}$ **c)** $\tan^{-1}(\frac{-1}{\sqrt{3}})$

3. **a)** $\sin^{-1}(\frac{-1}{2})$ **b)** $\sin^{-1}(\frac{1}{\sqrt{2}})$ **c)** $\arcsin(\frac{-\sqrt{3}}{2})$

4. **a)** $\sin^{-1}(\frac{1}{2})$ **b)** $\arcsin(\frac{-1}{\sqrt{2}})$ **c)** $\sin^{-1}(\frac{\sqrt{3}}{2})$

5. **a)** $\cos^{-1}(\frac{1}{2})$ **b)** $\cos^{-1}(\frac{-1}{\sqrt{2}})$ **c)** $\arccos(\frac{\sqrt{3}}{2})$

6. **a)** $\csc^{-1}(\sqrt{2})$ **b)** $\text{arccsc}(\frac{-2}{\sqrt{3}})$ **c)** $\csc^{-1} 2$

7. **a)** $\text{arcsec}(-\sqrt{2})$ **b)** $\sec^{-1}(\frac{2}{\sqrt{3}})$ **c)** $\sec^{-1}(-2)$

8. **a)** $\cot^{-1}(-1)$ **b)** $\text{arccot}(\sqrt{3})$ **c)** $\cot^{-1}(\frac{-1}{\sqrt{3}})$

2. Find the values in following.

a) $sin\left(cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ **b)** $sec\left(arccos\left(\frac{1}{2}\right)\right)$

c) $tan\left(arcsin\left(-\frac{1}{2}\right)\right)$ **d)** $cot\left(sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

3. Find the limits in following. (If in doubt, look at the function's graph.)

a) $\lim_{x\to 1^-} \sin^{-1}x$ **b)** $\lim_{x\to 1^+} \cos^{-1}x$ **c)** $\lim_{x\to\infty} \tan^{-1}x$

d) $\lim_{x\to-\infty} \tan^{-1}x$ **e)** $\lim_{x\to 1^-} \lim_{x\to\infty} \sec^{-1}x$ **f)** $\lim_{x\to-\infty} \sec^{-1}x$

g) $\lim_{x\to\infty} \csc^{-1}x$ **h)** $\lim_{x\to-\infty} \csc^{-1}x$

4. In following, find the derivative of y with respect to the appropriate variable.

a) $y = cos^{-1}(x^2)$ **b)** $y = cos^{-1}(1/x)$ **c)** $y = \arcsin\sqrt{2}t$

d) $y = sin^{-1}(1-t)$ **e)** $y = sec^{-1}(2s+1)$ **f)** $y = sec^{-1}5s$

g) $y = csc^{-1}(x^2+1), x > 0$ **h)** $y = csc^{-1}\frac{x}{2}$

i) $y = \arcsin\frac{3}{t^2}$ **j)** $y = sec^{-1}\frac{1}{t}, 0 < t < 1$ **k)** $y = \text{arccot}\sqrt{t}$

l) $y = \cot^{-1}\sqrt{t-1}$ **m)** $y = \ln(\tan^{-1}x)$ **n)** $y = \tan^{-1}(ln\, x)$

o) $y = \text{arccsc}(e^t)$ **p)** $y = \sqrt{\arcsin x}$ **q)** $y = e^{\text{arcsec}\, x}$

r) $y = \arccos(e^{-t})$ **s)** $y = s\sqrt{1-s^2} + cos^{-1}s$ **t)** $y = \sqrt{s^2-1} - sec^{-1}s$

u) $y = tan^{-1}\sqrt{x^2 - 1} - csc^{-1}x, x > 1$

v) $y = cot^{-1}\frac{1}{x} - tan^{-1}x$

w) $y = x \arcsin x + \sqrt{1 - x^2}$

x) $y = \frac{x}{1 + \arctan x}$

y) $y = \ln(x^2 + 4) - x \arctan(\frac{x}{2})$

z) $y = (\text{arccot}(x^3))^3$

aa) $y = \cos(x - \arccos x)$

bb) $y = \log_2 \text{arccsc}\sqrt{x}$

5. For following use implicit differentiation to find $\frac{dy}{dx}$ at the given point P.

a) $3 \arctan x + \arcsin y = \frac{\pi}{4}; P(1, -1).$

b) $\arcsin(x + y) + \arccos(x - y) = \frac{5\pi}{6}; P\left(0, \frac{1}{2}\right).$

c) $ycos^{-1}(xy) = \frac{-3\sqrt{2}}{4}\pi; P\left(\frac{1}{2}, -\sqrt{2}\right).$

d) $16(tan^{-1}3y)^2 + 9(tan^{-1}2x)^2 = 2\pi; P\left(\frac{\sqrt{3}}{2}, \frac{1}{3}\right).$

Calculus I

First Semester

Lecturer 11

Dr. Ban Jaffar AL-Taiy

Taghreed Hussein Abed

3.9 Extreme Values of Functions on Closed Intervals

This section shows how to locate and identify extreme (maximum or minimum) values of a function from its derivative.

Definition:

Let f be a function with domain D. Then f has an ***absolute maximum*** value on D at a point c if

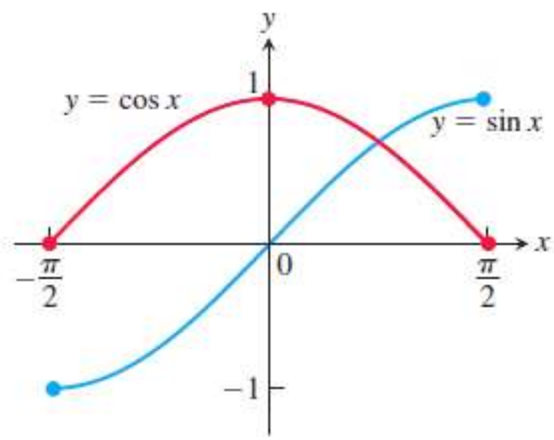
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an ***absolute minimum*** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Remark:

1. Maximum and minimum values are called **extreme values** of the function f. Absolute maxima or minima are also referred to as ***global maxima or minima***. For example, on the closed interval $[-\pi/2, \ \pi/2]$ the function $f(x) = cos\ x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice).



On the same interval, the function $g(x) = sin\ x$ takes on a maximum value of 1 and a minimum value of -1.

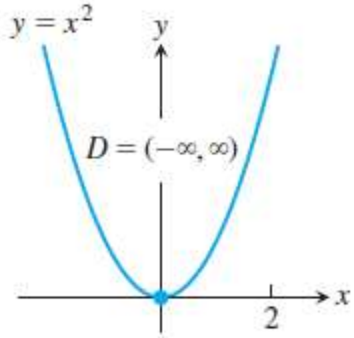
2. Functions defined by the same equation or formula can have different extrema (maximum or minimum values), depending on the domain. A function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint. We see this in the following example.

Example:

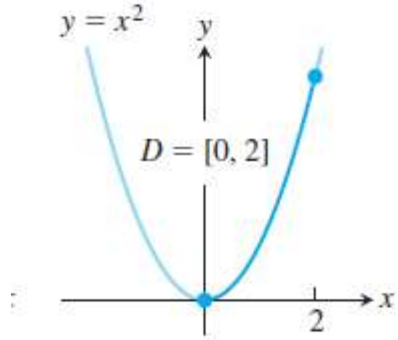
Find the absolute maximum and minimum values of $y = x^2$ on the intervals **a)** $(-\infty, \infty)$, **b)** $[0,2]$, **c)** $(0,2]$, **d)** $(0,2)$.

Solution:

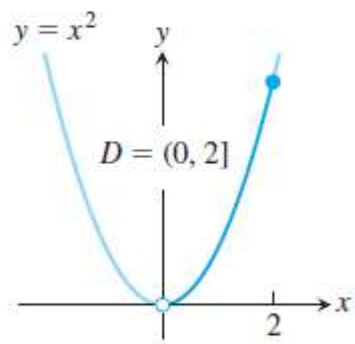
a) On $(-\infty, \infty)$, $y = x^2$ has no absolute maximum and has absolute minimum of 0 at $x = 0$.



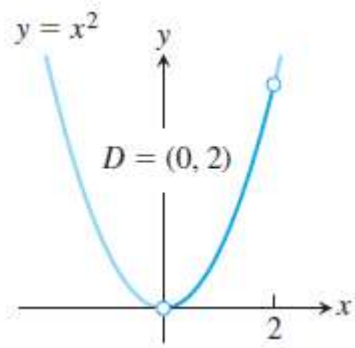
b) On $[0,2]$, $y = x^2$ has absolute maximum of 4 at $x = 2$ and absolute minimum of 0 at $x = 0$.



c) On $(0,2]$, $y = x^2$ has absolute maximum of 4 at $x = 2$ and has no absolute minimum.



d) On $(0,2)$, $y = x^2$ has no absolute extrema



Remark:

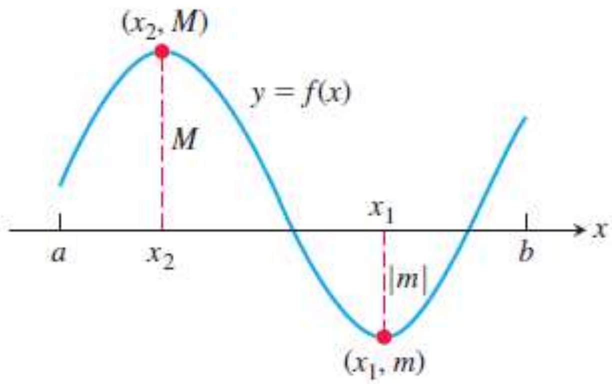
The following theorem asserts that a function which is continuous over (or on) a finite closed interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We look for these extreme values when we graph a function.

Theorem (The Extreme Value Theorem):

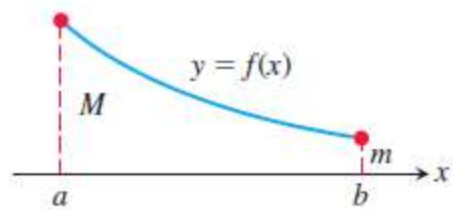
If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \le f(x) \le M$ for every other x in $[a, b]$.

Remark:

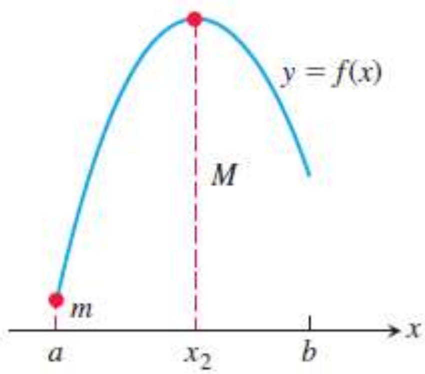
The following Figures illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$.



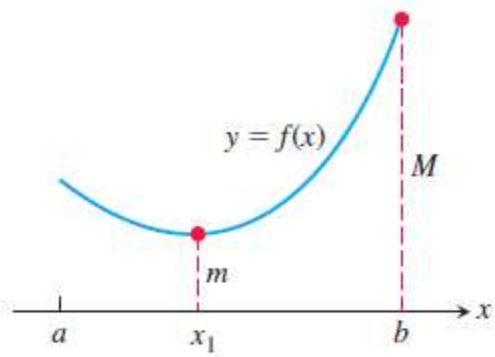
Maximum and minimum
at interior points



Maximum and minimum
at endpoints



Maximum at interior point,
minimum at endpoint



Minimum at interior point,
maximum at endpoint

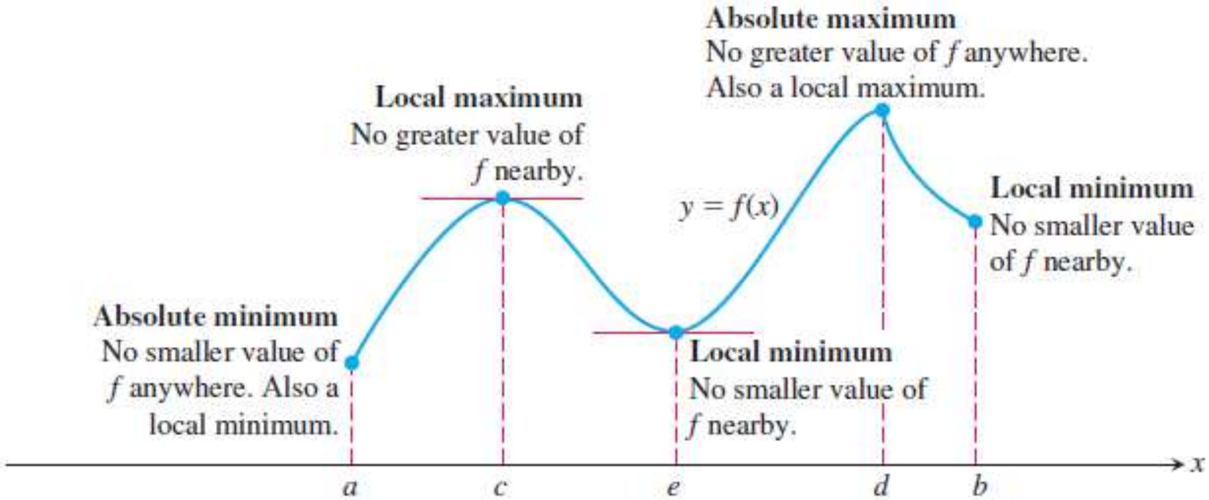
Definition:

A function f has a ***local maximum value*** at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum value** at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Remark:

The following figure shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point nearby. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d. The function f has local maxima at c and d and local minima at a, e, and b. We now define what we mean by local extrema.



Remark:

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

Theorem (The First Derivative Theorem for Local Extreme Values):

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Definition:

An interior point of the domain of a function f where f is zero or undefined is a ***critical point*** of f.

Remark:

To Find the Absolute Extrema of a Continuous Function f on a finite closed interval we use the following steps:

1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

Example:

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2,1]$.

Solution:

The function is differentiable over its entire domain, so the only critical point occurs where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

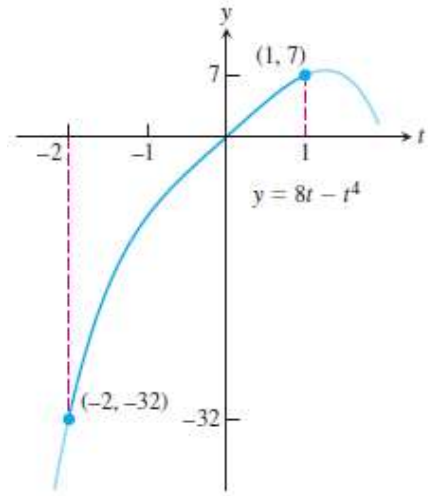
Endpoint values: $f(-2) = 4$

$f(1) = 1.$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Example:

Find the absolute maximum and minimum values of $g(t) = 8t - t^4$ on [-2,1].



Solution:

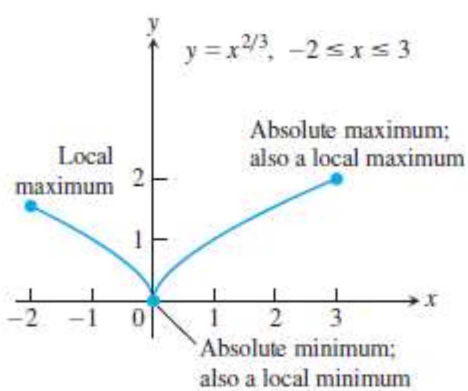
The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0 \quad or \quad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum).

Example:

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on $[-2,3]$.



Solution:

We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

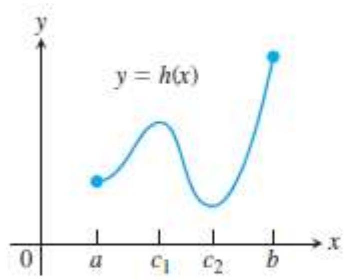
$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} = 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp.

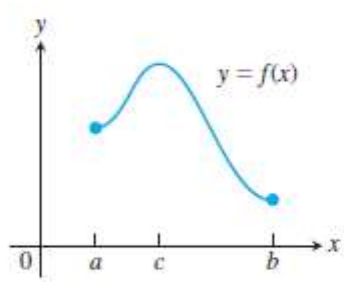
Exercises:

1. In following, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with The Extreme Value Theorem.

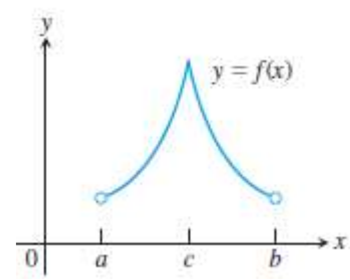
a)



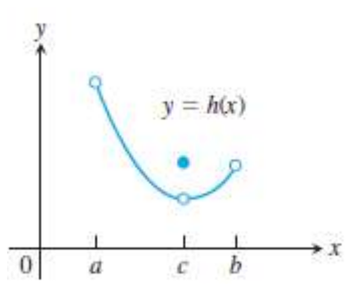
b)



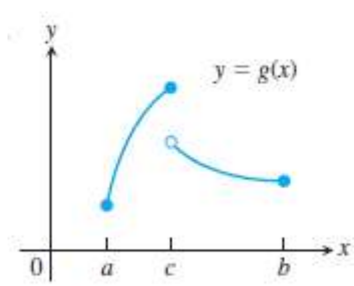
c)



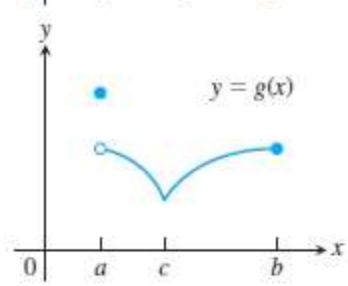
d)



e)

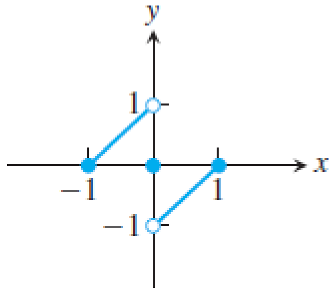


f)

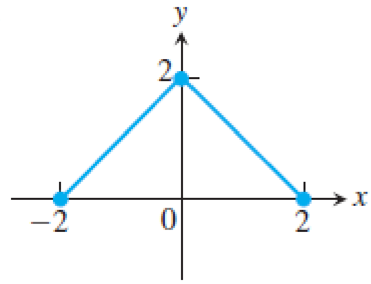


2. In following, find the absolute extreme values and where they occur.

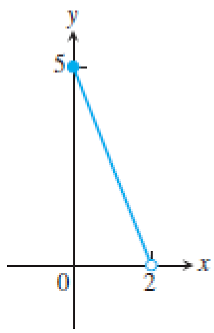
a)



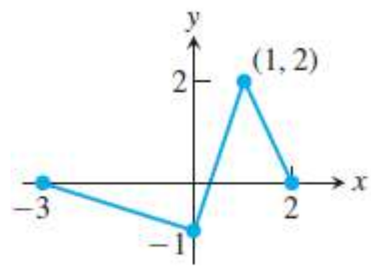
b)



c)



d)



e)

3. In following, match the table with a graph

a)

x	$f'(x)$
a	0
b	0
c	5

b)

x	$f'(x)$
a	0
b	0
c	-5

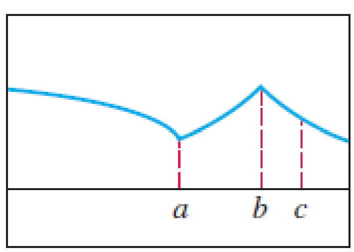
c)

x	$f'(x)$
a	does not exist
b	0
c	-2

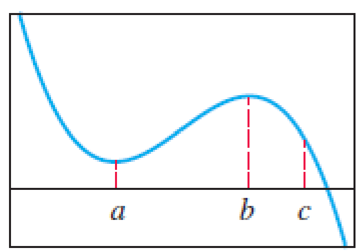
d)

x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7

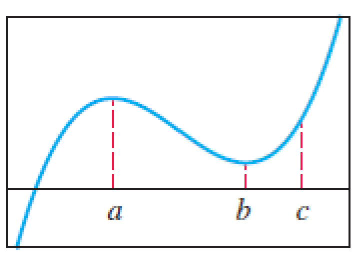
I)



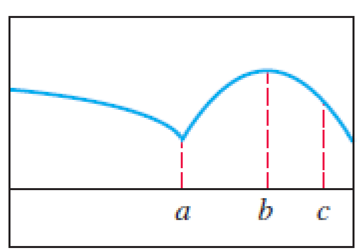
II)



III



IV



4. In following, sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with The Extreme Value Theorem.

a) $f(x) = |x|, -1 < x < 2.$ **b)** $y = 2-x^2, -1 < x < 1.$

c) $g(x) = \begin{cases} -x & 0 \le x < 1 \\ x-1 & 1 \le x \le 2 \end{cases}.$ **d)** $h(x) = \begin{cases} \frac{1}{x} & -1 \le x < 0 \\ \sqrt{x} & 0 \le x \le 4 \end{cases}.$

e) $y = 3\sin x, \quad 0 < x < 2\pi.$ **f)** $f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ \cos x, & 0 < x \le \frac{\pi}{2} \end{cases}.$

5. In following, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

a) $f(x) = x^2 - 1, \ -1 \le x \le 2.$ **b)** $f(x) = 4 - x^3, \ -2 \le x \le 1.$

c) $F(x) = -\frac{1}{x^2}. 0.5 \le x \le 2.$ **d)** $F(x) = -\frac{1}{x}. -2 \le x \le -1.$

e) $h(x) = \sqrt[3]{x}, -1 \le x \le 8.$ **f)** $h(x) = -3x^{2/3}, -1 \le x \le 1.$

g) $g(x) = \sqrt{4 - x^2}, -2 \le x \le 1.$ **h)** $g(x) = -\sqrt{5 - x^2}, -\sqrt{5} \le x \le 0.$

i) $f(\theta) = \sin\theta, -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}.$ **j)** $f(\theta) = \tan\theta, -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}.$

k) $g(x) = \csc x, \frac{\pi}{3} \le x \le \frac{2\pi}{3}.$ **l)** $g(x) = \sec x, -\frac{\pi}{3} \le x \le \frac{\pi}{6}.$

m) $f(t) = 2 - |t|, -1 \le t \le 3.$ **n)** $f(t) = |t - 5|, 4 \le t \le 7.$

6. In following, find the function's absolute maximum and minimum values and say where they occur.

a) $f(x) = x^{4/3}, -1 \le x \le 8.$ **b)** $f(x) = x^{5/3}, -1 \le x \le 8.$

c) $g(\theta) = \theta^{3/5}, -32 \le \theta \le 1.$ **d)** $h(\theta) = 3\theta^{2/3}, -27 \le \theta \le 8.$

7. In following, determine all critical points for each function.

a) $y = x^2 - 6x + 7$ **b)** $f(x) = 6x^2 - x^3$ **c)** $f(x) = x(4-x)^3$

d) $g(x) = (x-1)^2(x-3)^2$ **e)** $y = x^2 + \frac{2}{x}$ **f)** $f(x) = \frac{x^2}{x-2}$

g) $y = x^2 - 32\sqrt{x}$ **h)** $g(x) = \sqrt{2x - x^2}$ **i)** $y = x - 3x^{2/3}$

j) $y = x^3 + 3x^2 - 24x + 7$

8. In following, find the critical points and domain endpoints for each function. Then find the value of the function at each of these points and identify extreme values (absolute and local).

a) $y = x^{2/3}(x+2)$

b) $y = x^{2/3}(x^2-4)$

c) $y = x\sqrt{4-x^2}$

d) $y = x^2\sqrt{3-x}$

e) $y = \begin{cases} 4-2x & x \le 1 \\ x+1 & x > 1 \end{cases}$

f) $y = \begin{cases} 3-x & x < 0 \\ 3+2x & x \ge 0 \end{cases}$

g) $y = \begin{cases} -x^2-2x+4 & x \le 1 \\ -x^2+6x-4 & x > 1 \end{cases}$

h) $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4} & x \le 1 \\ x^3 - 6x^2 + 8x & x > 1 \end{cases}$

9. In following, give reasons for your answers.

I. Let $f(x) = (x-2)^{2/3}$.

a) Does $f'(2)$ exist?

b) Show that the only local extreme value of f occurs at $x = 2$.

c) Does the result in part (b) contradict the Extreme Value Theorem?

d) Repeat parts (a) and (b) for $f(x) = (x-a)^{2/3}$, replacing 2 by a.

II. Let $f(x) = |x^3 - 9x|$.

a) Does $f'(0)$ exist?

b) Does $f'(3)$ exist?

c) Does $f'(-3)$ exist?

d) Determine all extrema of f.

10. In following, show that the function has neither an absolute minimum nor an absolute maximum on its natural domain

a) $y = x^{11} + x^3 + x - 5$.

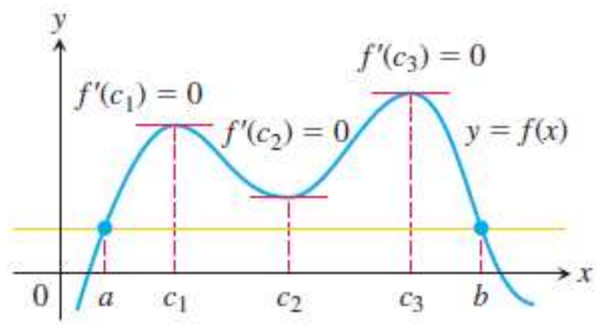
b) $y = 3x + \tan x$.

3.10 The Mean Value Theorem:

We know that constant functions have zero derivatives, but could there be a more complicated function whose derivative is always zero? If two functions have identical derivatives over an interval, how are the functions related? We answer these and other questions by applying the Mean Value Theorem. First, we introduce a special case, known as Rolle's Theorem, which is used to prove the Mean Value Theorem.

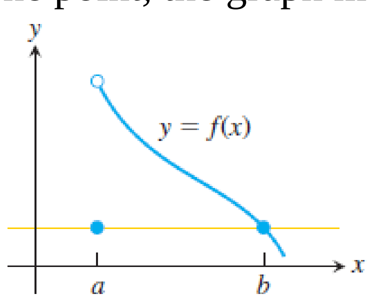
Theorem (Rolle's Theorem):

Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

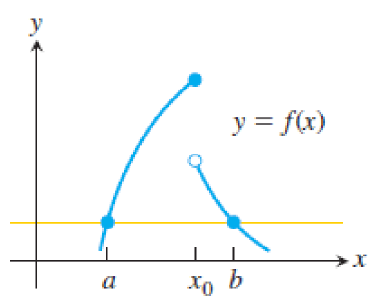


Remark:

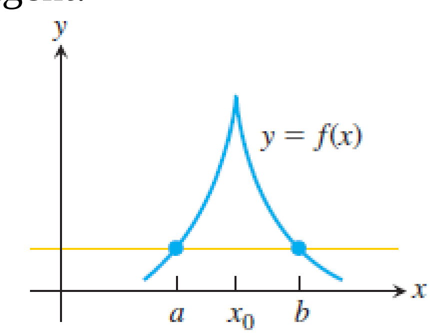
The hypotheses of Rolle's Theorem are essential. If they fail at even one point, the graph may not have a horizontal tangent.



Discontinuous at an endpoint of $[a, b]$



Discontinuous at an interior point of $[a, b]$



Continuous on [a, b] but not differentiable at an interior point

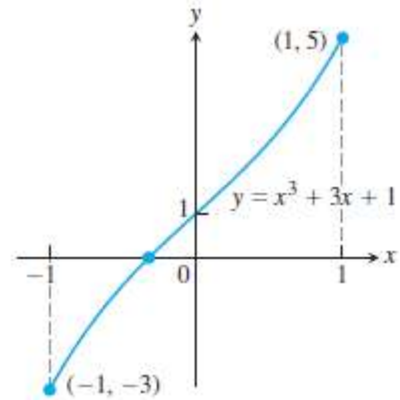
Example:

Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Solution:

We define the continuous function $f(x) = x^3 + 3x + 1$. Since

$f(-1) = -3$ and $f(0) = 1$, the Intermediate Value Theorem tells us that the graph of f crosses the x-axis somewhere in the open interval (-1, 0). Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. However, the derivative

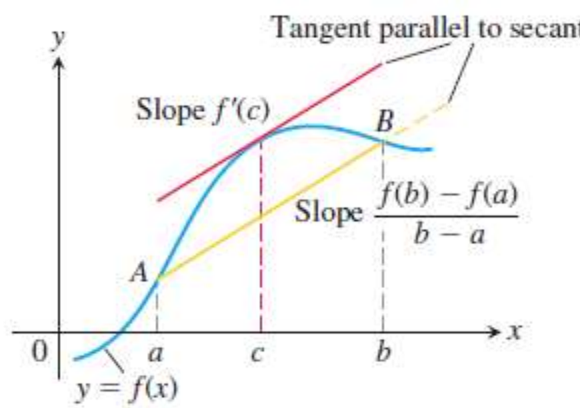


$$f'(x) = 3x^2 + 3,$$

is never zero (because it is always positive). Therefore, f has no more than one zero.

Remark:

Our main use of Rolle's Theorem is in proving the Mean Value Theorem. The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem. The Mean Value Theorem guarantees that there is a point where the tangent line is parallel to the secant line that joins A and B.



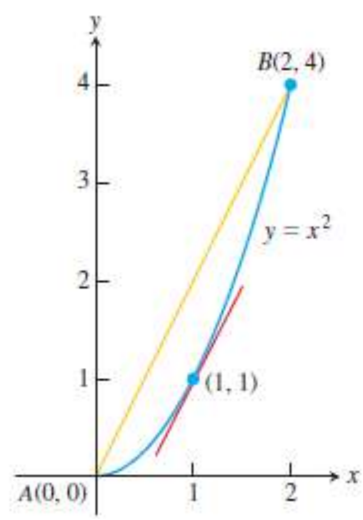
Theorem (The Mean Value Theorem):

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Example:

The function $f(x) = x^2$ is continuous for $0 \leq x \leq 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4-0)/(2-0) = 2$. In this case we can identify c by solving the equation $2c = 2$ to get



$c = 1$. However, it is not always easy to find c algebraically, even though we know it always exists.

Remark:

At the beginning of the section, we asked what kind of function has a zero derivative over an interval. The first corollary of the Mean Value Theorem provides the answer that only constant functions have zero derivatives.

Corollary (1):

If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Remark:

At the beginning of this section, we also asked about the relationship between two functions that have identical derivatives over an interval. The next corollary tells us that their values on the interval have a constant difference.

Corollary (2):

If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f' - g'$ is a constant function on (a, b).

Example:

Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point (0, 2).

Solution:

Since the derivative of $g(x) = -\cos x$ is $g'(x) = \sin x$, we see that f and g have the same derivative. Corollary 2 then says that $f(x) = -\cos x + C$ for some constant C. Since the graph of f passes through the point (0, 2), the value of C is determined from the condition that $f(0) = 2$:

$$f(0) = -\cos(0) + C, \text{ so } C = 3.$$

The function is $f(x) = -\cos x + 3$.

Exercises:

1. Find the value or values of c that satisfy the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$ in the conclusion of the Mean Value Theorem for the functions and intervals in following.

a) $f(x) = x^2 + 2x - 1,\ [0,1]$

b) $f(x) = x^{2/3},\ [0,1]$

c) $f(x) = x + \frac{1}{x},\ [\frac{1}{2}, 2]$

d) $f(x) = \sqrt{x-1},\ [1,3]$

e) $f(x) = x^3 - x^2,\ [-1,2]$

f) $g(x) = \begin{cases} x^3, & -2 \le x \le 0 \\ x^2, & 0 < x \le 2 \end{cases}$

2. Which of the functions in following satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

a) $f(x) = x^{2/3},\ [-1,8]$

b) $f(x) = x^{4/5},\ [0,1]$

c) $f(x) = \sqrt{x(1-x)},\ [0,1]$

d) $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$

e) $f(x) = \begin{cases} x^2 - x, & -2 \le x \le -1 \\ 2x^2 - 3x - 3, & -1 < x \le 0 \end{cases}$

f) $f(x) = \begin{cases} 2x - 3, & 0 \le x \le 2 \\ 6x - x^2 - 7, & 2 < x \le 3 \end{cases}$

3. The function $f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$ is zero at $x = 0$ and $x = 1$ and differentiable on (0, 1), but its derivative on (0, 1) is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in (0, 1)? Give reasons for your answer.

4. For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

5.

a. Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

I) $y = x^2 - 4$

II) $y = x^2 + 8x + 15$

III) $y = x^3 - 3x^2 + 4 = (x+1)(x-2)^2$

IV) $y = x^3 - 33x^2 + 216x = x(x-9)(x-24)$

b. Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there lies a zero of

$$nx^{n-1} + (n-1)\, a_{n-1}x^{n-2} + \cdots + a_1.$$

6. Show that the functions in following have exactly one zero in the given interval

a) $f(x) = x^4 + 3x + 1,\ [-2,-1]$ **b)** $f(x) = x^3 + \frac{4}{x^2} + 7,\ (-\infty, 0)$

c) $g(t) = \sqrt{t} + \sqrt{1+t} - 4,\ (0,\infty)$ **d)** $g(t) = \frac{1}{1-t} + \sqrt{1+t} - 3.1,\ (-1,1)$

e) $r(\theta) = \theta + \sin^2(\frac{\theta}{3}) - 8,\ (-\infty,\infty)$ **f)** $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2},\ (-\infty,\infty)$

g) $r(\theta) = \sec\theta - \frac{1}{\theta^3} + 5,\ (0,\frac{\pi}{2})$ **h)** $r(\theta) = \tan\theta - \cot\theta - \theta,\ (0,\frac{\pi}{2})$

7. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

8. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

9. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a) $f(0) = 0$ **b)** $f(1) = 0$ **c)** $f(-2) = 3$

10. In following, find all possible functions with the given derivative.

a) $y' = x$ **b)** $y' = 3x^2 + 2x - 1$ **c)** $y' = 1 - \frac{1}{x^2}$

d) $y' = 4x - \frac{1}{\sqrt{x}}$ **e)** $y' = \sin 2t + \cos\frac{t}{2}$ **f)** $y' = \sqrt{\theta} - \sec^2\theta$

3.11 Monotonic Functions and the First Derivative Test:

In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

As another corollary to the Mean Value Theorem, we show that functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be monotonic on the interval.

Corollary (3):

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

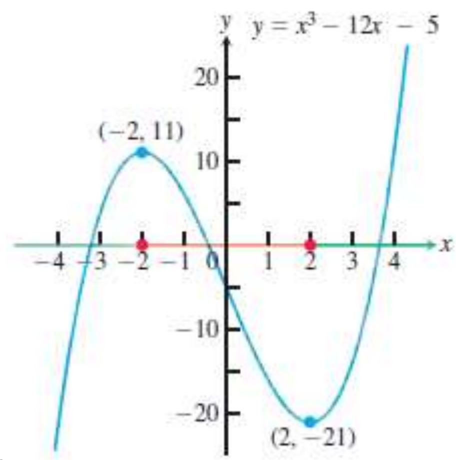
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Example:

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution:

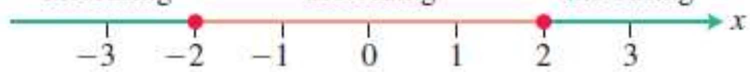


The function f is everywhere continuous and differentiable. The first derivative

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x - 2)(x + 2),$$

is zero at x = -2 and x = 2. These critical points subdivide the domain of f to create nonoverlapping open intervals $(-\infty, -2), (-2, 2)$, and $(2, \infty)$ on which f is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. We evaluate f' at $x = -3$ in the first interval, $x = 0$ in the second interval and $x = 3$ in the third, since f' is relatively easy to compute at these points. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure above.

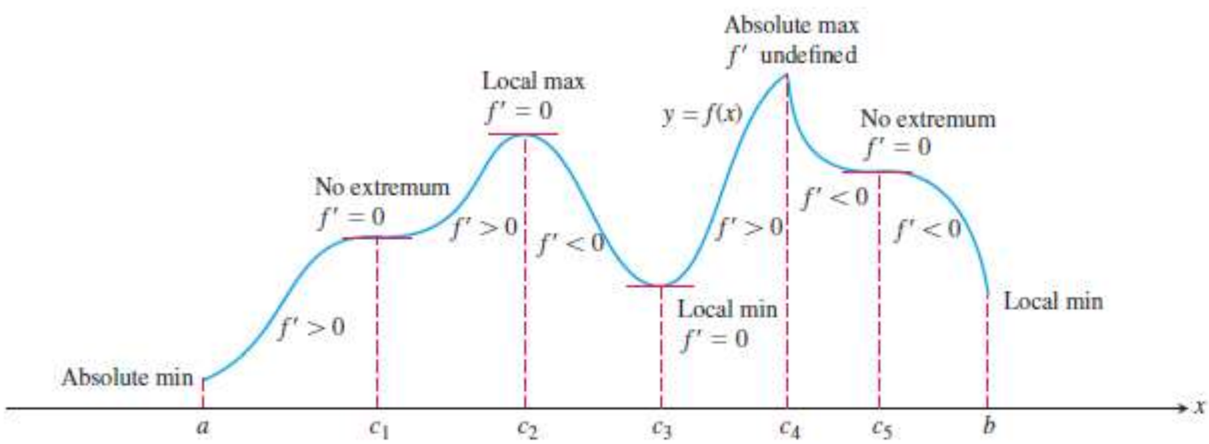
Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



Remark:

We used "strict" less-than inequalities to identify the intervals in the summary table for previous Example, since open intervals were specified. Corollary 3 says that we could use inequalities as well. That is, the function f in the example is increasing on $-\infty < x \leq -2$, decreasing on $-2 \leq x \leq 2$, and increasing on $2 \leq x < \infty$. We do not talk about whether a function is increasing or decreasing at a single point.

Remark:



In above Figure, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

Remark (First Derivative Test for Local Extrema):

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a ***local minimum*** at c;
2. if f' changes from positive to negative at c, then f has a ***local maximum*** at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no ***local extremum*** at c.

The test for local extrema at endpoints is similar, but there is only one side to consider in determining whether f is increasing or decreasing, based on the sign of f'.

Example:

Find the critical points of $f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$. Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution:

The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x-4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}.$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into open intervals on which f is either positive or negative. The sign pattern of f reveals the behavior of f between and at the critical points, as summarized in the following table.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

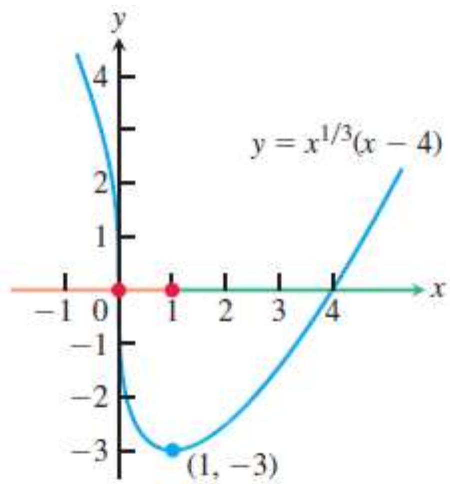
Corollary 3 to the Mean Value Theorem implies that f decreases on $(-\infty, 0)$, decreases on (0, 1), and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is

$$f(1) = 1^{1/3}\,(1 - 4) = -3.$$

This is also an absolute minimum since f is decreasing on $(-\infty, 0)$ and increasing on $(1, \infty)$.

Note that $\lim_{x\to 0} f'(x) = -\infty$, so the graph of f has a vertical tangent at the origin.



Example:

Within the interval $0 \le x \le 2\pi$, find the critical points of

$$f(x) = sin^2 x - sin\, x - 1.$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution:

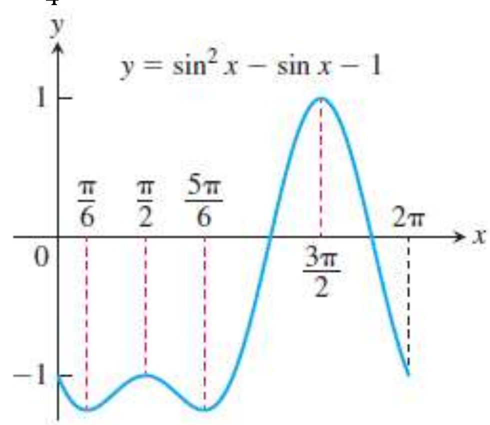
The function f is continuous over $[0,2\pi]$ and differentiable over $(0,2\pi)$, so the critical points occur at the zeros of f' in $(0,2\pi)$. We find

$$f'(x) = 2\sin x\, cos\, x - cos\, x = (2\sin x - 1)(\cos x).$$

The first derivative is zero if and only if $sin\ x = \frac{1}{2}$ or $cos\ x = 0$. So, the critical points of f in $(0,2\pi)$ are $x = \pi/6$, $x = \frac{5\pi}{6}$, $x = \pi/2$, and $x = 3\pi/2$. They partition $[0,2\pi]$ into open intervals as follows.

Interval	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Sign of f'	$-$	$+$	$-$	$+$	$-$
Behavior of f	dec	inc	dec	increasing	decreasing

The table displays the open intervals on which f is increasing and decreasing. We can deduce from the table that there is a local minimum value of $f\left(\frac{\pi}{6}\right) = \frac{1}{4} - \frac{1}{2} - 1 = \frac{5}{4}$, a local maximum value of $f\left(\frac{\pi}{2}\right) = 1 - 1 - 1 = -1$, another local minimum value of $f\left(\frac{5\pi}{6}\right) = -\frac{5}{4}$, and another local maximum value of $f\left(\frac{3\pi}{2}\right) = 1 - (-1) - 1 = 1$. The endpoint values are $f(0) = f(2\pi) = -1$. The absolute minimum in $[0,2\pi]$ is $-\frac{5}{4}$ occurring at $x = \pi/6$ and $x = \frac{5\pi}{6}$; the absolute maximum is 1 occurring at $x = 3\pi/2$.



Exercises:

1. Answer the following questions about the functions whose derivatives are given in Exercises *a*)–I):

I. What are the critical points of *f*?

II. On what open intervals is *f* increasing or decreasing?

III. At what points, if any, does *f* assume local maximum and minimum values?

a) $f'(x) = x(x-1)$

b) $f'(x) = (x-1)(x+2)$

c) $f'(x) = (x-1)^2(x+2)$

d) $f'(x) = (x-1)^2(x+2)^2$

e) $f'(x) = (x-7)(x+1)(x+5)$

f) $f'(x) = \frac{x^2(x-1)}{(x+2)}, x \neq -2$

g) $f'(x) = \frac{(x-2)(x+4)}{(x+1)(x-3)}, x \neq -1,3$

h) $f'(x) = 1 - \frac{4}{x^2}, x \neq 0$

i) $f'(x) = 3 - \frac{6}{\sqrt{x}}, x \neq 0$

j) $f'(x) = x^{-1/3}(x+2)$

k) $f'(x) = (\sin x - 1)(2\cos x + 1), \quad 0 \leq x \leq 2\pi$

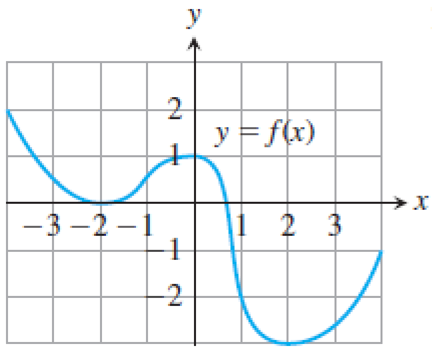
l) $f'(x) = (\sin x + \cos x)(\sin x - \cos x), \quad 0 \leq x \leq 2\pi$

2. In following

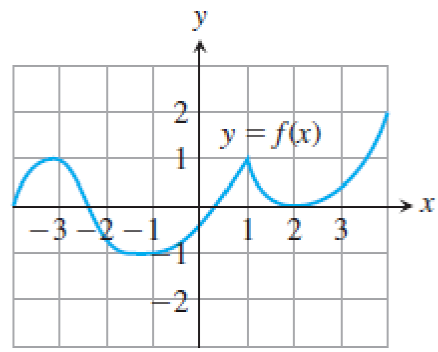
I. Find the open intervals on which the function is increasing and decreasing.

II. Identify the function's local and absolute extreme values, if any, saying where they occur.

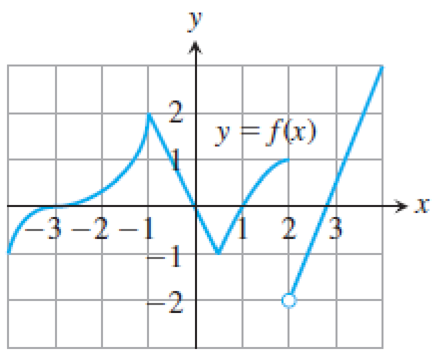
a)



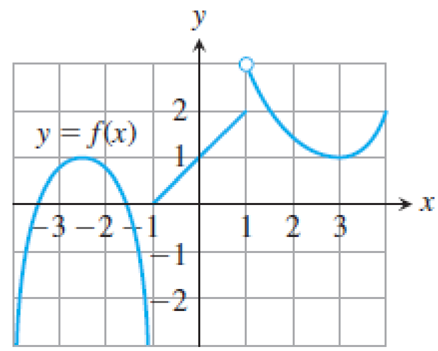
b)



c)



d)



e) $g(t) = -t^2 - 3t + 3$

f) $h(x) = -x^3 + 2x^2$

g) $f(\theta) = 6\theta - \theta^3$

h) $h(r) = (r+7)^3$

i) $f(x) = x^4 - 8x^2 + 16$

j) $H(t) = \frac{3}{2}t^4 - t^6$

k) $f(x) = x - 6\sqrt{x-1}$

l) $g(x) = 4\sqrt{x} - x^3 + 3$

m) $g(x) = x\sqrt{8 - x^2}$

n) $g(x) = x^2\sqrt{5 - x}$

o) $f(x) = \frac{x^2-3}{x-2}, \quad x \neq 2$

p) $f(x) = \frac{x^3}{3x^2+1}$

q) $f(x) = x^{1/3}(x+8)$ **r)** $k(x) = x^{2/3}(x^2-4)$

3. In following

I. Identify the function's local extreme values in the given domain, and say where they occur.

II. Which of the extreme values, if any, are absolute?

III. Support your findings with a graphing calculator or computer grapher.

a) $f(x) = 2x - x^2, -\infty < x \le 2$
b) $f(x) = (x+1)^2, -\infty < x \le 0$
c) $g(x) = x^2 - 4x + 4, 1 \le x < \infty$
d) $g(x) = -x^2 - 6x - 9, -4 \le x < \infty$
e) $f(t) = 12t - t^3, -3 \le t < \infty$
f) $f(t) = t^3 - 3t^2, -\infty < t \le 3$
g) $h(x) = \frac{x^3}{3} - 2x^2 + 4x, 0 \le x < \infty$
h) $k(x) = x^3 + 3x^2 + 3x + 1, -\infty < x \le 0$
i) $f(x) = \sqrt{25 - x^2}, -5 \le x \le 5$
j) $f(x) = \sqrt{x^2 - 2x - 3}, 3 \le x < \infty$
k) $g(x) = \frac{x-2}{x^2-1}, 0 \le x < 1$
l) $g(x) = \frac{x^2}{4-x^2}, -2 < x \le 1$

4. In following

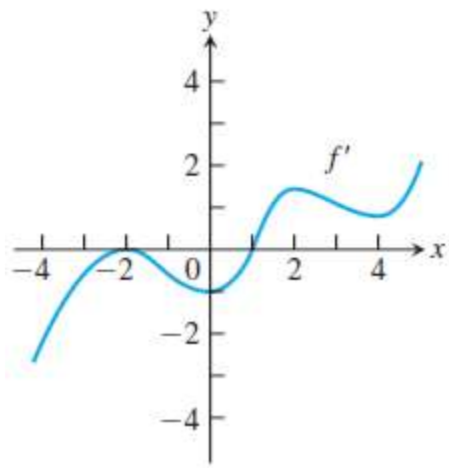
I. Find the local extrema of each function on the given interval, and say where they occur.

II. Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

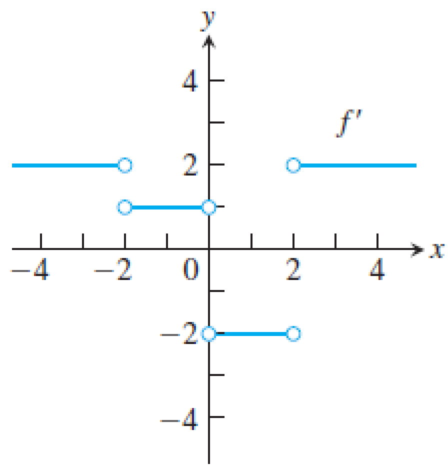
a) $f(x) = \sin 2x, 0 \le x \le \pi$
b) $f(x) = \sin x - \cos x, 0 \le x \le 2\pi$
c) $f(x) = \sqrt{3}\cos x + \sin x, 0 \le x \le 2\pi$
d) $f(x) = -2x + \tan x, \frac{-\pi}{2} \le x \le \frac{\pi}{2}$
e) $f(x) = \frac{x}{2} - 2\sin\frac{x}{2}, 0 \le x \le 2\pi$
f) $f(x) = -2\cos x - \cos^2 x, -\pi \le x \le \pi$
g) $f(x) = \csc^2 x - 2\cot x, 0 < x < \pi$
h) $f(x) = \sec^2 x - 2\tan x, \frac{-\pi}{2} < x < \frac{\pi}{2}$

5. In following, the graph of f' is given. Assume that f is continuous and determine the x-values corresponding to local minima and local maxima.

a)



b)



6. Sketch the graph of a differentiable function $y = f(x)$ that has

a) a local minimum at (1, 1) and a local maximum at (3, 3);

b) a local maximum at (1, 1) and a local minimum at (3, 3);

c) local maxima at (1, 1) and (3, 3);

d) local minima at (1, 1) and (3, 3).

7. Sketch the graph of a continuous function $y = g(x)$ such that

a) $g(2) = 2, 0 < g' < 1$ for $x < 2, g'(x) \to 1^-$ as $x \to 2^-$, $-1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

b) $g(2) = 2, g' < 0$ for $x < 2, g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$;

8. Sketch the graph of a continuous function $y = h(x)$ such that

a) $h(0) = 0, -2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

b) $h(0) = 0, -2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$;

9. Sketch the graph of a differentiable function $y = f(x)$ through the point (1, 1) if $f'(1) = 0$ and

a) $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

b) $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

c) $f'(x) > 0$ for $x \neq 1$;

d) $f'(x) < 0$ for $x \neq 1$.

Calculus I

First Semester

Lecturer 12

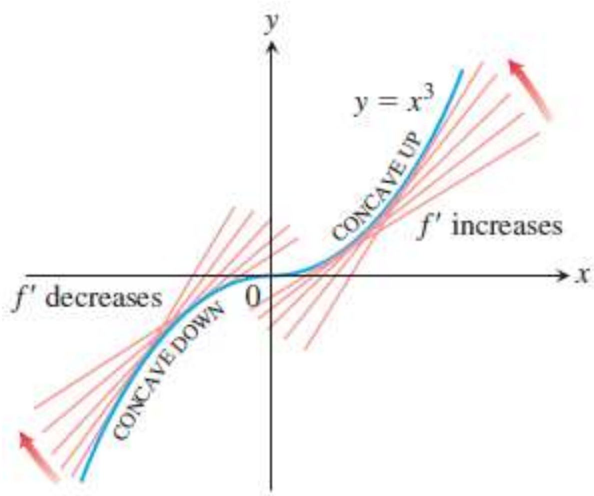
Dr. Ban Jaffar AL-Taiy

Taghreed Hussein Abed

3.12 Concavity and Curve Sketching:

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns.

As we can see, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the concavity of the curve.



Definition:

The graph of a differentiable function $y = f(x)$ is

a) ***concave up*** on an open interval I if f' is increasing on I;

b) ***concave down*** on an open interval I if f' is decreasing on I.

Remark:

A function whose graph is concave up is also often called ***convex***. If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to the first derivative function. We conclude that f' increases if $f'' > 0$ on I, and decreases if $f'' < 0$.

Remark(The Second Derivative Test for Concavity):

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.

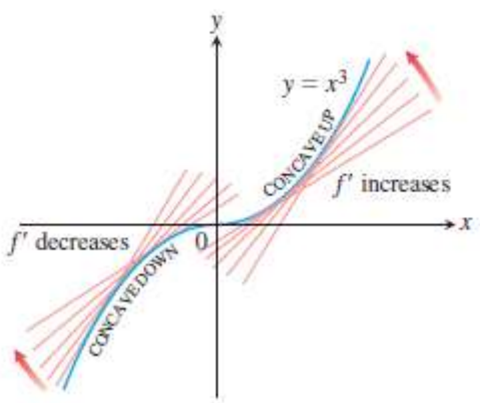
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Remark:

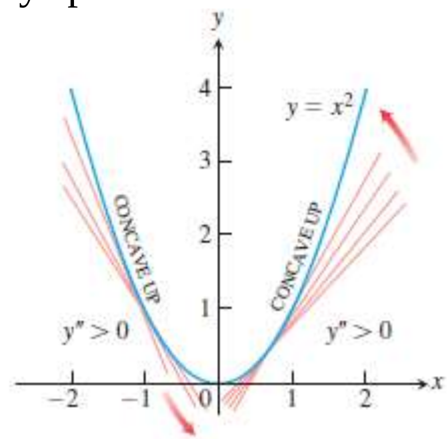
If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

Example:

a) The curve $y = x^3$ is concave down on $(-\infty, 0)$, where $y'' = 6x < 0$, and concave up on $(0, \infty)$, where $y'' = 6x > 0$.



b) The curve $y = x^2$ is concave up on $(-\infty, \infty)$, because its second derivativewhere $y'' = 2$ is always positive.

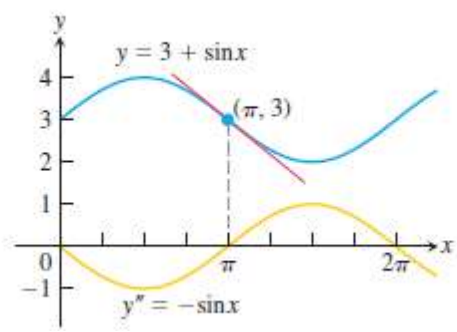


Example:

Determine the concavity of $y = 3 + sin\ x$ on $[\ 0, 2\pi]$.

Solution:

The first derivative of $y = 3 + sin\ x$ is $y' = cos\ x$, and the second derivative is $y'' = -sin\ x$. The graph of $y = 3 + sin\ x$ is concave down on $(0, \pi)$, where $y'' = -sin\ x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -sin\ x$ is positive.



Remark:

The curve $y = 3 + sin\ x$ in previous example changes concavity at the point $(\pi, 3)$. Since the first derivative $y' = cos\ x$ exists for all x, we see that the curve has a tangent line of slope -1 at the point $(\pi, 3)$. This point is called ***a point of inflection*** of the curve. Notice that the graph crosses its tangent line at this point and that the second derivative $y'' = -sin\ x$ has value 0 when $x = \pi$. In general, we have the following definition.

Definition:

A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a ***point of inflection***.

Remark:

We observed that the second derivative of $f(x) = 3 + \sin x$ is equal to zero at the inflection point $(\pi, 3)$. Generally, if the second derivative exists at a point of inflection (c, f (c)), then $f''(c) = 0$. This follows immediately from the Intermediate Value Theorem whenever f'' is continuous over an interval containing $x = c$ because the second derivative changes sign moving across this interval. Even if the continuity assumption is dropped, it is still true that $f''(c) = 0$, provided the second derivative exists (although a more advanced argument is required in this noncontinuous case). Since a tangent line must exist at the point of inflection, either the first derivative $f'(c)$ exists (is finite) or the graph has a vertical tangent line at the point. At a vertical tangent, neither the first nor second derivative exists. In summary, one of two things can happen at a point of inflection.

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Example:

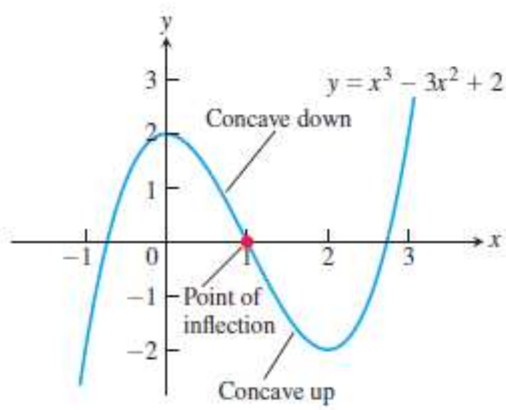
Determine the concavity and find the inflection points of the function

$$f(x) = x^3 - 3x^2 + 2.$$

Solution:

We start by computing the first and second derivatives.

$$f'(x) = 3x^2 - 6x, \quad f''(x) = 6x - 6.$$



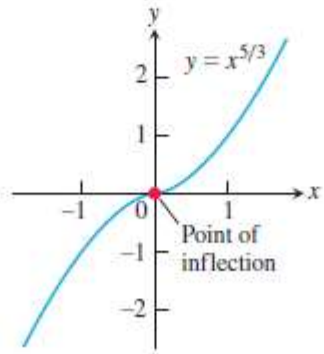
To determine concavity, we look at the sign of the second derivative $f''(x) = 6x - 6$. The sign is negative when $x < 1$, is 0 at $x = 1$, and is positive when $x > 1$. It follows that the graph of f is concave down on $(-\infty, 1)$, is concave up on $(1, \infty)$, and has an inflection point at the point (1, 0) where the concavity changes.

Remark:

The next example illustrates that a function can have a point of inflection where the first derivative exists but the second derivative fails to exist.

Example:

The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = (5/3)\, x^{2/3} = 0$ when $x = 0$. However, the second derivative,



$$f''(x) = \frac{d}{dx}\left((5/3)\, x^{2/3}\right) = \frac{10}{9}\, x^{-1/3},$$

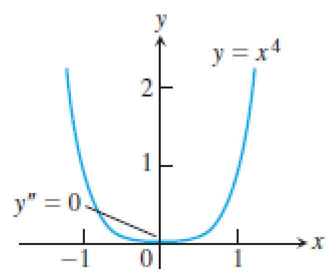
fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin.

Remark:

The following example shows that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

Example:

The curve $y = x^4$ has no inflection point at $x = 0$. Even though the second derivative $y'' = 12\, x^2$ is zero there, it does not change sign. The curve is concave up everywhere.

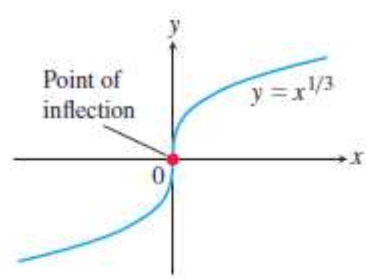


Remark:

In the next example, a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.

Example:

The graph of $y = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:



$$y'' = \frac{d^2}{dx^2} x^{1/3} = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = \frac{1}{3}x^{-2/3}$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there.

Remark:

We show that the function $f(x) = x^{2/3}$ does not have a second derivative at $x = 0$ and does not have a point of inflection there (there is no change in concavity at $x = 0$). Combined with the behavior of the function $f(x) = x^{2/3}$, we see that when the second derivative does not exist at $x = c$, an inflection point may or may not occur there. So, we need to be careful about interpreting functional behavior whenever first or second derivatives fail to exist at a point. At such points the graph can have vertical tangents, corners, cusps, or various discontinuities.

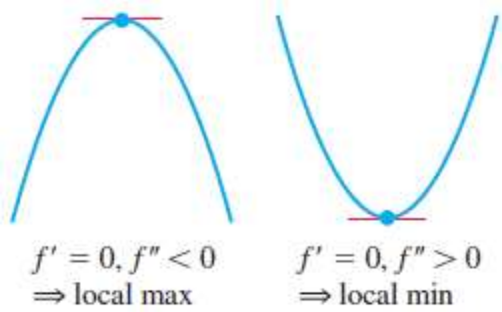
Remark:

Instead of looking for sign changes in f at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

Theorem (Second Derivative Test for Local Extrema):

Suppose f'' is continuous on an open interval that contains $x = c$.

1. ***If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.***
2. ***If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.***
3. ***If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.***



Example:

Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10,$$

using the following steps.

a) Identify where the extrema of f occur.

b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

c) Find where the graph of f is concave up and where it is concave down.

d) Sketch the general shape of the graph for f.

e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution:

The function f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f . Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	-	-	+
Behavior of f	decreasing	decreasing	increasing

a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and [0, 3], and increasing on $[3, \infty)$.

c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where the graph of f is concave up or concave down.

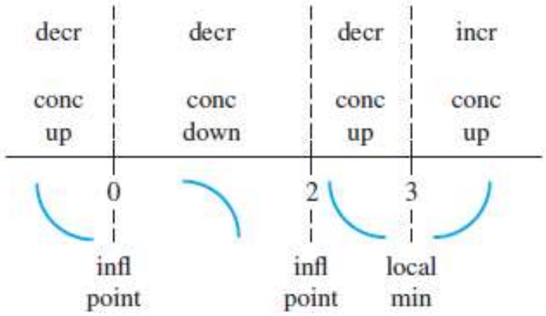
Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f'	+	-	+
Behavior of f	concave up	concave down	concave up

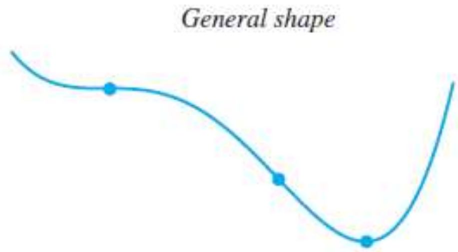
We see that the graph of f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on (0, 2).

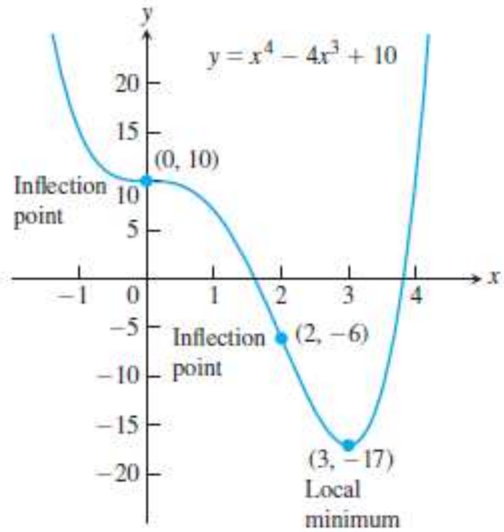
d) Summarizing the information in the last two tables, we obtain the following.

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

The general shape of the curve is shown in the accompanying figure.







e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.)

Remark:

The steps in previse example give a procedure for graphing the key features of a function.

Remark (Procedure for Graphing $y = f\ (x)$):

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives f' and f''.
3. Find the critical points of f , if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

Example:

Sketch a graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

Solution:

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin.

2. Find f' and f''

x-intercept at $x = -1$,
y-intercept at $y = 1$

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2}$$
$$= \frac{(x+1)[(2+2x^2)-(2x^2+2x)]}{(1+x^2)^2}$$
$$= \frac{(x+1)[2+2x^2-2x^2-2x]}{(1+x^2)^2}$$
$$= \frac{2(x+1)(1-x)}{(1+x^2)^2}$$
$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$
$$= \frac{-4x(1+x^2)[(1+x^2)+2(1-x^2)]}{(1+x^2)^4}$$
$$= \frac{-4x(1+x^2)(1+x^2+2-2x^2)}{(1+x^2)^4}$$
$$= \frac{-4x(1+x^2)(3-x^2)}{(1+x^2)^4}$$
$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

3. Behavior at critical points. The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1, f''(-1) = 1 > 0$, yielding a relative minimum by the Second Derivative Test. At $x = 1, f''(1) = -1 < 0$, yielding a relative maximum by the Second Derivative test.

4. Increasing and decreasing. We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.

5. Inflection points. Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $(-\infty, -\sqrt{3})$, positive on $(-\sqrt{3}, 0)$, negative on $(0, \sqrt{3})$, and positive again on $(\sqrt{3}, \infty)$. Thus, each point is a point of inflection. The curve is concave down on the interval $(-\infty, -\sqrt{3})$, concave up on $(-\sqrt{3}, 0)$, concave down on $(0, \sqrt{3})$, and concave up again on $(\sqrt{3}, \infty)$.

6. **Asymptotes**. Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

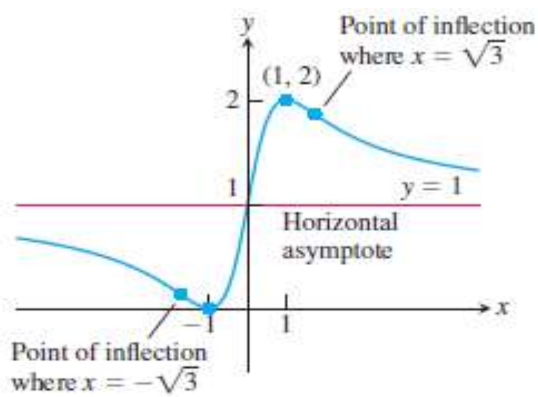
$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \textbf{Expanding numerator}$$

$$= \frac{1+\left(\frac{2}{x}\right)+\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)+1} \qquad \textbf{Dividing by } \boldsymbol{x^2}$$

We see that $f(x) \to 1^+$ as $x \to \infty$ and that $f(x) \to 1^-$ as $x \to -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on (-1, 1), we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So, the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \leq y \leq 2$).

7. The graph of f is sketched. Notice how the graph is concave down as it approaches the horizontal asymptote y = 1 as $x \to -\infty$, and concave up in its approach to $y = 1$ as $x \to \infty$.



Example:

Sketch a graph of $f(x) = \frac{x^2+4}{2x}$.

Solution:

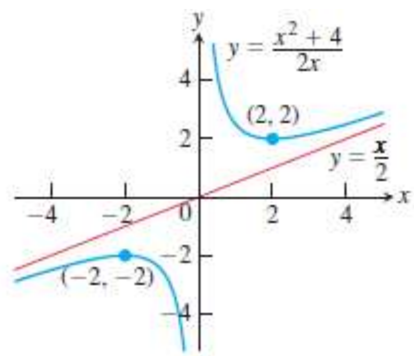
1. The domain of f is all nonzero real numbers. There are no intercepts because neither x nor $f(x)$ can be zero. Since $f(-x) = -f(x)$, we note that f is an odd function, so the graph of f is symmetric about the origin.
2. We calculate the derivatives of the function, but we first rewrite it in order to simplify our computations:

$f(x) = \frac{x^2+4}{2x} = \frac{x}{2} + \frac{2}{x}$ **Function simplified for differentiation**

$f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2-4}{2x^2}$ **Combine fractions to solve easily $f'(x) = 0$.**

$f''(x) = \frac{4}{x^3}$ **Exists throughout the entire domain of f.**

3. The critical points occur at $x = \pm 2$ where $f'(x) = 0$. Since $f''(-2) < 0$ and $f''(2) > 0$, we see from the Second Derivative Test that a relative maximum occurs at $x = -2$ with $f(-2) = -2$, and a relative minimum occurs at $x = 2$ with $f(2) = 2$.
4. On the interval $(-\infty, -2)$ the derivative f' is positive because $x^2 - 4 > 0$ so the graph is increasing; on the interval $(-2, 0)$ the derivative is negative and the graph is decreasing. Similarly, the graph is decreasing on the interval $(0, 2)$ and increasing on $(2, \infty)$.
5. There are no points of inflection because $f''(x) < 0$ whenever $x < 0$, $f''(x) > 0$ whenever $x > 0$, and f'' exists everywhere and is never zero throughout the domain of f. The graph is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.
6. From the rewritten formula for $f(x)$, we see that $\lim_{x \to 0^+}\left(\frac{x}{2} + \frac{2}{x}\right) = +\infty$ and $\lim_{x \to 0^+}\left(\frac{x}{2} + \frac{2}{x}\right) = -\infty$, so, the y-axis is a vertical asymptote. Also, as $x \to \infty$ or as $x \to -\infty$, the graph of $f(x)$ approaches the line $y = x/2$. Thus $y = x/2$ is an oblique asymptote.



Example:

Sketch a graph of $f(x) = \cos x - \frac{\sqrt{2}}{2}x$ over $0 \le x \le 2\pi$.

Solution:

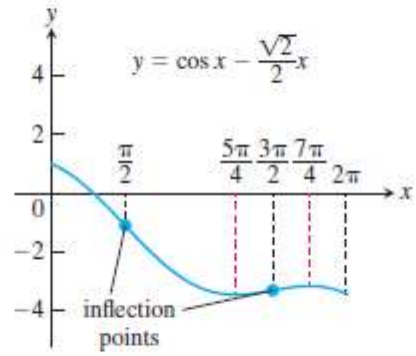
The derivatives of f are

$$f'(x) = -\sin x - \frac{\sqrt{2}}{2} \quad \text{and} \quad f''(x) = -\cos x.$$

Both derivatives exist everywhere over the interval $(0, 2\pi)$. Within that open interval, the first derivative is zero when $\sin x = -\sqrt{2}/2$, so the critical points are $x = 5\pi/4$ and $x = 7\pi/4$. Since $f''\left(\frac{5\pi}{4}\right) = -\cos\frac{5\pi}{4} =$

$\frac{\sqrt{2}}{2} > 0$, the function has a local minimum value of $f\ (5\pi/\ 4\ )\ \approx\ -3.48$ (evaluated with a calculator) by the Second Derivative Test. Also, $f''(7\pi/$- $4) = -\cos\frac{7\pi}{}4 = -\frac{\sqrt{2}}{2} < 0$ so the function has a local maximum value of $f\ (7\pi/\ 4)\ \approx\ -3.18$.

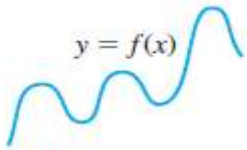
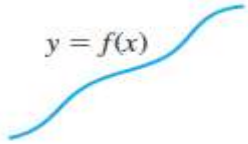
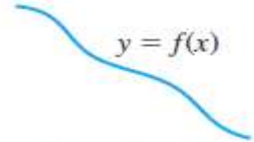
Examining the second derivative, we find that $f'' = 0$ when $x = \pi/\ 2$ or $x = 3\pi/\ 2$. Since f'' changes sign at these two points, we conclude that $(\pi/\ 2, f(\ \pi/\ 2))\ \approx\ (\pi/\ 2, -1.11)$ and $(3\pi/\ 2, f\ (3\pi/\ 2))\ \approx\ (3\pi/\ 2, -3.33)$ are points of inflection.



Finally, we evaluate f at the endpoints of the interval to find $f(0) = 1$ and $f(2\pi)\ \approx -3.44$. Therefore, the values $f(0) = 1$ and $f\ (\frac{5\pi}{4})\ \approx\ -3.48$ are the absolute maximum and absolute minimum values of f over the closed interval [0, 2]. The graph of f is sketched.

Remark:

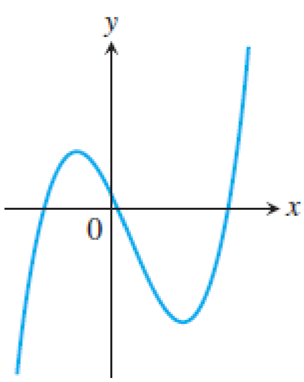
As we saw in previous examples, we can learn much about a twice-differentiable function $y =\ f(x)$ by examining its first derivative. We can find where the function's graph rises and falls and where any local extrema are located. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. Together with information about the function's asymptotes and its value at some key points, such as intercepts, this information about the derivatives helps us determine the shape of the function's graph. The following figure summarizes how the first derivative and second derivative affect the shape of a graph.

 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall or both	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall or both	 y'' changes sign at an inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

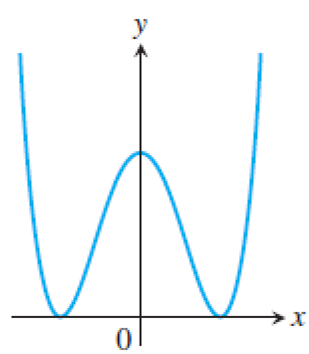
Exercises:

1. Identify the inflection points and local maxima and minima of the functions graphed in following. Identify the open intervals on which the functions are differentiable and the graphs are concave up and concave down.

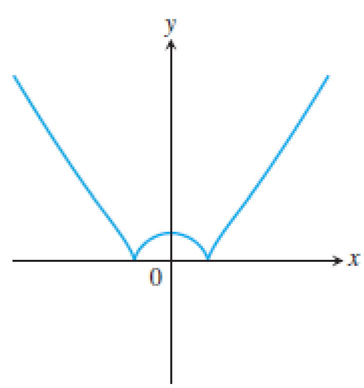
a) $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



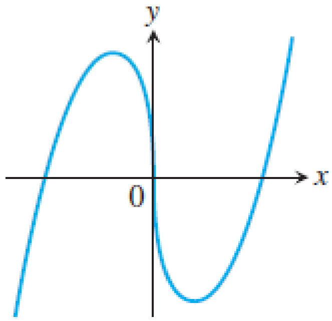
b) $y = \frac{x^4}{4} - 2x^2 + 4$



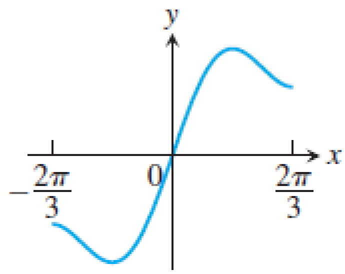
c) $y = \frac{3}{4}(x^2 - 1)^{2/3}$



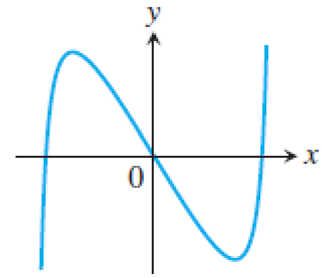
d) $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



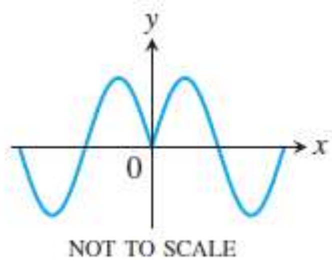
e) $y = x + \sin 2x\,, -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



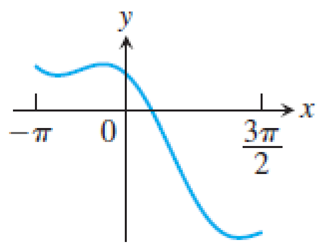
f) $y = \tan x - 4x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



g) $y = \sin|x|\,, -2\pi \le x \le 2\pi$



h) $y = 2\cos x - \sqrt{2}x, -\pi \le x \le \frac{3\pi}{2}$



2. In following, identify the coordinates of any local and absolute extreme points and inflection points. Graph the function.

a) $y = x^2 - 4x + 3$

b) $y = (x-2)^3 + 1$

c) $y = -x^4 + 6x^2 - 4$

d) $y = x(\frac{x}{2} - 5)^4$

e) $y = \sqrt{3}x - 2\cos x,\ \ 0 \le x \le 2\pi$

f) $y = \frac{4}{3}x - \tan x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

g) $y = x^{1/5}$

h) $y = \frac{\sqrt{1-x^2}}{2x+1}$

i) $y = x^{2/3}(\frac{5}{2} - x)$

j) $y = \sqrt[3]{x^3 + 1}$

k) $y = |x^2 - 1|$

l) $y = \sqrt{|x|} = \begin{cases} \sqrt{-x} & x < 0 \\ \sqrt{x} & x \ge 0 \end{cases}$

m) $y = \sqrt{|x-4|}$

n) $y = \frac{x}{9-x^2}$

o) $y = \frac{x^2}{1-x}$

p) $y = \sin x \cos x\,, 0 \le x \le \pi$

3. Each of following gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use Steps 2–4 of the graphing procedure on page 7 to sketch the general shape of the graph of f.

a) $y' = 2 + x - x^2$

b) $y' = x^2(2 - x)$

c) $y' = x(x^2 - 12)$

d) $y' = (x^2 - 2x)(x - 5)^2$

e) $y' = \sec^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

f) $y' = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

g) $y' = (x + 1)^{-2/3}$

h) $y' = 1 - \cot^2 \theta \, , 0 < \theta < \pi$

i) $y' = x^{-2/3}(x - 1)$

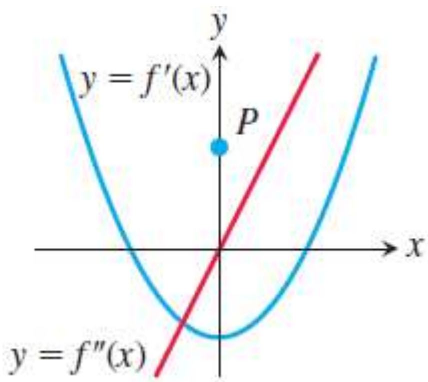
j) $y' = x^{-4/5}(x + 1)$

k) $y' = 2|x| = \begin{cases} -2x & x \leq 0 \\ 2x & x > 0 \end{cases}$

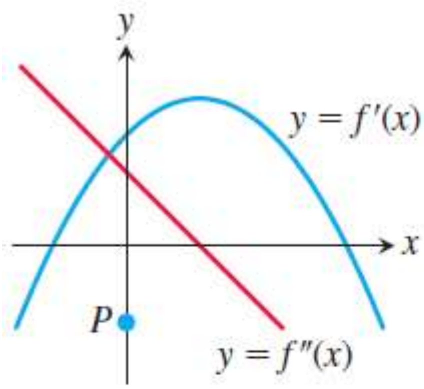
l) $y' = \begin{cases} -x^2 & x \leq 0 \\ x^2 & x > 0 \end{cases}$

4. Each of following shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

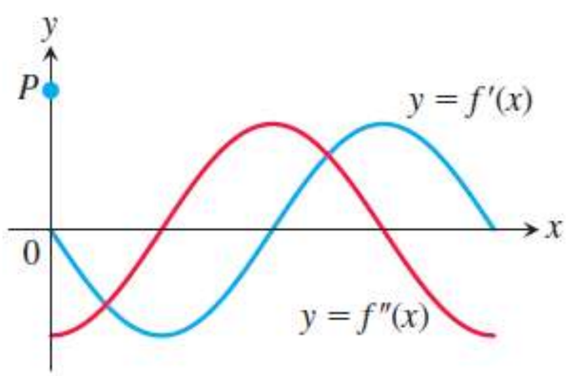
a)



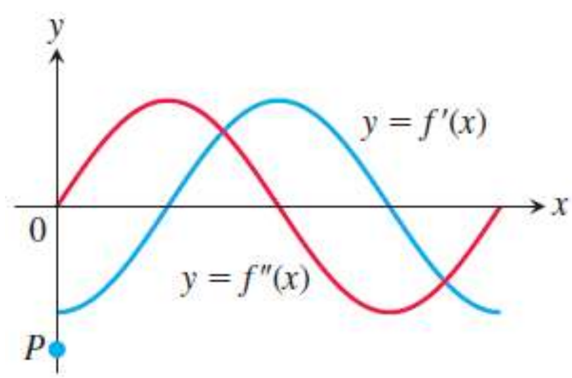
b)



c)



d)



5. Graph the rational functions in following using all the steps in the graphing procedure on page 7.

a) $y = \frac{2x^2+x-1}{x^2-1}$

b) $y = \frac{x^2-49}{x^2+5x-14}$

c) $y = \frac{x^2-4}{x^2-2}$

d) $y = \frac{x^4+1}{x^2}$

e) $y = \frac{x^2-4}{2x}$

f) $y = \frac{1}{x^2-1}$

g) $y = -\frac{x^2-2}{x^2-1}$

h) $y = \frac{x^2}{x+1}$

i) $y = \frac{x^2-x+1}{x-1}$

j) $y = \frac{x^3-3x^2+3x-1}{x^2+x-2}$

k) $y = \frac{x^3+x-2}{x-x^2}$

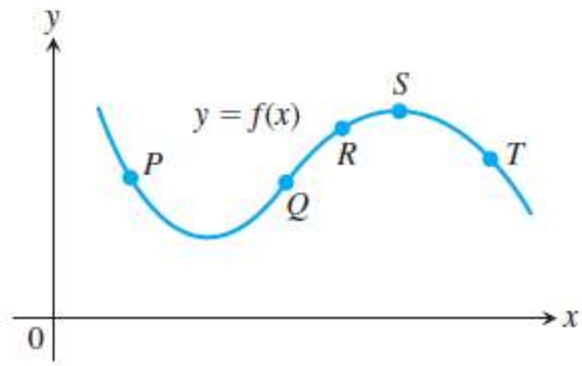
l) $y = \frac{x}{x^2-1}$

m) $y = \frac{x-1}{x^2(x-2)}$

n) $y = \frac{8}{x^2+4}$

o) $y = \frac{4x}{x^2+4}$

6. The accompanying figure shows a portion of the graph of a twice differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



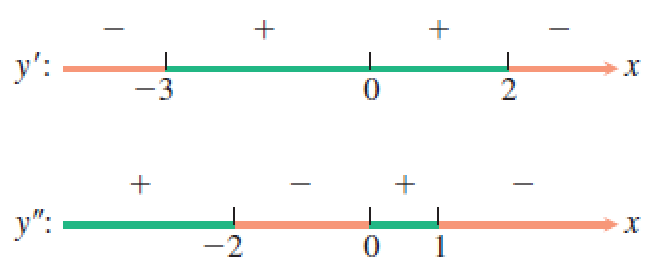
7. Sketch a smooth connected curve $y = f(x)$ with

$f(-2) = 8,$	$f'(2) = f'(-2) = 0,$
$f(0) = 4,$	$f'(x) < 0$ for $\|x\| < 2,$
$f(2) = 0,$	$f''(x) < 0$ for $x < 0,$
$f'(x) > 0$ for $\|x\| > 2,$	$f''(x) > 0$ for $x > 0,$

8. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

x	y	Derivatives
$x < 2$		$y' < 0, y'' > 0$
2	1	$y' = 0, y'' > 0$
$2 < x < 4$		$y' > 0, y'' = 0$
4	4	$y' > 0, y'' = 0$
$4 < x < 6$		$y' > 0, y'' < 0$
6	7	$y' = 0, y'' < 0$
$x > 6$		$y' < 0, y'' < 0$

9. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points (-3, -2), (-2, 0), (0, 1), (1, 2), and (2, 3) and whose first two derivatives have the following sign patterns.



3.13 Indeterminate Forms and L'Hôpital's Rule:

Consider the four limits

$$\lim_{x\to 0}\frac{x^3}{x} = \lim_{x\to 0} x^2 = 0,$$
$$\lim_{x\to 0}\frac{x}{x^3} = \lim_{x\to 0}\frac{1}{x^2} = \infty,$$
$$\lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1,$$
$$\lim_{x\to 0}\frac{x}{2x} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2},$$

In each case both the numerator and the denominator approach zero as $x \to 0$, even though these limits ultimately lead to completely different results: 0, ∞,1, and $\frac{1}{2}$. We say the expression

$$\lim_{x\to a}\frac{f(x)}{g(x)} \text{ when } \lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0. \qquad (1)$$

involves an indeterminate form 0/0. The expression "0/ 0" has the form of a number, but it is not a meaningful quantity. Stating that both the numerator and the denominator approach zero does not provide sufficient information to obtain the limit of the ratio. We have to examine the behavior of the expression in more detail by performing algebraic manipulation or by applying methods that we will introduce in this section.

Other forms exhibit behavior similar to Equation (1). For instance, if both the numerator and the denominator approach $+\infty$ or $-\infty$, then the limit of the ratio leads to an indeterminate form ∞/∞. Additional indeterminate forms we consider in this section are $\infty, 0, \infty - \infty, 1^\infty, 0^0$, and ∞^0. Their purpose is to summarize the behavior of certain types of limits.

John (Johann) Bernoulli discovered a rule for using derivatives to calculate limits of fractions whose numerators and denominators both approach zero or $\pm\infty$. The rule is known today as l'Hôpital's Rule, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print. Limits involving transcendental functions often require some use of this rule.

It is important to understand that the notation "0/0" is not intended to imply numerically dividing 0 by 0. Instead, the indeterminate form 0/0 refers to a limit of a ratio of two functions, each of which approaches zero. L'Hôpital's rule can help us evaluate such limits.

Theorem (L'Hôpital's Rule):

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, ***that f and g are differentiable on an open interval I containing a, and that*** $g'(x) \neq 0$ ***on I if*** $x \neq a$***. Then***

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

Remark:

Theorem L'Hôpital's rule also applies if $x = \pm\infty$ or when $\frac{f'(x)}{g'(x)} \to \pm\infty$. Also to find $\lim_{x\to a} \frac{f(x)}{g(x)}$ by use l'Hôpital's Rule, we continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

Example:

The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

a) $\lim_{x\to 0} \frac{3x-\sin x}{x} =$ **The numerator and the denominator are both approaching 0; apply l'Hôpital's Rule.**

$\lim_{x\to 0} \frac{3-\cos x}{1} =$ **Not $0/0$**

$\frac{3-\cos 0}{1} = 2$ **Limit is found.**

b) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

c) $\lim_{x\to 0} \frac{\sqrt{1+x}-1-1/2}{x^2} =$ **$0/0$; apply l'Hôpital's Rule.**

$\lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2}-x/2}{2x} =$ **Still $0/0$; apply l'Hôpital's Rule again.**

$\lim_{x\to 0} \frac{-(1/4)(1+x)^{-1/2}}{2} = -\frac{1}{8}$ **Not $0/0$; limit is found.**

d) $\lim_{x\to 0} \frac{x-\sin x}{x^3} =$ **$0/0$; apply l'Hôpital's Rule.**

$\lim_{x\to 0} \frac{1-\cos x}{3x^2} =$ **Still $0/0$; apply l'Hôpital's Rule again.**

$\lim_{x\to 0} \frac{\sin x}{6x} =$ **Still $0/0$; apply l'Hôpital's Rule again.**

$\lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$ **Not $0/0$; limit is found.**

Remark:

We have to be careful to apply l'Hôpital's Rule correctly as the following example:

Example:

$\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \lim_{x\to 0} \frac{\sin x}{1+2x}$. It is tempting to try to apply l'Hôpital's Rule again, which would result in $\lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$, but this is not the correct limit. L'Hôpital's Rule can be applied only to limits that give indeterminate forms, and $\lim_{x\to 0} \frac{\sin x}{1+2x}$ does not give an indeterminate form. Instead, this limit is $\frac{0}{1} = 0$, and the correct answer for the original limit is 0.

Remark:

L'Hôpital's Rule applies to one-sided limits as well. In following example, the one-sided limits are different.

Example:

a) $\lim_{x\to 0^+} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$= \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty$ **positive for $x > 0$**

b) $\lim_{x\to 0^-} \frac{\sin x}{x^2}$ $\qquad \frac{0}{0}$

$= \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$ **Negative for $x < 0$**

Remark:

Sometimes when we try to evaluate a limit as $x \to a$, we get an indeterminate form like $\infty/\infty, \infty \cdot 0, \infty - \infty$, instead of $0/0$. We first consider the form ∞/∞ .

More advanced treatments of calculus prove that l'Hôpital's Rule applies to the indeterminate form ∞/∞, as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided the limit on the right exists or approaches ∞ or $-\infty$. In the notation $x \to a$, a may be either finite or infinite. Moreover, $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

Example:

Find the limits of these ∞/∞ form $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$.

Solution:

The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x}$ **$\frac{\infty}{\infty}$ from left, apply l'Hôpital's Rule**

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x} = \lim_{x\to(\pi/2)^-} \sin x = 1.$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

Remark:

Next, we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or ∞/∞ form. Here again, we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

Example:

Find the limits of these $\infty \cdot 0$ form $\lim_{x\to\infty} x \sin\frac{1}{x}$.

Solution:

$$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \qquad \infty \cdot 0 \text{ converted to } \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{(\cos(1/x))(-1/x^2)}{-1/x^2} \qquad \text{L'Hôpital's Rule applied}$$

$$= \lim_{x\to\infty} \left(\cos\frac{1}{x}\right) = 1.$$

Example:

Find the limits of these $\infty - \infty$ form $\lim_{x\to0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Solution:

If $x \to 0^+$, then $\sin x \to 0^+$ and $\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty$.
Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$ **Common denominator is x sin x.**

Then we apply l'Hôpital's Rule to the result:

$$\lim_{x\to0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to0}\frac{x-\sin x}{x\sin x} \qquad \frac{0}{0}$$

$$= \lim_{x\to0}\frac{1-\cos x}{\sin x + x\cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x\to0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0 .$$

Remark:

Limits that lead to the indeterminate forms $1^\infty, 0^0$, and ∞^0 can sometimes be handled by first taking the logarithm of the function. We use l'Hôpital's Rule to find the limit of the logarithm expression and then exponentiate the result to find the original function limit. This procedure is justified by the continuity of the exponential function it is formulated as follows. (The formula is also valid for one-sided limits.)

If $\lim_{x\to a} ln\, f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

Example:

Apply l'Hôpital's Rule to show that $\lim_{x\to0^+}(1+x)^{1/x} = e$.

Solution:

The limit leads to the indeterminate form 1^∞. We let $f(x) = (1+x)^{1/x}$ and find $\lim_{x\to0^+} ln\, f(x)$. Since

$$ln\, f(x) = ln\,(1+x)^{1/x} = \frac{1}{x} ln\,(1+x),$$

l'Hôpital's Rule now applies to give

$$\lim_{x\to 0^+} ln\, f(x) = \lim_{x\to 0^+} \frac{ln\,(1+x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} \qquad \text{L'Hôpital's Rule applied}$$

$$= \frac{1}{1} = 1.$$

Therefore, $\lim_{x\to 0^+}(1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{ln\, f(x)} = e^1 = e$.

Example:

Find $\lim_{x\to\infty} x^{1/x}$.

Solution:

The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} ln\, f(x)$. Since

$$ln\, f(x) = ln\, x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\lim_{x\to\infty} ln\, f(x) = \lim_{x\to\infty} \frac{ln\, x}{x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} \qquad \text{L'Hôpital's Rule applied}$$

$$= \frac{1}{1} = 0.$$

Therefore, $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{ln\, f(x)} = e^0 = 1$.

Theorem (Cauchy's Mean Value Theorem):

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f\prime(c)}{g\prime(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Exercises:

1. In following, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2

a) $\lim_{x\to 2} \frac{x-2}{x^2-4}$ **b)** $\lim_{x\to 0} \frac{\sin 5x}{x}$ **c)** $\lim_{x\to\infty} \frac{5x^2-3x}{7x^2+1}$

d) $\lim_{x\to 1} \frac{x^3-1}{4x^3-x-3}$ **e)** $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ **f)** $\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1}$

2. Use l'Hôpital's rule to find the limits in following.

a) $\lim_{x\to -2} \frac{x+2}{x^2-4}$ b) $\lim_{x\to -5} \frac{x^2-25}{x+5}$ c) $\lim_{t\to -3} \frac{t^3-4t+15}{t^2-t-12}$

d) $\lim_{t\to -1} \frac{3t^3+3}{4t^3-t+3}$ e) $\lim_{x\to \infty} \frac{5x^3-2x}{7x^3+3}$ f) $\lim_{x\to \infty} \frac{x-8x^2}{12x^2+5x}$

g) $\lim_{t\to 0} \frac{\sin t^2}{t}$ h) $\lim_{t\to 0} \frac{\sin 5t}{2t}$ i) $\lim_{x\to 0} \frac{8x^2}{\cos x-1}$

j) $\lim_{x\to 0} \frac{\sin x-x}{x^3}$ k) $\lim_{\theta\to \pi/2} \frac{2\theta-\pi}{\cos(2\pi-\theta)}$ l) $\lim_{\theta\to -\pi/3} \frac{3\theta+\pi}{\sin(\theta+(\pi/3))}$

m) $\lim_{\theta\to \pi/6} \frac{\sin\theta-\frac{1}{2}}{\theta-\pi/6}$ n) $\lim_{\theta\to \pi/4} \frac{\tan\theta-1}{\theta-\pi/4}$ o) $\lim_{t\to 0} \frac{t\sin t}{1-\cos t}$

p) $\lim_{\theta\to \pi/2} \frac{1-\sin\theta}{1+\cos 2\theta}$ q) $\lim_{t\to 0} \frac{t(1-\cos t)}{t-\sin t}$ r) $\lim_{x\to (\pi/2)^-} \left(\frac{\pi}{2}-x\right)\tan x$

s) $\lim_{x\to (\pi/2)^-} \left(x-\frac{\pi}{2}\right)\sec x$ t) $\lim_{y\to 0} \frac{\sqrt{5y+25}-5}{y}$ u) $\lim_{y\to 0} \frac{\sqrt{ay+a^2}-a}{y}$, $a>0$

v) $\lim_{x\to 0^+} \left(\frac{3x+1}{x}-\frac{1}{\sin x}\right)$ w) $\lim_{x\to 0^+} (\csc x-\cot x+\cos x)$ x) $\lim_{x\to 0} \frac{x-\sin x}{x\tan x}$

y) $\lim_{\theta\to 0} \frac{\theta-\sin\theta\cos\theta}{\tan\theta-\theta}$ z) $\lim_{x\to 0} \frac{\sin 3x-3x+x^2}{\sin x\sin 2x}$

3. Find the limits in following.

a) $\lim_{x\to 1^+} x^{1/(1-x)}$ b) $\lim_{x\to 1^+} x^{1/(x-1)}$ c) $\lim_{x\to \infty} (\ln x)^{1/x}$

d) $\lim_{x\to e^+} (\ln x)^{1/(x-e)}$ e) $\lim_{x\to 0^+} x^{-1/\ln x}$ f) $\lim_{x\to \infty} x^{1/\ln x}$

g) $\lim_{x\to \infty} (1+2x)^{1/(2\ln x)}$ h) $\lim_{x\to 0} (e^x+x)^{1/x}$ i) $\lim_{x\to 0^+} x^x$

j) $\lim_{x\to 0^+} (1+\frac{1}{x})^x$ k) $\lim_{x\to \infty} (\frac{x+2}{x-1})^x$ l) $\lim_{x\to \infty} (\frac{x^2+1}{x+2})^{1/x}$

m) $\lim_{x\to 0^+} x^2\ln x$ n) $\lim_{x\to 0^+} x(\ln x)^2$ o) $\lim_{x\to 0^+} x\tan(\frac{\pi}{2}-x)$

p) $\lim_{x\to 0^+} \sin x\cdot\ln x$

4. Find all values of c that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a) $f(x) = x,\quad g(x) = x^2,\quad (a,b) = (-2,0)$

b) $f(x) = x,\quad g(x) = x^2,\quad (a,b)$ arbitrary

c) $f(x) = \frac{x^3}{3}-4x,\quad g(x) = x^2,\quad (a,b) = (0,3)$