Calculus I

First Semester

Lecturer 1

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Contents

References

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[2] Thomas, G. B., Weir, M. D., Hass, J. & Heil, G., "*Thomas' calculus* "(Vol. 14). Boston: Pearson, 2018.

[3] Thomas, G. B. "Calculus and analytic geometry." Massachusetts Institute of Technology, Massachusetts, USA, Addison-Wesley Publishing Company, ISBN: 0-201-60700-X (1992).

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Chapter One: Real Functions.

1.1 Functions and Their Graphs:

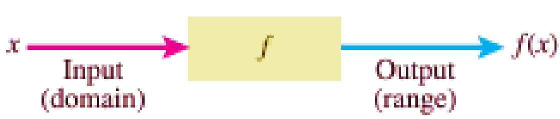
1.1.1 Domain and Range

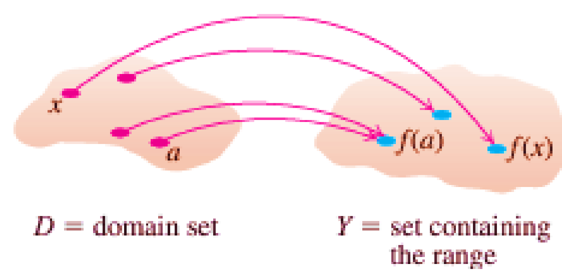
Definition:

A ***function*** f from a set D to a set Y ($f: D \to Y$) is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.

Remark:

1. The set D of all possible input values is called the ***domain*** of the function.
2. The set of all values of f(x) as x varies throughout D is called the ***range*** of the function.
3. The range may not include every element in the set Y.
4. The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers interpreted as points of a coordinate line.





Example:

Find the domain and range of each function:

1. $f(x) = x^2$
2. $f(x) = 1/x$
3. $f(x) = \sqrt{x}$
4. $f(x) = \sqrt{4 - x}$
5. $f(x) = \sqrt{1 - x^2}$

Solution:

1. The function $f(x) = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $f(x) = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, for $y = (\sqrt{y})^2$ for $y \geq 0$.

2. The function $f(x) = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, we cannot divide any number by zero. The range of $f(x) = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

3. The function $f(x) = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $f(x) = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

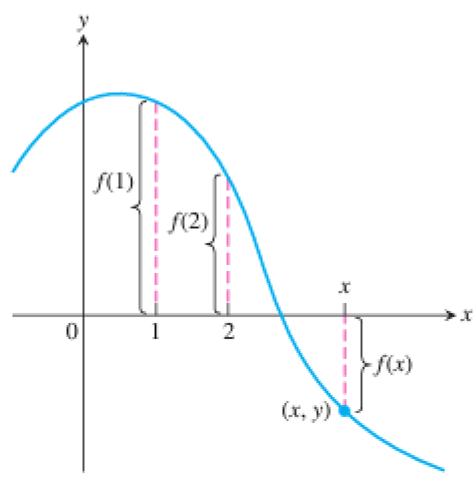
4. In $f(x) = \sqrt{4 - x}$ the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$ or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $f(x) = \sqrt{4 - x}$ is $[0, \infty)$ the set of all nonnegative numbers.

5. The function $f(x) = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from -1 to 1. Outside this domain, $1 - x^2$ is negative and its square root is not a real number. The values of $1 - x^2$ vary from 0 to 1 on the given domain, and the square roots of these values do the same. The range of $f(x) = \sqrt{1 - x^2}$ is [0, 1].

1.1.2 Graphs of Functions

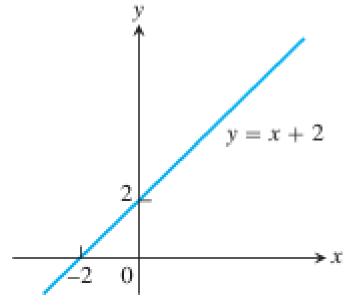
If f is a function with domain D, its ***graph*** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is $\{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}): \boldsymbol{x} \in \boldsymbol{D}\}$.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above the point x. The height may be positive or negative, depending on the sign of $f(x)$.



Example:

Graph the function $f(x) = x + 2$.



Solution:

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched.

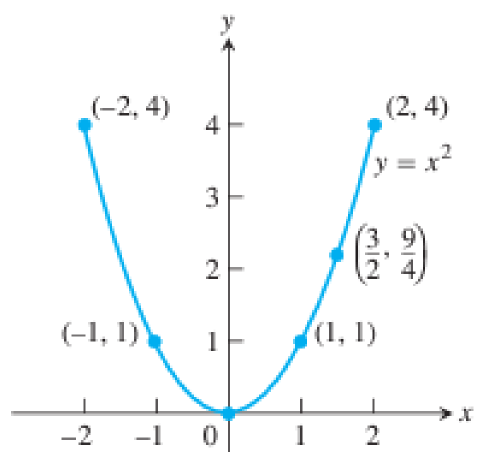
Example:

Graph the function $f(x) = x^2$. over the interval $[-2,2]$.

Solution:

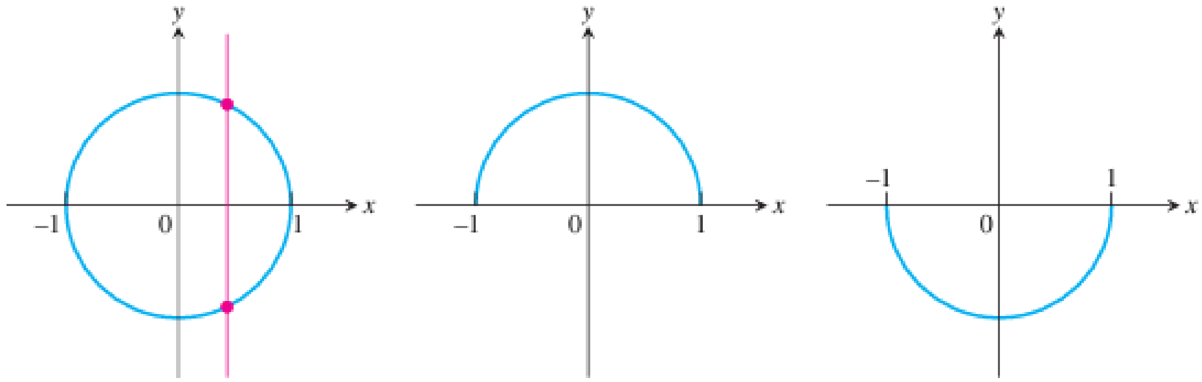
Fist we make a table of xy-pairs that satisfy the equation $y = x^2$. Then we plot the points (x, y) whose coordinates appear in the table, and we draw a smooth curve (labeled with its equation) through the plotted points.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



Remark:

Not every curve in the coordinate plane can be the graph of a function. A function f can have only one value $f(x)$ for each x in its domain, so no vertical line can intersect the graph of a function more than once. If a is in the domain of the function f, then the vertical line will intersect the graph of f at the single point $(a, f(a))$.



$x^2 + y^2 = 1$ $\qquad$ $f(x) = \sqrt{1 - x^2}$ $\qquad$ $g(x) = \sqrt{1 - x^2}$

A circle $x^2 + y^2 = 1$ cannot be the graph of a function since some vertical lines intersect the circle twice. The circle $x^2 + y^2 = 1$, however, does contain the graphs of two functions of *x*: the upper semicircle defined by the function defined by the function $f(x) = \sqrt{1 - x^2}$ and the lower semicircle $g(x) = \sqrt{1 - x^2}$.

Exercises:

1. Find the domain and range of each function

a. $f(x) = 1 + x^2$

b. $f(x) = 1 - \sqrt{x}$

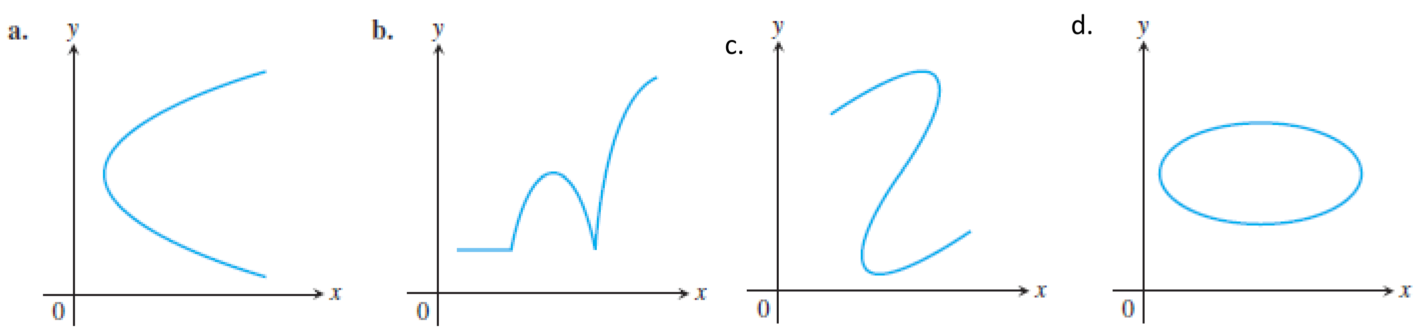
c. $F(x) = \sqrt{5x + 10}$

d. $g(x) = \sqrt{x^2 - 3x}$

e. $f(t) = 4/(3 - t)$

i. $G(t) = 2/(t^2 - 16)$

2. Which of the graphs are graphs of functions of *x*, and which are not? Give reasons for your answers.

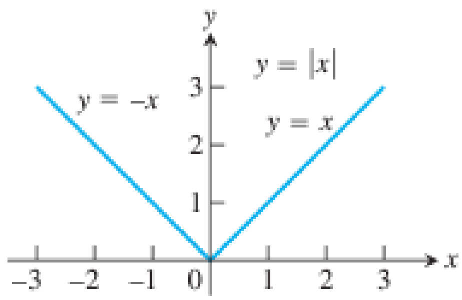


Remark(Piecewise-Defined Functions):

Sometimes a function is described by using different formulas different parts of its domain. For example, the function **absolute value function**

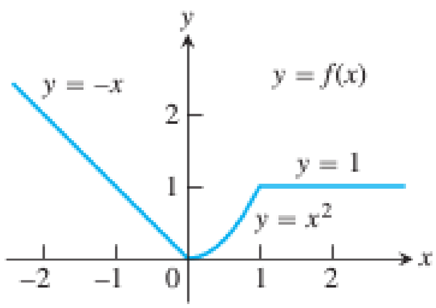
$$f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases},$$

has domain $(-\infty, \infty)$ and range $[0, \infty)$ whose graph is



Example:

The graph of the function $f(x) = \begin{cases} -x & if\ x < 0 \\ x^2 & if\ 0 \leq x \leq 1 \\ 1 & if\ x > 1 \end{cases}$ is



Example:

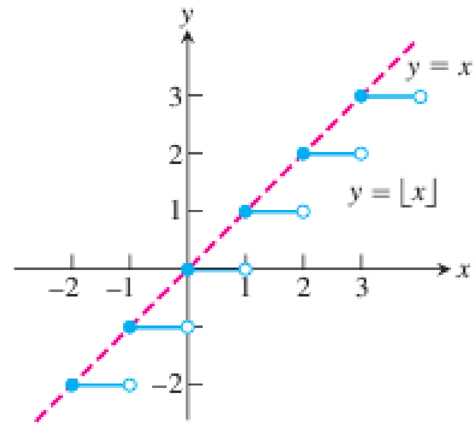
Graph the function **greatest integer function** $f(x) = \lfloor x \rfloor$.

Solution:

The function $f(x) = \lfloor x \rfloor$ is the function whose value at any number x is the greatest integer less than or equal to x.

$$\lfloor 2.4 \rfloor = 2 \quad \lfloor 1.9 \rfloor = 1 \quad \lfloor 0 \rfloor = 0 \quad \lfloor -1.2 \rfloor = -2$$
$$\lfloor 2 \rfloor = 2 \quad \lfloor 0.2 \rfloor = 0 \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2$$

The graph of the greatest integer function $f(x) = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for *x*.

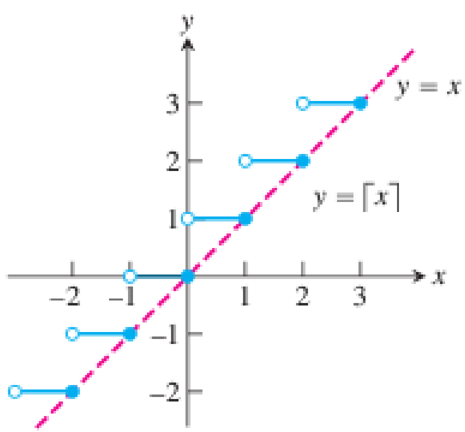


Example:

Graph the function **least integer function** $f(x) = \lceil x \rceil$.

Solution:

The function $f(x) = \lceil x \rceil$is the function whose value at any number *x* is the smallest integer greater than or equal to *x*. The graph of the least integer function $f(x) = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for *x*.



Exercises:

1. Find the natural domain and graph of the following functions

a. $f(x) = 5 - 2x$

b. $f(x) = 1 - 2x - x^2$

c. $g(x) = \sqrt{|x|}$

d. $g(x) = \sqrt{-x}$

e. $f(t) = t/|t|$

i. $G(t) = 1/|t|$

2. Find the domain of $f(x) = \frac{x+3}{4-\sqrt{x^2-9}}$.

3. Find the range of $f(x) = 2 + \frac{x^2}{x^2+4}$.

3. Graph the following equations and explain why they are not graphs of functions of x.

a. $|y| = x$

b. $y^2 = x^2$

4. Graph the following equations and explain why they are not graphs of functions of x.

a. $|x| + |y| = 1$

b. $|x + y| = 1$

5. Graph the following functions

a. $f(x) = \begin{cases} x, & if\ 0 \le x \le 1 \\ 2 - x, & if\ 1 < x \le 2 \end{cases}$

b. $g(x) = \begin{cases} 1 - x, & if\ 0 \le x \le 1 \\ 2 - x, & if\ 1 < x \le 2 \end{cases}$

c. $F(x) = \begin{cases} 4 - x^2, & if\ x \le 1 \\ x^2 - 2x, & if\ x > 1 \end{cases}$

d. $G(x) = \begin{cases} 1/x, & if\ x < 0 \\ x, & if\ 0 \le x \end{cases}$

6. For what values of x is

a. $\lfloor x \rfloor = 0$?

b. $\lceil x \rceil = 0$?

7. What real numbers x satisfies the equation $\lfloor x \rfloor = \lceil x \rceil$?

8. Does $\lceil -x \rceil = -\lfloor x \rfloor$; for all real x? Give reasons for your answer.

9. Graph the function $f(x) = \begin{cases} \lfloor x \rfloor, & if\ x \ge 0 \\ \lceil x \rceil, & if\ x < 0 \end{cases}$.

Definition:

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$whenever $x_1 < x_2$ then f is said to be ***increasing*** on I.

2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$ then $f$is said to be ***decreasing*** on I.

Example:

The function $f(x) = \begin{cases} -x & if\ x < 0 \\ x^2 & if\ 0 \le x \le 1 \\ 1 & if\ x > 1 \end{cases}$ is decreasing on $(-\infty, 0]$ and increasing on [0, 1]. The function is neither increasing nor decreasing on the interval $[1, \infty)$ because of the strict inequalities used to compare the function values in the definitions.

Definition:

A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,

odd function of x if $f(-x) = -f(x)$,

for every *x* in the function's domain.

Remark:

1. The graph of an even function is ***symmetric about the y-axis***. Since $f(-x) = f(x)$ a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph. A reflection across the y-axis leaves the graph unchanged.
2. The graph of an odd function is ***symmetric about the origin***. Since $f(-x) = -f(x)$ a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph. Equivalently, a graph is symmetric about the origin if a rotation of 180°about the origin leaves the graph unchanged.
3. Notice that the definitions imply that both x and $-x$ must be in the domain of f.

Example:

Specify whether the function is even, odd, or neither. Give reasons for your answer

a. $f(x) = x^2$

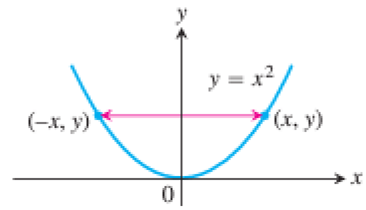
b. $f(x) = x^2 + 1$

d. $g(x) = x$

e. $f(x) = x + 1$

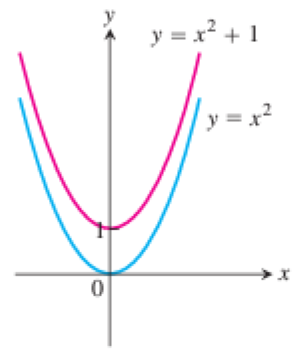
Solution:

a. $f(x) = x^2$



Since $f(-x) = (-x)^2 = f(x)$ then f is an even function for all x; symmetry about y-axis.

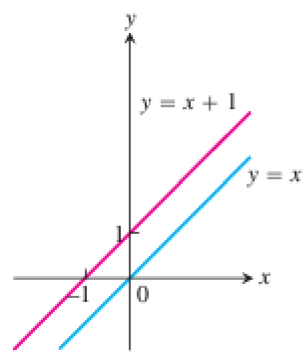
b. $f(x) = x^2 + 1$



Since $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$

then f is an even function for all x; symmetry about y-axis.

d. $f(x) = x$



Since $f(-x) = -x = -f(x)$ then f is an odd function for all x; symmetry about origin.

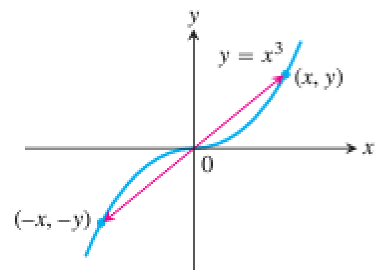
e. $f(t) = x + 1$

Since $f(-x) = -x + 1$ but $-f(x) = -x - 1$. The two are not equal. then f is not odd function for all x. Not even since $f(-x) = -x + 1 \neq f(x) = x + 1$ for all $x \neq 0$.

Remark:

The names even and odd come from powers of x. If $f(x)$ is an even power of x, as in $f(x) = x^2$ or $f(x) = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$.

If $f(x)$ is an odd power of x, as in $f(x) = x$ or $f(x) = x^3$, it is an odd function of x because $(-x)^1 = -x$ and $(-x)^3 = -x^3$



Exercises:

1. Graph the following functions. What symmetries, if any, do the graphs have? Specify the intervals over which the function is increasing and the intervals where it is decreasing.

a. $f(x) = -x^3$
b. $f(x) = -1/x^2$
c. $f(x) = -1/x$
d. $f(x) = 1/|x|$
e. $f(x) = \sqrt{|x|}$
f. $f(x) = \sqrt{-x}$
g. $f(x) = x^3/8$
h. $f(x) = -4\sqrt{x}$
i. $f(x) = -x^{3/2}$
j. $f(x) = (-x)^{2/3}$

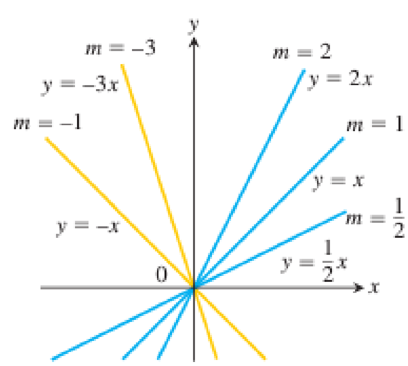
2. Whether the following functions is even, odd, or neither. Give reasons for your answer

a. $f(x) = 3$
b. $f(x) = x^{-5}$
c. $f(x) = x^2 + 1$
d. $f(x) = x^2 + x$
e. $g(x) = x^3 + x$
f. $g(x) = x^4 + 3x^2 - 1$
g. $g(x) = 1/(x^2 - 1)$
h. $g(x) = x/(x^2 - 1)$
i. $h(t) = 1/(t - 1)$
k. $h(t) = |t^3|$
g. $h(t) = 2t + 1$
h. $h(t) = 2|t| + 1$

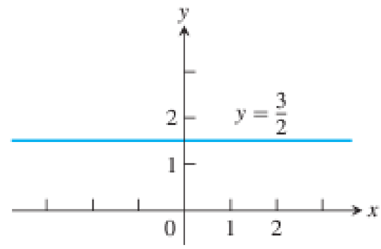
1.1.3 Common Functions

A variety of important types of functions are frequently encountered in calculus. We identify and briefly describe them here.

1. **Linear Functions:** A function of the form $f(x) = mx + b$, for constants m and b, is called a ***linear function***. When $b = 0$,the array of lines $f(x) = mx$ is lines pass through the origin.



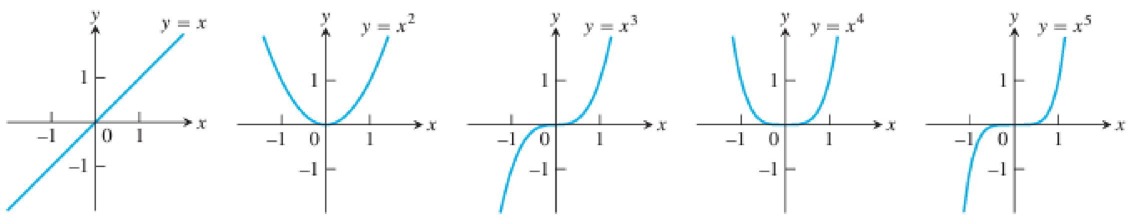
The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the ***identity function***. The function $f(x) = b$ where and $m = 0$ is called the ***Constant functions.***



A linear function with positive slope whose graph passes through the origin is called a ***proportionality*** relationship.

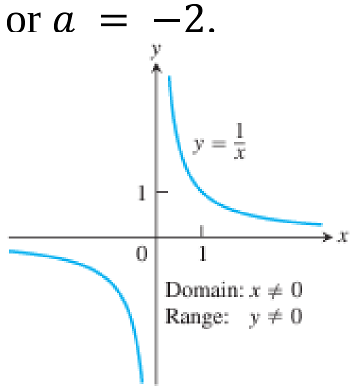
2. **Power Functions:** A function $f(x) = x^a$, where a is a constant, is called a ***power function***. There are several important cases to consider.

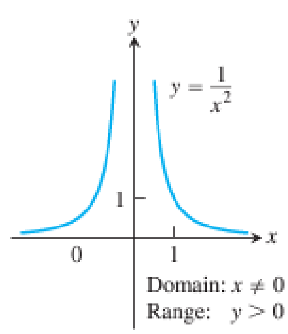
(a) *a* = *n*, a positive integer.



The graphs of $y = f(x) = x^n$, for n = 1, 2, 3, 4, 5 are defined for all real values of x (i.e., $-\infty < x < \infty$). Notice that as the power n gets larger, the curves tend to flatten toward the x-axis on the interval (-1, 1), and to rise more steeply for $|x| > 1$. Each curve passes through the point (1, 1) and through the origin. The graphs of functions with even powers are symmetric about the y-axis; those with odd powers are symmetric about the origin. The even-powered functions are decreasing on the interval $(-\infty, 0]$ and increasing on $[0, \infty)$; the odd-powered functions are increasing over the entire real line $(-\infty, \infty)$.

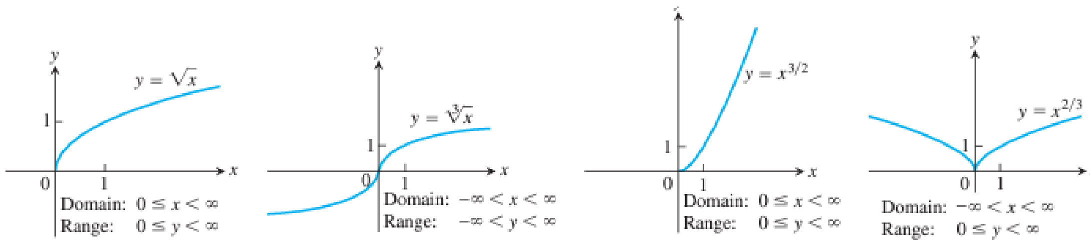
(b) $a = -1$ or $a = -2$.





The functions $f(x) = x^{-1} = \frac{1}{x}$ and $g(x) = x^{-2} = \frac{1}{x^2}$ are defined for all $x \neq 0$ (you can never divide by zero). The graph of $f(x) = \frac{1}{x}$ is the hyperbola $xy = 1$, which approaches the coordinate axes far from the origin. The graph of $g(x) = \frac{1}{x^2}$ also approaches the coordinate axes. The graph of the function f is symmetric about the origin; f is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$. The graph of the function g is symmetric about the y-axis; g is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

(c) $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ and $\frac{2}{3}$

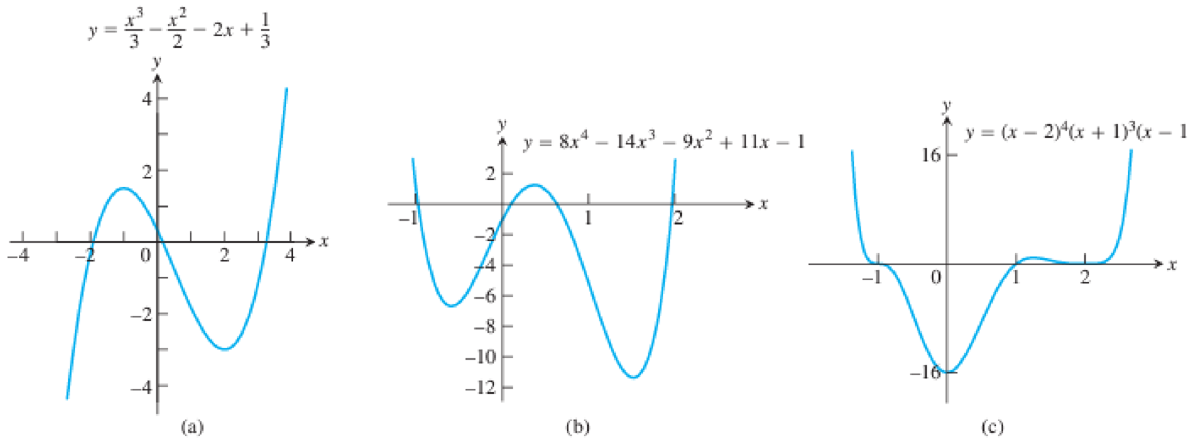


The functions $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ and $g(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ are the ***square root*** and ***cube root*** functions, respectively. The domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x. Their graphs are displayed above, along with the graphs of $f(x) = x^{\frac{3}{2}}$ and $f(x) = x^{\frac{2}{3}}$. (Recall that $x^{\frac{3}{2}} = \left(x^{\frac{1}{2}}\right)^3$ and $x^{\frac{2}{3}} = \left(x^{\frac{1}{3}}\right)^2$.

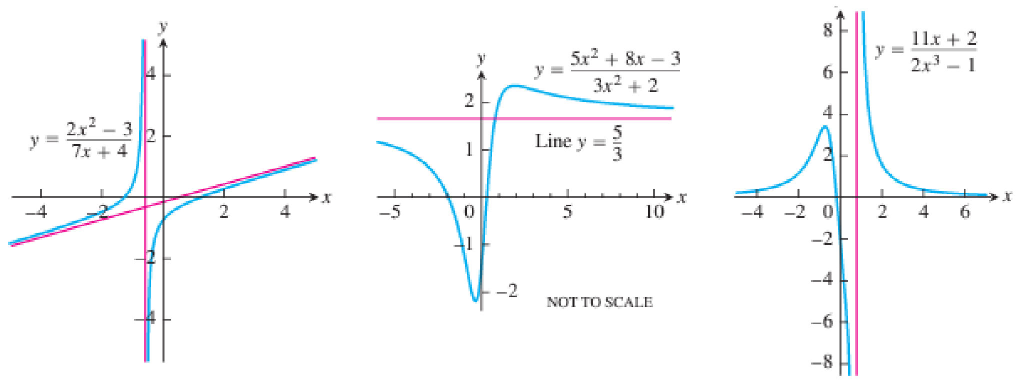
3. Polynomials: A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

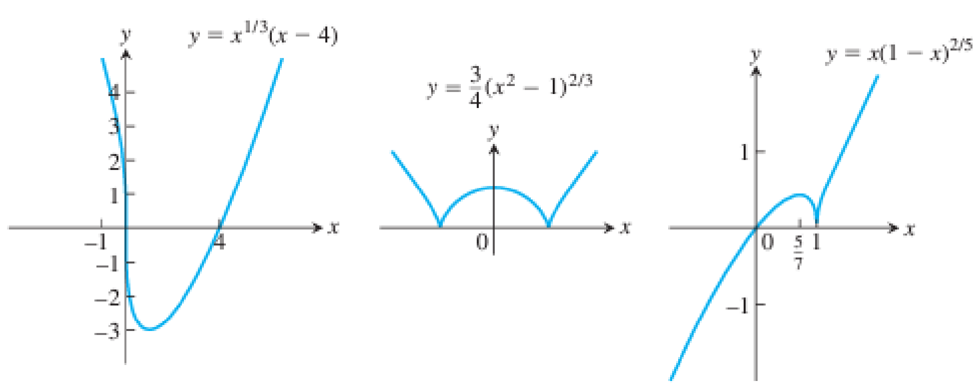
where n is a nonnegative integer and the numbers $a_0, a_1, \cdots, a_n$, are real constants (called the ***coefficients*** of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n > 0$, then n is called the ***degree*** of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x) = ax^2 + bx + c$, are called ***quadratic functions***. Likewise, ***cubic functions*** are polynomials $p(x) = ax^3 + bx^2 + cx + d$ of degree 3.



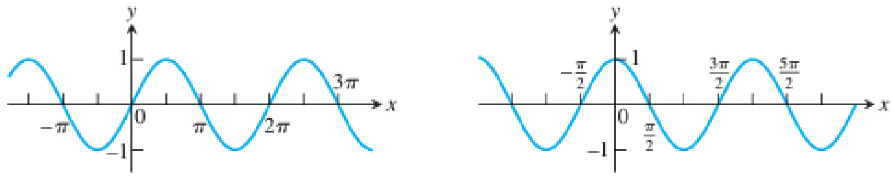
4. **Rational Functions:** A rational function is a quotient or ratio $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$.



5. **Algebraic Functions**: Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) lies within the class of ***algebraic functions***. All rational functions are algebraic, but also included are more complicated functions (such as those satisfying an equation like $y^3 - 9xy + x^3 = 0$.

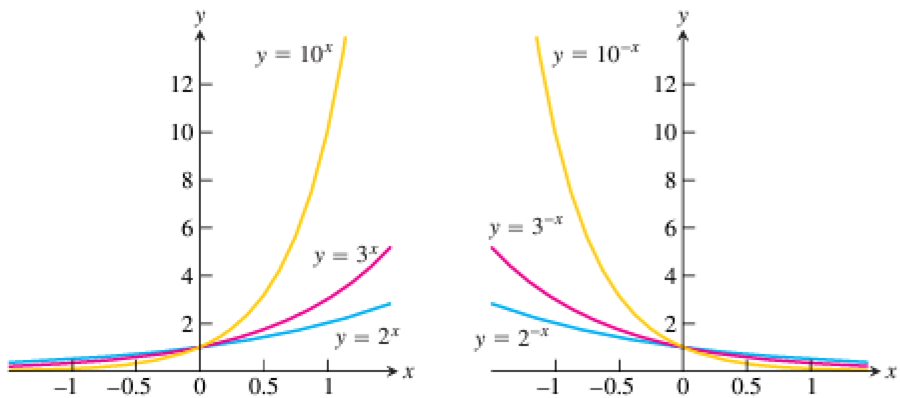


6. **Trigonometric Functions**: The six basic trigonometric functions are sine: sin, cosine, tan, cosecant, secant, cotan.



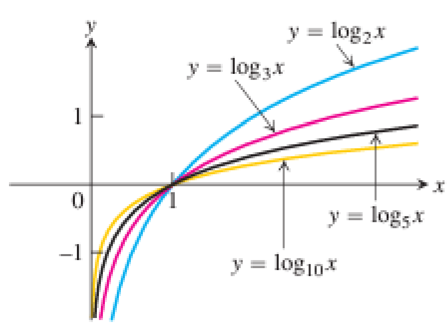
(a) $f(x) = \sin x$ (b) $f(x) = \cos x$

7. **Exponential Functions:** Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called exponential functions. All ***exponential functions*** have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0.

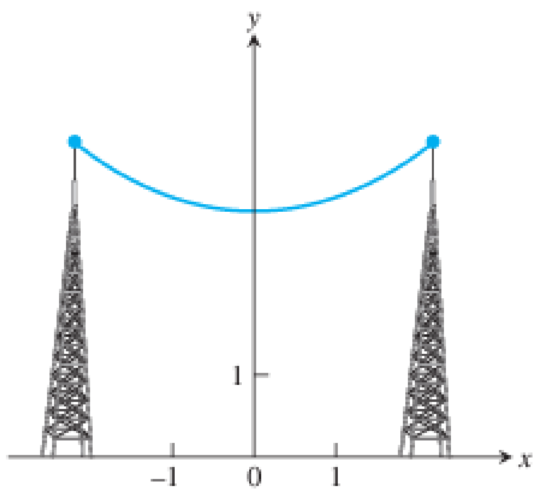


(a) (b)

8. **Logarithmic Functions:** These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the inverse functions of the exponential functions. The following Figure shows the graphs of four logarithmic functions with various bases. In each case the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.



9. **Transcendental Functions:** These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well. A particular example of a transcendental function is a ***catenary*** (The Latin word catena means "chain"). Its graph has the shape of a cable, like a telephone line or electric cable, strung from one support to another and hanging freely under its own weight.



Calculus I

First Semester

Lecturer 2

Dr. Ban Jaffar AL-Taiy

Taghreed Hussein Abed

1.2 Combining Functions; Shifting and Scaling Graphs

1.2.1 Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

If f and g are functions, then for every x that belongs to the domains of both f and g (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, f - g, and $f \cdot g$ by the formulas

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x).$$

Notice that the + sign on the left-hand side of the first equation represents the operation of addition of functions, whereas the + on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function f/g by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \qquad \text{(Where } g(x) \neq 0\text{).}$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by

$$(cf)(x) = cf(x).$$

Example:

Find the domains and ranges of $f, g, f + g, f - g, g - f, f \cdot g, f/g$, and g/f defined by the formulas $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$.

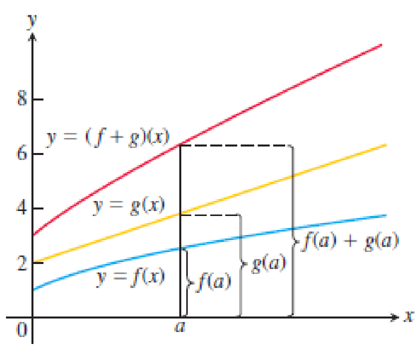
Solution:

The functions defined by the formulas $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$

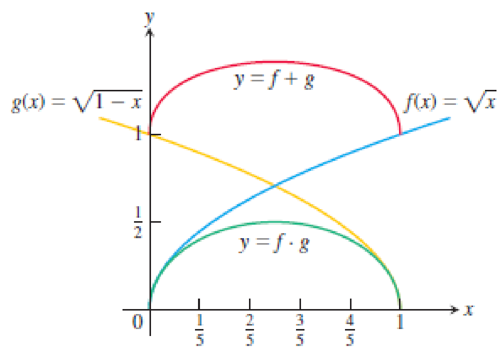
have domains $D(f) = [0,\infty)$ and $D(g) = (-\infty,1]$. The points common to these domains are the points $[0,\infty) \cap (-\infty,1] = [0,1]$.

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions.

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0,1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g - f,$	$(g - f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f \cdot g,$	$(f \cdot g)(x) = f(x) \cdot g(x) = \sqrt{x(1-x)},$	$[0,1]$
$f/g,$	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x = 1$ excluded)
g/f	$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x = 0$ excluded)



The graph of the function $f + g$



The graph of the function $f \cdot g$

Exercises:

1. Find the domains and ranges of $f, g, f + g$ and $f \cdot g$ defined by the formulas

a. $f(x) = x$, $g(x) = \sqrt{x-1}$ b. $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{x-1}$

2. Find the domains and ranges of $f, g, f/g,$ and g/f defined by the formulas

a. $f(x) = 2$, $g(x) = x^2 + 1$ b. $f(x) = 1$, $g(x) = 1 + \sqrt{x}$

1.2.2 Composite Functions

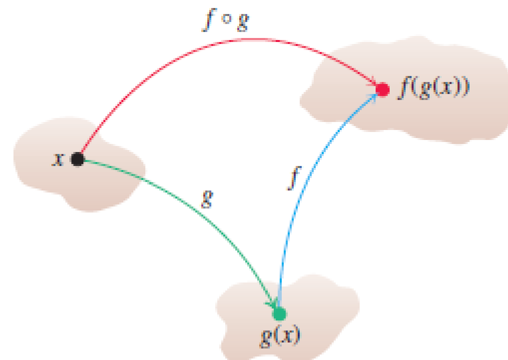
Definition:

If f and g are functions, the composite function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f.

A composite function $f \circ g$ uses the output $g(x)$ of the first function g as the input for the second function f.



Arrow diagram for $f \circ g$. If x lies in the domain of g and $g(x)$ lies in the domain of f, then the functions f and g can be composed to form $(f \circ g)(x)$.

Remark:

1. To evaluate the composite function $g \circ f$ (when defined), we find f(x) first and then $g(f(x))$. The domain of $g \circ f$ is the set of numbers x in the domain of f such that f(x) lies in the domain of g.
2. The functions $f \circ g$ and $g \circ f$ are usually quite different.

Example:

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$, **(b)** $(g \circ f)(x)$, **(c)** $(f \circ f)(x)$, **(d)** $(g \circ g)(x)$.

Solution:

Composite | **Domain**

Composite	Domain
(a) $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
(b) $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c) $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d) $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x+1) + 1 = x + 2$	$(-\infty, \infty)$

Remark:

1. In previous example the domain of $f \circ g$ is $[-1, \infty)$, since $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$.
2. If $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

Exercises:

1. If $f(x) = x + 5$ and $g(x) = x^2 - 3$, Find the following.

a) $f(g(0))$ **b)** $g(f(0))$ **c)** $f(g(x))$ **d)** $g(f(x))$
e) $f(f(-5))$ **f)** $g(g(2))$ **g)** $f(f(x))$ **h)** $g(g(x))$

2. If $f(x) = x - 1$ and $g(x) = 1/(x+1)$, Find the following.

a) $f(g(1/2))$ **b)** $g(f(1/2))$ **c)** $f(g(x))$ **d)** $g(f(x))$
e) $f(f(2))$ **f)** $g(g(2))$ **g)** $f(f(x))$ **h)** $g(g(x))$

3. For the following write a formula for $f \circ g \circ h$.

a) $f(x) = x + 1, g(x) = 3x$, $h(x) = 4 - x$ **b)** $f(x) = 3x + 4, g(x) = 2x - 1$, $h(x) = x^2$

c) $f(x) = \sqrt{x+1}, g(x) = \frac{1}{x+4}$, $h(x) = \frac{1}{x}$ **d)** $f(x) = \frac{x+2}{3-x}, g(x) = \frac{x^2}{x^2+1}$, $h(x) = \sqrt{2-x}$

4. Let $f(x) = x - 3, g(x) = \sqrt{x}, h(x) = x^3$, and $j(x) = 2x$. Express each of the functions in following as a composite involving one or more of f, g, h, and j.

1. **a)**$y = \sqrt{x} - 3$ **b)**$y = 2\sqrt{x}$ **c)**$y = x^{1/4}$
d)$y = 4x$ **e)**$y = \sqrt{(x-3)^3}$ **f)**$y = (2x-6)^3$

2. **a)** $y = 2x - 3$ **b)** $y = x^{3/2}$ **c)** $y = x^9$
d) $y = x - 6$ **e)** $y = 2\sqrt{x-3}$ **f)** $y = \sqrt{x^3 - 3}$

5. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x-7$	$\sqrt{x}$	?
b.	$x+2$	$3x$	?
c.	?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
d.	$\frac{x}{x-1}$	$\frac{x}{x-1}$	?
e.	?	$1+\frac{1}{x}$	x
f.	$\frac{1}{x}$	?	x

6. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$\frac{1}{x-1}$	$\lvert x\rvert$	?
b.	?	$\frac{x-1}{x}$	$\frac{x}{x+1}$
c.	?	$\sqrt{x}$	$\lvert x\rvert$
d.	$\sqrt{x}$	?	$\lvert x\rvert$

7. Evaluate each expression using the given table of values:

x	-2	-1	0	1	2
$f(x)$	1	0	-2	1	2
$g(x)$	2	1	0	-1	0

a) $f(g(-1))$ **b)** $g(f(0))$ **c)** $f(f(-1))$
d) $g(g(2))$ **e)** $g(f(-2))$ **f)** $f(g(1))$

8. Evaluate each expression using the functions

$$f(x) = 2 - x, \qquad g(x) = \begin{cases} -x & -2 \le x < 0, \\ x-1 & 0 \le x \le 2. \end{cases}$$

a) $f(g(0))$ **b)** $g(f(3))$ **c)** $g(g(-1))$
d) $f(f(2))$ **e)** $g(f(0))$ **f)** $f(g(1/2))$

9. write formulas for $f \circ g$ and $g \circ f$ and find the domain and range of each.

a. $f(x) = \sqrt{x+1}$, $g(x) = 1/x$ b. $f(x) = x^2$, $g(x) = 1 - \sqrt{x}$.

10. Let $f(x) = \frac{x}{x-2}$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x$.

11. Let $f(x) = 2x^3 - 4$. Find a function $y = g(x)$ so that $(f \circ g)(x) = x + 2$.

1.2.3 Shifting a Graph of a Function:

A common way to obtain a new function from an existing one is by adding a constant to each output of the existing function, or to its input variable. The graph of the new function is the graph of the original function shifted vertically or horizontally, as follows.

Shift Formulas

Vertical Shifts

$y = f(x) + k$ — Shifts the graph of f up k units if $k > 0$
Shifts it down $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ — Shifts the graph of f left h units if $h > 0$
Shifts it right $|h|$ units if $h < 0$

Example:

(a) Shifts the graph of $f(x) = x^2$ up 1 unit.

(b) Shifts the graph of $f(x) = x^2$ down 2 unit.

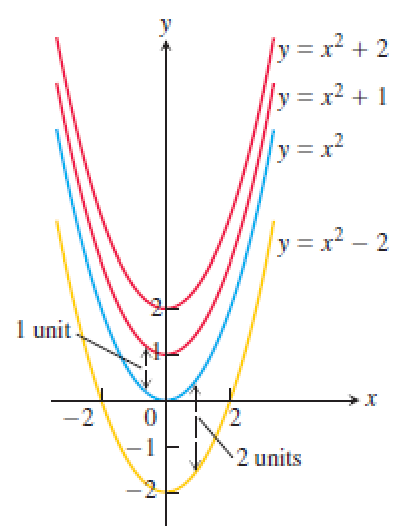
(c) Shifts the graph of $f(x) = x^2$ left 3 unit.

(d) Shifts the graph of $f(x) = x^2$ right 2 unit.

(e) Shifts the graph of $f(x) = |x|$ right 2 unit and down 2 unit.

Solution:

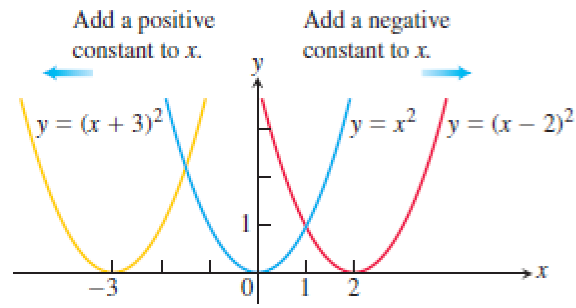
(a) Adding 1 to the right-hand side of the formula $f(x) = x^2$ to get $f(x) = x^2 + 1$ shifts the graph up 1 unit.



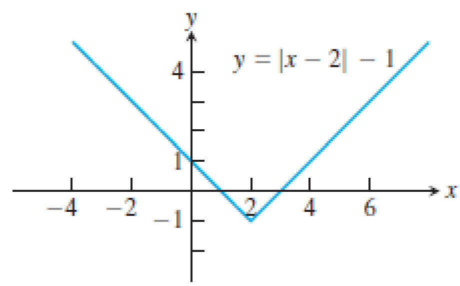
(b) Adding -2 to the right-hand side of the formula $f(x) = x^2$ to get $f(x) = x^2 - 2$ shifts the graph down 2 units.

(c) Adding 3 to x in $f(x) = x^2$ to get $f(x) = (x+3)^2$shifts the graph 3 units to the left.

(d) Adding -2 to x in $f(x) = x^2$ to get $f(x) = (x-2)^2$shifts the graph 2 units to the right.



(e) Adding -2 to x in $f(x) = |x|$, and then adding -1 to the result, gives $f(x) = |x-2| - 1$ and shifts the graph 2 units to the right and 1 unit down.



1.2.4 Scaling and Reflecting a Graph of a Function:

To scale the graph of a function y = f(x) is to stretch or compress it, vertically or horizontally. This is accomplished by multiplying the function f, or the independent variable x, by an appropriate constant c. Reflections across the coordinate axes are special cases where $c = 1$.

Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c}f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = cf(x/c)$	Stretches the graph of f horizontally by a factor of c.

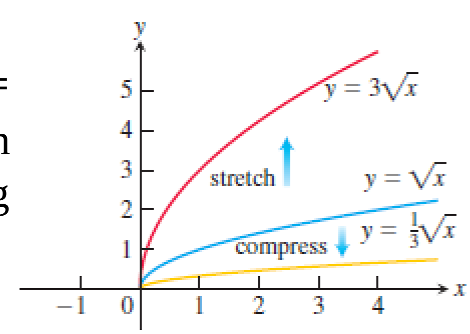
For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

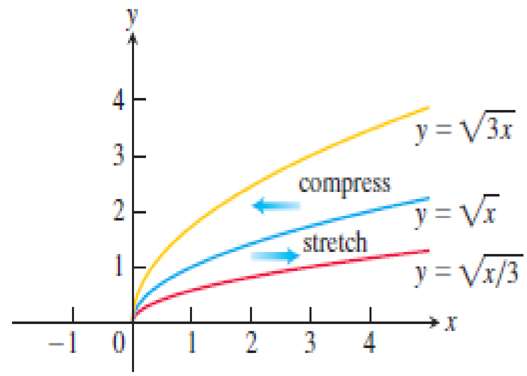
Example:

Here we scale and reflect the graph of $y = \sqrt{x}$.

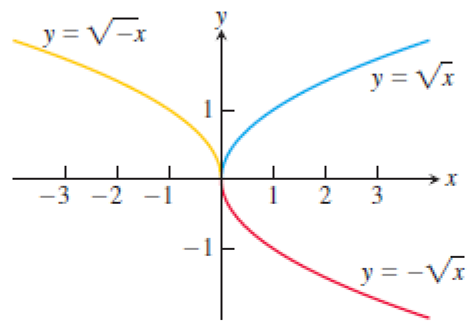
(a) Vertical: Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get y = 32x stretches the graph vertically by a factor of 3, whereas multiplying by 1/3 compresses the graph by a factor of 3.



(b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3. Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression may correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

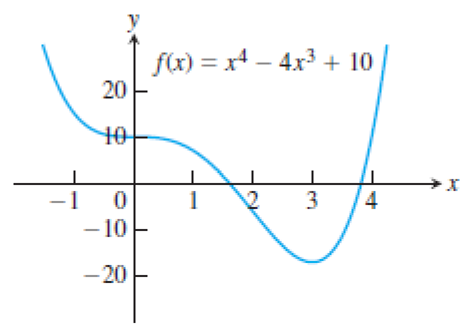


(c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$. is a reflection across the y-axis.



Example:

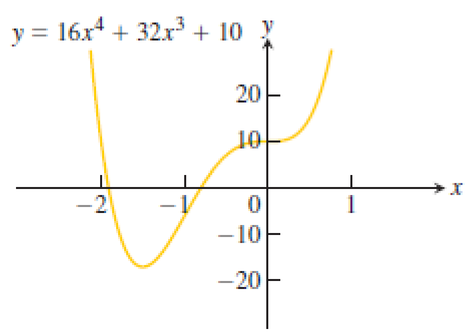
Given the function $f(x) = x^4 - 4x^3 + 10$, find formulas to



(a) Compress the graph horizontally by a factor of 2 followed by a reflection across the *y*-axis.

(b) Compress the graph vertically by a factor of 2 followed by a reflection across the *x*-axis.

Solution:

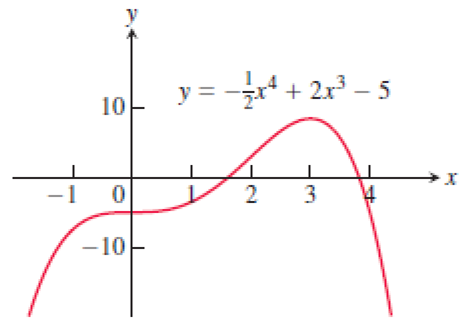


(a) We multiply x by 2 to get the horizontal compression, and by -1 to give reflection across the *y*-axis. The formula is obtained by substituting -2*x* for x in the right-hand side of the equation for f:

$$y = f(-2x) = (-2x)^4 - 4(-2x)^3 + 10$$
$$= 16x^4 + 32x^3 + 10.$$

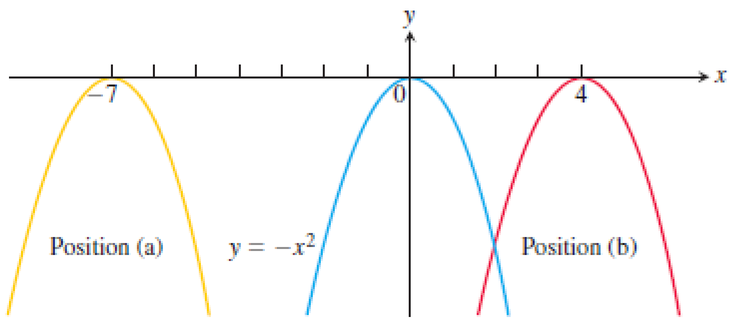
(b) The formula is

$$y = \tfrac{1}{2}f(x) = \frac{1}{2}x^4 + 2x^3 - 5.$$

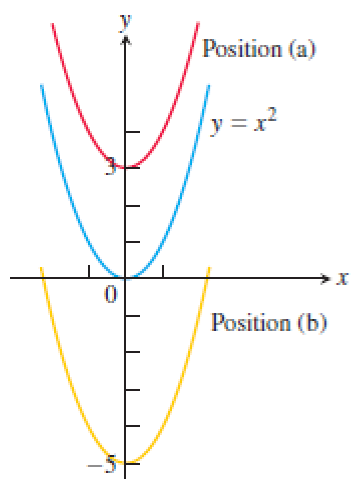


Exercises:

1. The accompanying figure shows the graph of $y = -x^2$ shifted to two new positions. Write equations for the new graphs.

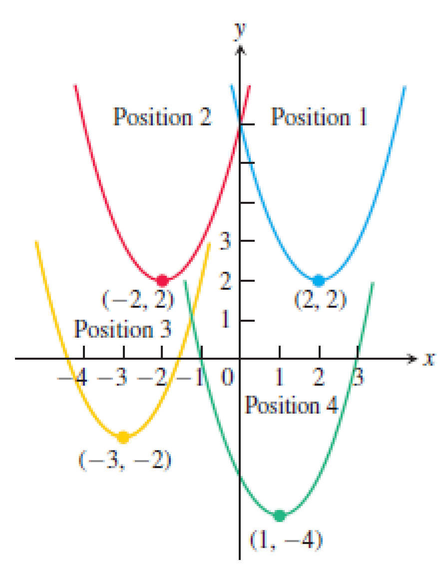


2. The accompanying figure shows the graph of $y = x^2$ shifted to two new positions. Write equations for the new graphs.

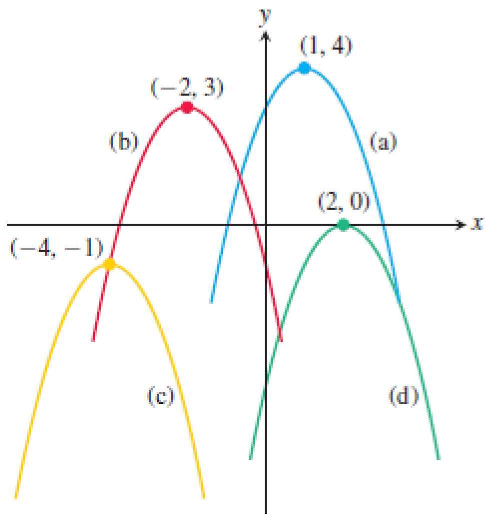


3. Match the equations listed in parts (a)–(d) to the graphs in the accompanying figure.

a. $y = (x - 1)^2 - 4$
b. $y = (x - 2)^2 + 2$
c. $y = (x + 2)^2 + 2$
d. $y = (x + 3)^2 - 2$



4. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



5. How many units and in what directions the graphs of the given equations are to be shifted. Give an equation for the shifted graph. Then sketch the original and shifted graphs together, labeling each graph with its equation.

a. $x^2 + y^2 = 49$ Down 3, left 2 **b.** $x^2 + y^2 = 25$ Up 3, left 4

c. $y = x^3$ Left 1, down 1 **d.** $y = x^{2/3}$ Right 1, down 1

e. $y = \sqrt{x}$ Left 0.81 **f.** $y = -\sqrt{x}$ Right 3

g. $y = 2x - 7$ Up 7 **h.** $y = \frac{1}{2}(x + 1) + 5$ Down 5, right 1

i. $y = 1/x$ Up 1, right 1 **j.** $y = 1/x^2$ Left 2, down 1

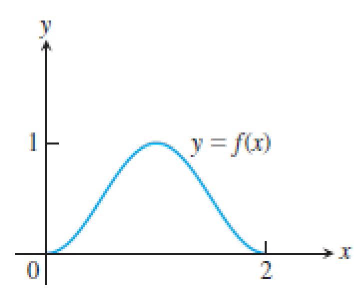
6. The accompanying figure shows the graph of a function $f(x)$ with domain $[0, 2]$ and range $[0, 1]$. Find the domains and ranges of the following functions, and sketch their graphs.

a. $f(x) + 2$ **b.** $f(x) - 1$

c. $2f(x)$ **d.** $-f(x)$

e. $f(x + 2)$ **f.** $f(x - 1)$

g. $f(-x)$ **h.** $-f(x + 1) + 1$



7. By what factor and direction, the graphs of the given functions are to be stretched or compressed. Give an equation for the stretched or compressed graph.

a. $y = x^2 - 1$, stretched vertically by a factor of 3
b. $y = x^2 - 1$, compressed horizontally by a factor of 2
c. $y = 1 + 1/x^3$, compressed vertically by a factor of 2
d. $y = 1 + 1/x^3$, stretched horizontally by a factor of 3
e. $y = \sqrt{x+1}$, compressed horizontally by a factor of 4
f. $y = \sqrt{x+1}$, stretched vertically by a factor of 3
g. $y = \sqrt{4 - x^2}$, stretched horizontally by a factor of 2
h. $y = \sqrt{4 - x^2}$, compressed vertically by a factor of 3
i. $y = 1 - x^3$, compressed horizontally by a factor of 3
j. $y = 1 - x^3$, stretched horizontally by a factor of 2

8. Assume that f is an even function, g is an odd function, and both f and g are defined on the entire real line $(-\infty, \infty)$. Which of the following (where defined) are even? odd?

a. $f \cdot g$	b. f/g	c. g/f
d. $f^2 = f \cdot f$	e. $g^2 = g \cdot g$	f. $f \circ g$
g. $g \circ f$	h. $f \circ f$	i. $g \circ g$

Calculus I

First Semester

Lecturer 3

Dr. Ban Jaffar AL-Taiy
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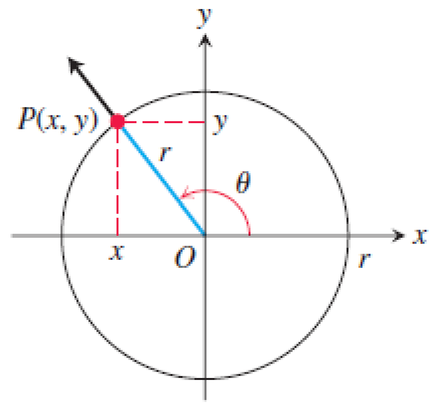
1.3 Trigonometric Functions

1.3.1 The Six Basic Trigonometric Functions

The trigonometric functions of a general angle θ are defined in terms of x, y, and r.



sine: $\sin\theta = \frac{y}{r}$ **cosecant:** $\csc\theta = \frac{r}{y}$

cosine: $\cos\theta = \frac{x}{r}$ **secant:** $\sec\theta = \frac{r}{x}$

tangent: $\tan\theta = \frac{y}{x}$ **cotangent:** $\cot\theta = \frac{x}{y}$

Notice also that

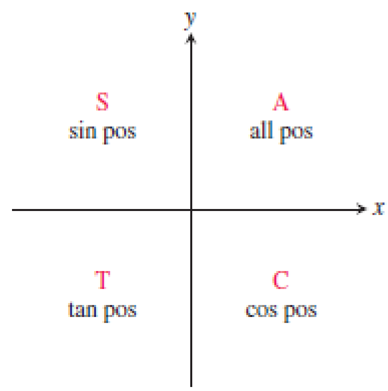
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Remark:

1. $\tan\theta$ and $\sec\theta$ are not defined if $x = \cos\theta = 0$. This means they are not defined if θ is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which y = 0, namely $\theta = \pm\pi, \pm 2\pi, \cdots$.

2. The CAST rule is useful for remembering when the basic trigonometric functions are positive or negative.



3. The following table shows the Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ.

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Exercises:

1. Copy and complete the following tables of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

a)

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

b)

θ	$-3\pi/2$	$-\pi/3$	$-\pi/6$	$\pi/4$	$5\pi/6$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

2. In following, one of $sin\ x$, $cos\ x$, and $tan\ x$ is given. Find the other two if x lies in the specified interval.

a) $sin\ x = \frac{3}{5}, x \in [\frac{\pi}{2}, \pi]$

b) $tan\ x = 2, x \in [0, \frac{\pi}{2}]$

c) $cos\ x = \frac{1}{3}, x \in [-\frac{\pi}{2}, 0]$

d) $cos\ x = -\frac{5}{13}, x \in [\frac{\pi}{2}, \pi]$

e) $tan\ x = \frac{1}{2}, x \in [\pi, \frac{3\pi}{2}]$

f) $sin\ x = -\frac{1}{2}, x \in [\pi, \frac{3\pi}{2}]$

1.3.2 Periodicity and Graphs of the Trigonometric Functions

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values: $sin\,(\theta + 2\pi) = sin\,\theta$, $tan\,(\theta + 2\pi) = tan\,\theta$, and so on. Similarly, $cos\,(\theta - 2\pi) = cos\,\theta$, $sin\,(\theta - 2\pi) = sin\,\theta$, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are periodic.

Definition:

A function $f(x)$ is periodic if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the period of f.

Remark:

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by x instead of θ. The tangent and cotangent functions have period $p = \pi$, and the other four functions have period 2π.

Periods of Trigonometric Functions

Period π	$tan\,(x + \pi) = tan\,x$
	$cot\,(x + \pi) = cot\,x$
Period 2π	$sin\,(x + 2\pi) = sin\,x$
	$cos\,(x + 2\pi) = cos\,x$
	$sec\,(x + 2\pi) = sec\,x$
	$csc\,(x + 2\pi) = csc\,x$

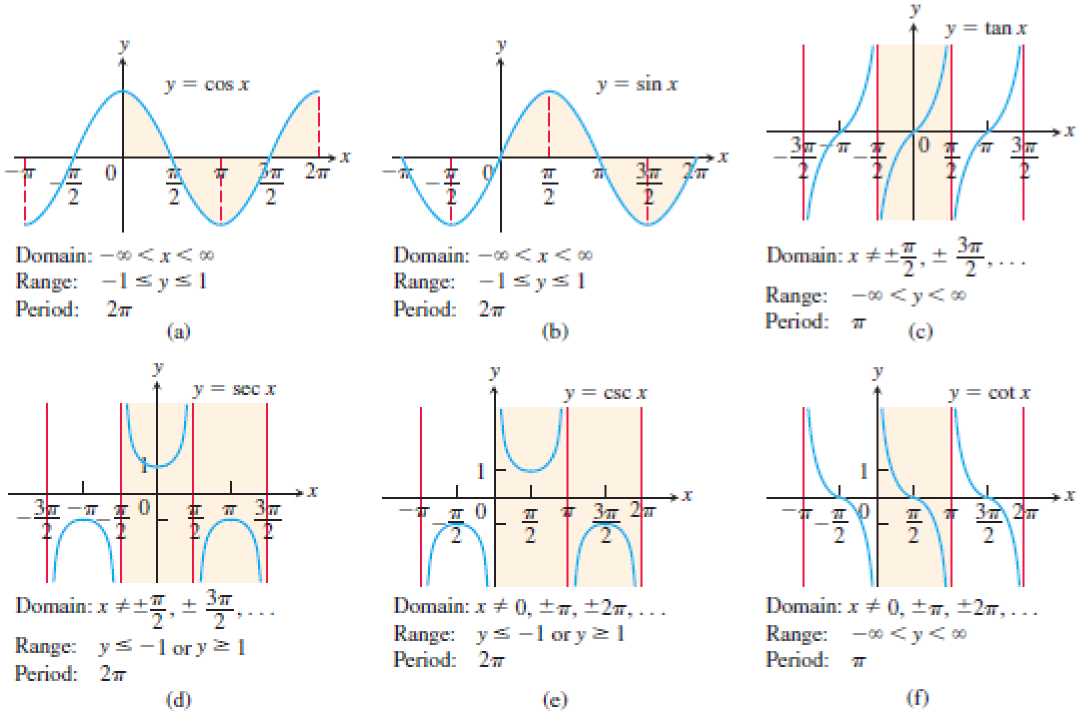
Also, the symmetries of their graphs reveal that the cosine and secant functions are even and the other four functions are odd.

Even
$cos\,(-x) = cos\,x$
$sec\,(-x) = sec\,x$

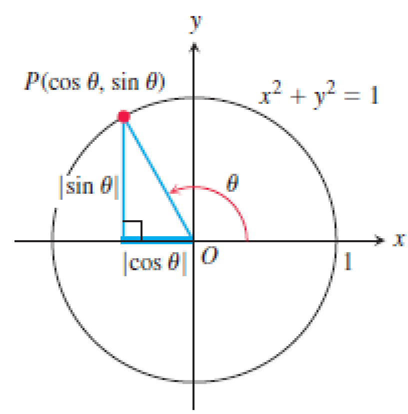
Odd
$sin\,(-x) = -sin\,x$
$tan\,(-x) = -tan\,x$
$csc\,(-x) = -csc\,x$
$cot\,(-x) = -cot\,x$

Remark:

The following graphs are graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.



1.3.3 Trigonometric Identities



The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle u that ray OP makes with the positive x-axis. Since $x/r = \cos\theta$ and $y/r = \sin\theta$, we have

$$\boldsymbol{x = r\cos\theta, y = r\sin\theta}.$$

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in and obtain the equation

$$\boldsymbol{\cos^2\theta + \sin^2\theta = 1.} \qquad \textbf{(1)}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta \qquad (2)$$
$$1 + \cot^2\theta = \csc^2\theta \qquad (3)$$

Remark:

The following formulas hold for all angles A and B.

Addition Formulas

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B \qquad (4)$$

Double-Angle Formulas

$$\cos(2A) = \cos^2 A - \sin^2 A$$
$$\sin(2A) = 2\sin A \cdot \cos A \qquad (5)$$

Half-Angle Formulas

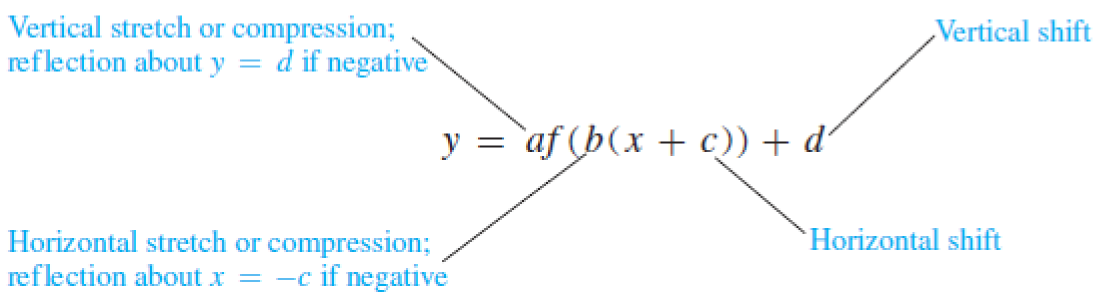
$$\cos^2 A = \frac{1+\cos(2A)}{2}$$
$$\sin^2 A = \frac{1-\cos(2A)}{2} \qquad (6)$$

For any angle θ measured in radians, the sine and cosine functions satisfy

$$-|\theta| \le \sin\theta \le |\theta| \text{ and } -|\theta| \le 1 - \cos\theta \le |\theta| .$$

Remark (Transformations of Trigonometric Graphs):

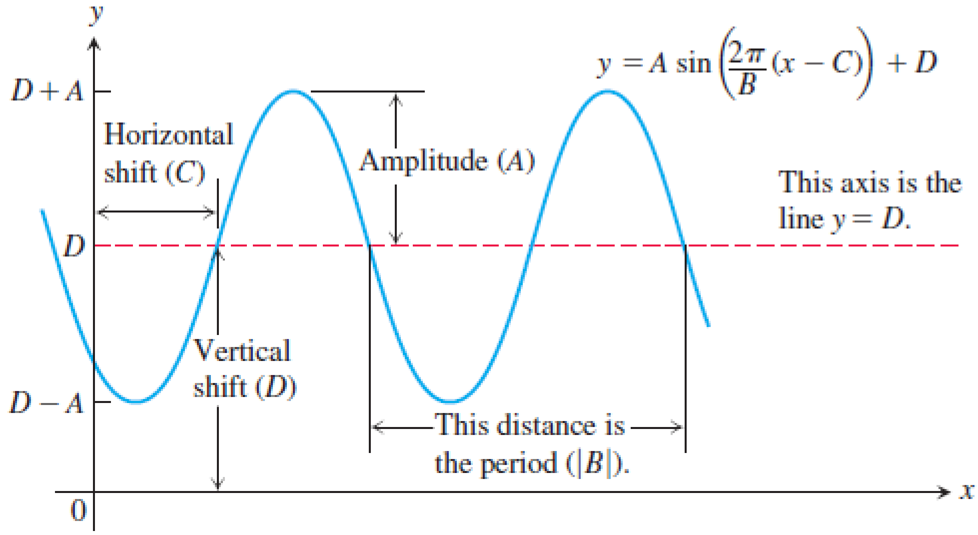
The rules for shifting, stretching, compressing, and reflecting the graph of a function summarized in the following diagram apply to the trigonometric functions.



The transformation rules applied to the sine function give the ***general sine function*** or ***sinusoid formula***

$$f(x) = A\sin\left(\frac{2\pi}{B}(x - C)\right) + D.$$

where $|A|$ is the ***amplitude***, B is the ***period***, C is the ***horizontal shift***, and D is the ***vertical shift***. A graphical interpretation of the various terms is given below.



Exercises:

1. Graph the following functions. What is the period of each function?

a) $\sin 2x$ b) $\sin (x/2)$ c) $\cos \pi x$

d) $\cos \frac{\pi x}{2}$ e) $-\sin\frac{\pi x}{3}$ f) $-\cos 2\pi x$

g) $\cos\left(x - \frac{\pi}{2}\right)$ h) $\sin\left(x + \frac{\pi}{6}\right)$ i) $\sin\left(x - \frac{\pi}{4}\right) + 1$

j) $\cos\left(x + \frac{2\pi}{3}\right) - 2$

2. Graph $y = \cos x$ and $y = \sec x$ together for $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$. Comment on the behavior of $\sec x$ in relation to the signs and values of $\cos x$.

3. Graph $y = \sin x$ and $y = \csc x$ together for $-\pi \leq x \leq 2\pi$. Comment on the behavior of $\csc x$ in relation to the signs and values of $\sin x$.

4. Use the addition formulas to derive the following identities

a) $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ **b)** $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$

c) $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ **d)** $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$

5. Express the given quantity in terms of $\sin x$ and $\cos x$.

a) $\cos(\pi + x)$ **b)** $\sin(2\pi - x)$

c) $\sin\left(\frac{3\pi}{2} - x\right)$ **d)** $\cos\left(\frac{3\pi}{2} + x\right)$

6. Evaluate $sin\frac{7\pi}{12}$ as $sin\ (\frac{\pi}{4}+\frac{\pi}{3})$.

7. Evaluate $cos\frac{11\pi}{12}$ as $cos\ (\frac{\pi}{4}+\frac{2\pi}{3})$.

8. Evaluate $cos\frac{\pi}{12}$.

9. Evaluate $sin\frac{5\pi}{12}$.

10. Find the function values using the Half-Angle Formulas

a) $cos^2\frac{\pi}{8}$ **b)** $cos^2\frac{5\pi}{12}$ **c)** $sin^2\ \frac{\pi}{12}$ **d)** $sin^2\ \frac{3\pi}{8}$

11. solve for the angle θ, where $0 \leq \theta \leq 2\pi$.

a) $sin^2\theta = \frac{3}{4}$ **b)** $sin^2\theta = cos^2\theta$

c) $sin2\theta - \cos\theta = 0$ **d)** $cos2\theta - \cos\theta = 0$

1.4 Exponential Functions

When a positive quantity P doubles, it increases by a factor of 2 and the quantity becomes $2P$. If it doubles again, it becomes $2(2P) = 2^2P$, and a third doubling gives $2(22P) = 2^3P$. Continuing to double in this fashion leads us to consider the function $f(x) = 2^x$. We call this an exponential function because the variable x appears in the exponent of 2^x. Functions such as $g(x) = 10^x$ and $h(x) = (1/\,2)^x$ are other examples of exponential functions. In general, if $a \neq 1$ is a positive constant, the function $\boldsymbol{f(x) = a^x}$ is the ***exponential function with base a***.

For integer and rational exponents, the value of an exponential function $f(x) = a^x$ is obtained arithmetically by taking an appropriate number of products, quotients, or roots. If $x = n$ is a positive integer, the number an is given by multiplying a by itself n times:

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}},$$

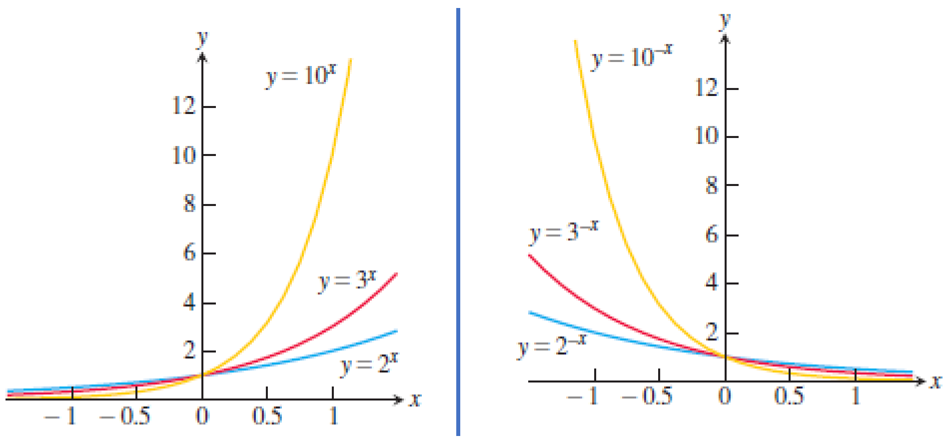
If $x = 0$, then we set $a^0 = 1$, and if $x = -n$ for some positive integer n, then $a^{-n} = \frac{1}{a^n} = (\frac{1}{a})^n$. If $x = 1/\,n$ for some positive integer n, then

$$a^{1/n} = \sqrt[n]{a},$$

which is the positive number that when multiplied by itself n times gives a. If $x = p/q$ is any rational number, then

$$a^{p/q} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p.$$

When x is irrational, the meaning of a^x is not immediately apparent. The value of a^x can be approximated by raising a to rational numbers that get closer and closer to the irrational number x.



Graphs of exponential functions.

Remark (Rules for Exponents):

If $a > 0$ and $b > 0$, the following rules hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$

2. $\frac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = (a^y)^x = a^{xy}$

4. $a^x \cdot b^x = (a \cdot b)^x$

5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Example:

Use the rules for exponents to simplify some numerical expressions.

1. $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8}$ **Rule 1**

2. $\frac{(\sqrt{10})^3}{\sqrt{10}} = \left(\sqrt{10}\right)^{3-1} = \left(\sqrt{10}\right)^2 = 10$ **Rule 2**

3. $\left(5^{\sqrt{2}}\right)^{\sqrt{2}} = 5^{\sqrt{2}\cdot\sqrt{2}} = 5^2 = 25$ **Rule 3**

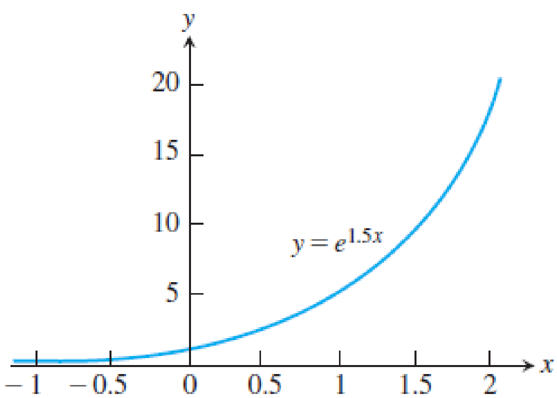
4. $7^\pi \cdot 8^\pi = (56)^\pi$ **Rule 4**

5. $\left(\frac{4}{9}\right)^{1/2} = \frac{4^{1/2}}{9^{1/2}} = \frac{2}{3}$ **Rule 5**

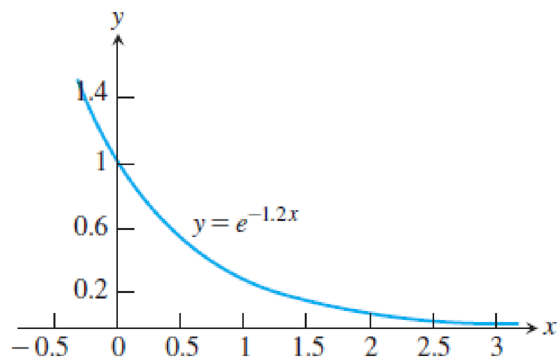
Remark:

The most important exponential function used for modeling natural, physical, and economic phenomena is the ***natural exponential function***, whose base is the special number e. The number e is irrational, and its value to nine decimal places is 2.718281828.

The function $y = y_0\, e^{kx}$, where k is a nonzero constant, is a model for ***exponential growth*** if $k > 0$ and a model for ***exponential decay*** if $k < 0$. Here y_0 is a positive constant that represents the value of the function when $x = 0$.



Graph of exponential growth, k = 1.5 > 0



Graph of exponential decay, k = −1.2 < 0

Exercises:

1. In following, sketch the given curves together in the appropriate coordinate plane, and label each curve with its equation

a) $y = 2^x, y = 4^x, y = 3^{-x}, y = (\frac{1}{5})^x$ **b)** $y = 2^{-t}$ *and* $y = -2^t$

c) $y = 3^x, y = 8^x, y = 2^{-x}, y = (\frac{1}{4})^x$ **d)** $y = 3^{-t}$ *and* $y = -3^t$

e) $y = e^x$ *and* $y = 1/e^x$ **f)** $y = -e^x$ *and* $y = -e^{-x}$

2. In each of following, sketch the shifted exponential curves.

a) $y = 2^x - 1$ *and* $y = 2^{-x} - 1$ **b)** $y = 3^x + 2$ *and* $y = 3^{-x} + 2$

c) $y = 1 - e^x$ *and* $y = 1 - e^{-x}$ **d)** $y = -1 - e^x$ *and* $y = -1 - e^{-x}$

3. Use the laws of exponents to simplify the expressions in following.

a) $16^2 \cdot 16^{-1.75}$ **b)** $9^{1/3} \cdot 9^{-1/6}$ **c)** $\frac{4^{4.2}}{4^{3.7}}$ **d)** $\frac{3^{5/3}}{3^{2/3}}$

e) $\left(25^{1/8}\right)^4$ **f)** $\left(13^{\sqrt{2}}\right)^{\sqrt{2}/2}$ **g)** $2^{\sqrt{3}} \cdot 7^{\sqrt{3}}$ **h)** $(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2}$

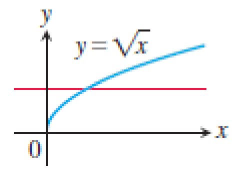
i) $\left(\frac{2}{\sqrt{2}}\right)^4$ **j)** $\left(\frac{\sqrt{6}}{3}\right)^2$

1.5 Inverse Functions and Logarithms

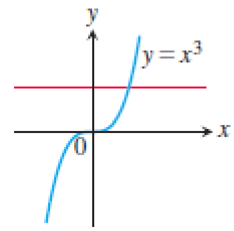
Definition:

A function $f(x)$ is ***one-to-one*** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

Example:

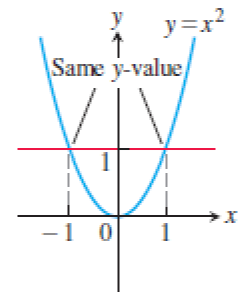


a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $f(x_1) = \sqrt{x_1} \neq f(x_2) = \sqrt{x_2}$ whenever $x_1 \neq x_2$.



b) $f(x) = x^3$ is one-to-one on their domain $(-\infty, \infty)$ because

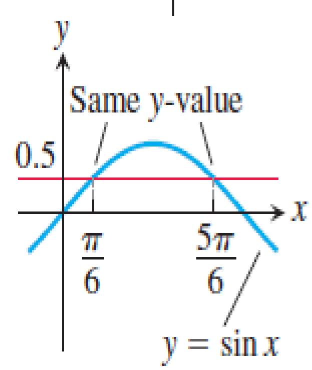
$f(x_1) = x_1^3 \neq f(x_2) = x_2^3$ whenever $x_1 \neq x_2$.



c) $g(x) = x^2$ is not one-to-one on interval $(-\infty, \infty)$ because

$f(-2) = (-2)^2 = 4 = 2^2 = f(2)$.

d) $g(x) = \sin x$ is not one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. In fact, for each element x_1 in the subinterval $[0, \pi/2)$ there is a corresponding element x_2 in the subinterval $(\pi/2, \pi]$ satisfying $\sin x_1 = \sin x_2$. The sine function is one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ and therefore gives distinct outputs for distinct inputs in that interval.



The Horizontal Line Test for One-to-One Functions:

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.

Definition:

Suppose that f is a one-to-one function on a domain D with range R. ***The inverse function*** f^{-1} is defined by

$$f^{-1}(b) = a \quad if \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

Example:

Suppose a one-to-one function $y = f(x)$ is given by a table of values

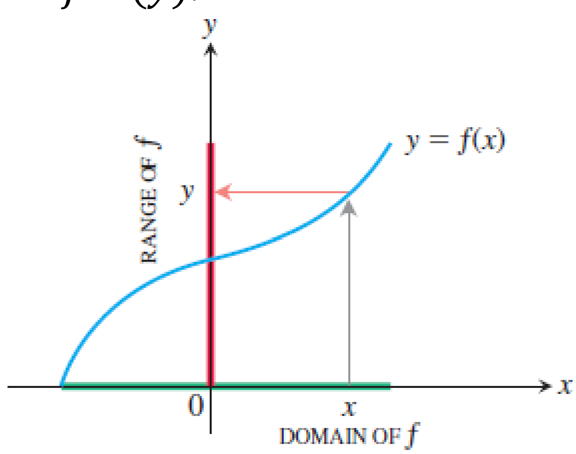
x	1	2	3	4	5	6	7	8
$f(x)$	3	4.5	7	10.5	15	20.5	27	34.5

A table for the values of $x = f^{-1}(y)$ can then be obtained by simply interchanging the values in each column of the table for f:

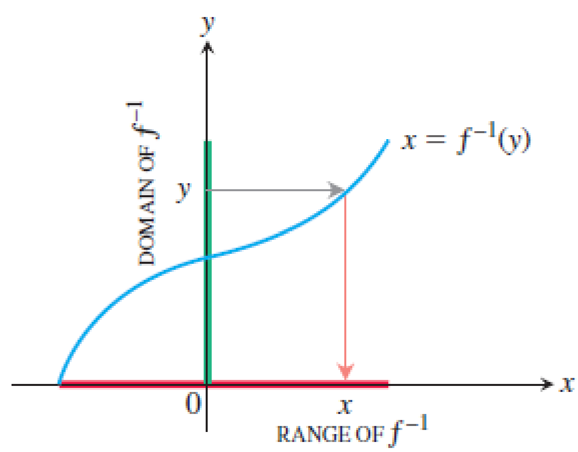
y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

Remark:

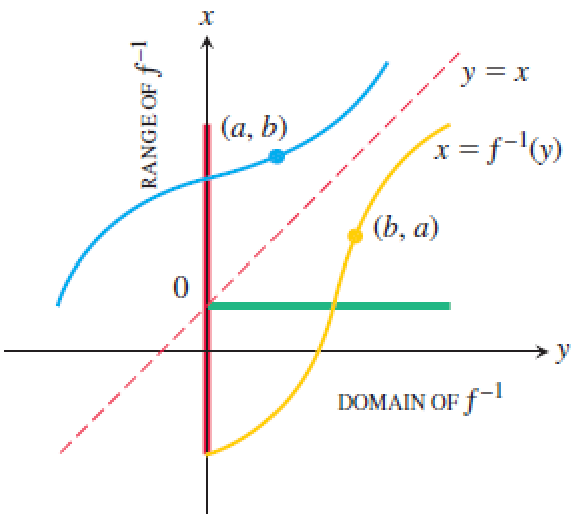
The graphs of a function and its inverse are closely related. To read the value of a function from its graph, we start at a point x on the x-axis, go vertically to the graph, and then move horizontally to the y-axis to read the value of y. The inverse function can be read from the graph by reversing this process. Start with a point y on the y-axis, go horizontally to the graph of $y = f(x)$, and then move vertically to the x-axis to read the value of $x = f^{-1}(y)$.



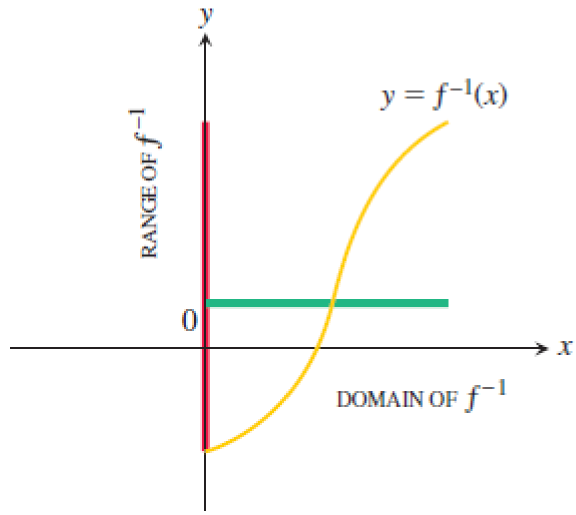
To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

Example:

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution:

The graph is a straight line satisfying the horizontal line test

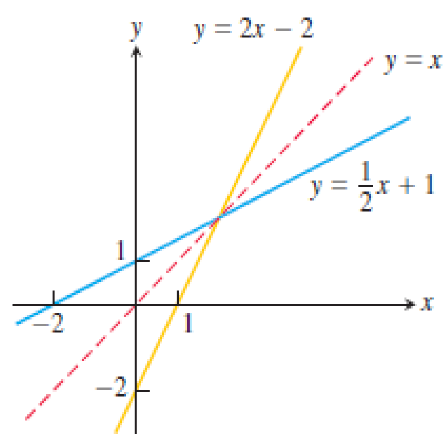
1. Solve for x in terms of y:

$$y = \frac{1}{2}x + 1$$
$$2y = x + 2$$
$$x = 2y - 2$$

2. Interchange x and y:

Expresses the function in the usual form where y is the dependent variable.

$$y = 2x - 2$$



The inverse of the function $f(x) = \frac{1}{2}x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both compositions give the identity function:

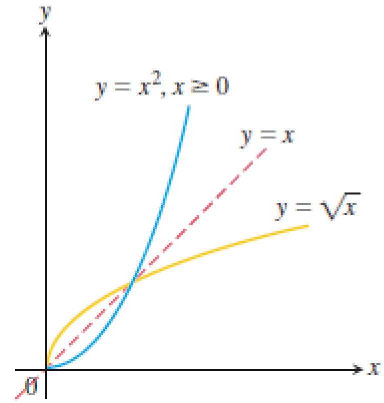
$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

Example:

Find the inverse of $y = x^2, x \geq 0$, expressed as a function of x.

Solution:



For $x \geq 0$, the graph satisfies the horizontal line test, so the function is one-to-one and has an inverse. To find the inverse, we first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad \mathbf{|x| = x \text{ because } x \geq 0}$$

We then interchange x and y, obtaining $y = \sqrt{x}$.

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$.

Remark:

Notice that the function $y = x^2, x \geq 0$, with domain *restricted* to the nonnegative real numbers, is one-to-one and has an inverse. On the other hand, the function $y = x^2$, with no domain restrictions, is not one-to-one and therefore has no inverse.

Remark:

If a is any positive real number other than 1, then the base a exponential function $f(x) = a^x$ is one-to-one. It therefore has an inverse. Its inverse is called the logarithm function with base a.

Definition:

The ***logarithm function with base a***, written $y = \log_a x$, is the inverse of the base a exponential function $y = a^x$ $(a > 0, a \neq 1)$.

Remark:

The graph of $y = a^x, a > 1$, increases rapidly for $x > 0$, so its inverse, $y = \log_a x$, increases slowly for $x > 1$. Logarithms with base e and base 10 are so important in applications that many calculators have special keys for them. They also have their own special notation and names:

$\log_e x$ is written as $\ln x$.

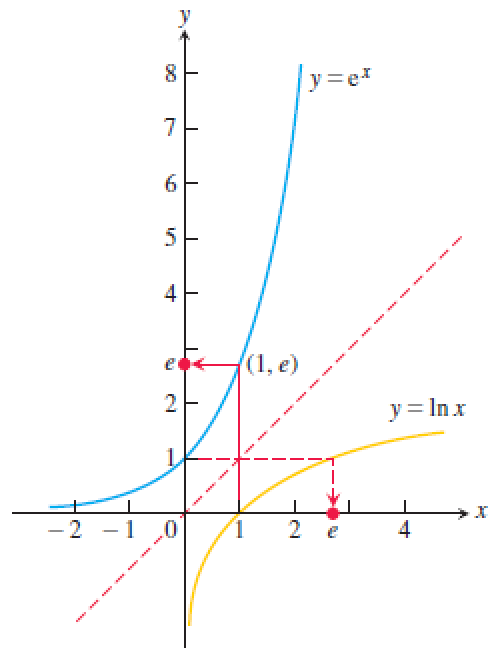
$\log_{10} x$ is written as $\log x$.

The function $y = ln\, x$ is called the ***natural logarithm function***, and $y = log\, x$ is often called the ***common logarithm function***. Since the logarithm is the inverse function of exponentiation, it follows that:

$$\log_a x = y \Leftrightarrow a^y = x$$

$$\ln x = y \Leftrightarrow e^y = x$$

In particular, because $e^1 = e$, we obtain $\ln e = 1$



Theorem (Algebraic Properties of the Natural Logarithm):

For any numbers b > 0 and x > 0, the natural logarithm satisfies the following rules:

1. ***Product Rule:*** $\boldsymbol{ln\, bx = ln\, b + ln\, x}$
2. ***Quotient Rule:*** $\boldsymbol{ln \frac{b}{x} = ln\, b - ln\, x}$
3. ***Reciprocal Rule*** $\boldsymbol{ln \frac{1}{x} = -ln\, x}$ ***Rule 2 with $b = 1$***
4. ***Power Rule:*** $\boldsymbol{ln\, x^r = r\, ln\, x}$

Example:

Use the properties in Theorem Algebraic Properties of the Natural Logarithm to rewrite three expressions.

a) $ln\, 4 + ln \sin x = \ln(4 \sin x)$ **Product Rule**

b) $ln \frac{x+1}{2x-3} = ln(x+1) - ln(2x-3)$ **Quotient Rule**

c) $ln \frac{1}{8} = -ln\, 8$ **Reciprocal Rule**

$= -ln\, 2^3 = -3 \ln 2$ **Power Rule**

Because a^x and $\log_a x$ are inverses, composing them in either order gives the identity function.

Remark (Inverse Properties for a^x and $\log_a x$):

1. Base $a (a > 0, a \neq 1)$: $a^{log_a x} = x, x > 0$

$$\log_a a^x = x$$

2. Base e: $e^{ln\, x} = x, x > 0$

$$ln\, e^x = x$$

Remark:

Every exponential function is a power of the natural exponential function.

$$a^x = e^{x\ln a}$$

That is, a^x is the same as e^x raised to the power $\ln a$: $a^x = e^{kx}$ for $k = \ln a$.For example $2^x = e^{(\ln 2)x} = e^{x\ln 2}$.

Remark (Change-of-Base Formula):

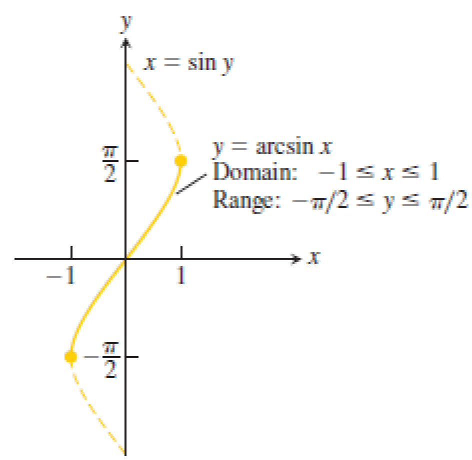
Since $\ln x = \ln(a^{\log_a x}) = (\log_a x)(\ln a)$ then every logarithmic function is a constant multiple of the natural logarithm.

$$\mathbf{\log_a x = \frac{\ln x}{\ln a}}, (a > 0, a \neq 1) .$$

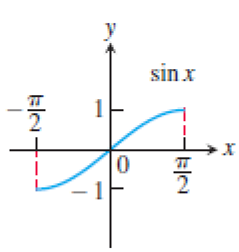
Remark (Inverse Trigonometric Functions):

The six basic trigonometric functions are not one-to-one (since their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one.

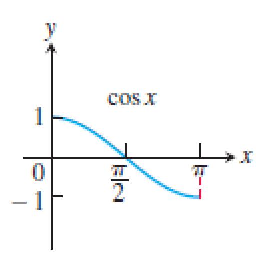
The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting its domain to the interval $[-\pi/2, \pi/2]$ we make it one-to-one, so that it has an inverse which is called $\arcsin x$. Similar domain restrictions can be applied to all six trigonometric functions.



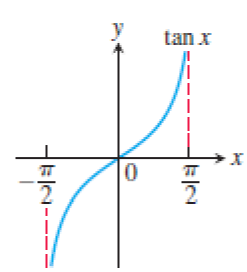
Now the domain restrictions that make the trigonometric functions one-to-one is shown in following:



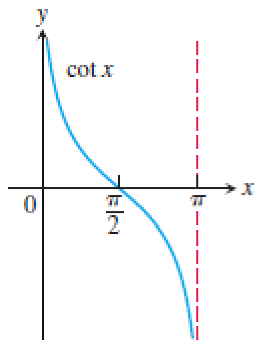
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



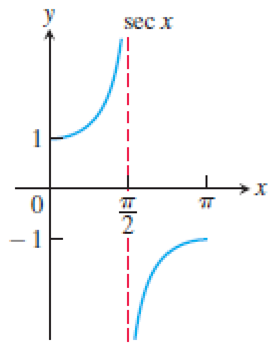
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



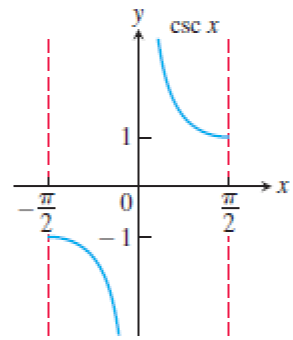
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = sin^{-1}x \text{ or } y = \arcsin x\,, \quad y = cos^{-1}x \text{ or } y = \arccos\ x$$

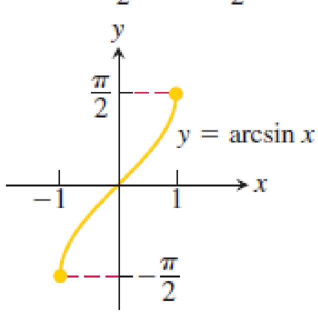
$$y = tan^{-1}x \text{ or } y = \arctan x\,, \quad y = cot^{-1}x \text{ or } y = \operatorname{arccot}\ x$$

$$y = sec^{-1}x \text{ or } y = \operatorname{arcsec} x\,, \quad y = csc^{-1}x \text{ or } y = \operatorname{arccsc}\ x$$

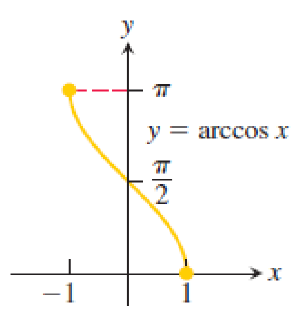
The -1 in the expressions for the inverse means "inverse." It does not mean reciprocal. For example, the reciprocal of $sin\ x$ is $(sin\ x\,)^{-1} = \frac{1}{sin\,x} = csc\ x$.

The graphs of the six inverse trigonometric functions are obtained by reflecting the graphs of the restricted trigonometric functions through the line $y = x$.

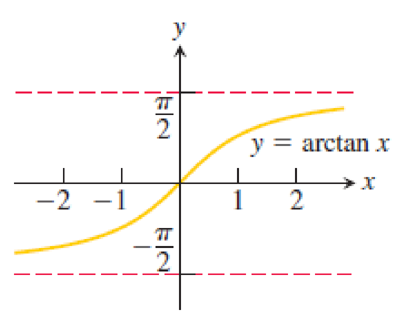
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

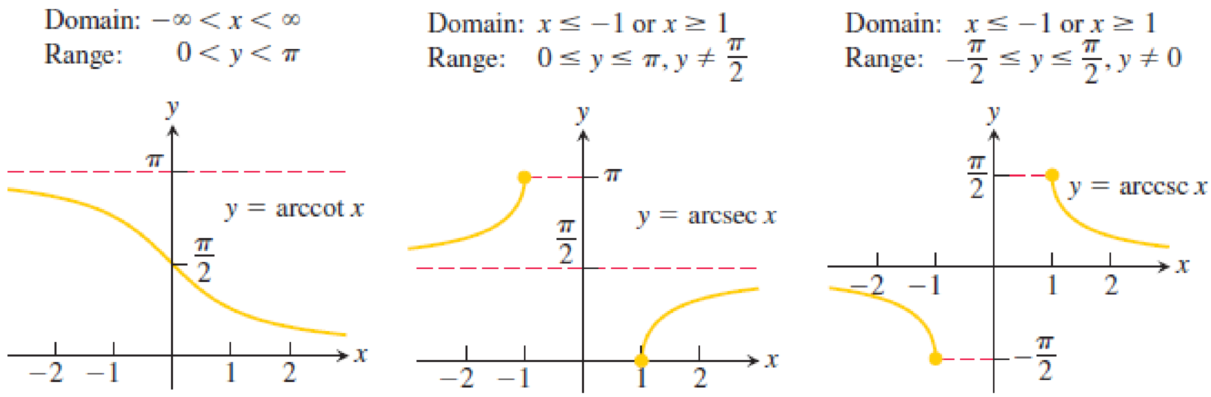


Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$





The graph of $y = arcsin\ x$ is symmetric about the origin. The arcsine is therefore an odd function: $arcsin(-x) = -arcsin\ x$. The graph of $y = arccos\ x$ has no such symmetry.

Definition:

$y = arcsin\ x$ is the number in $[-\pi/2, \pi/2]$for which $sin\ y = x$.

$y = arccos\ x$ is the number in $[\ 0, \pi\ ]$ for which $cos\ y = x$.

Example:

Evaluate **(a)** $arcsin\ (\frac{\sqrt{3}}{2})$ and **(b)** $\arccos(-\frac{1}{2})$.

Solution:

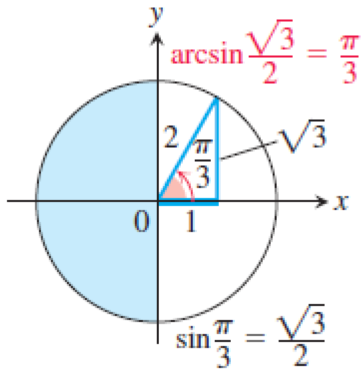
(a) We see that $arcsin\ \left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ because $sin(\ \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ belongs to the range$[-\pi/2, \pi/2]$ of the arcsine function.

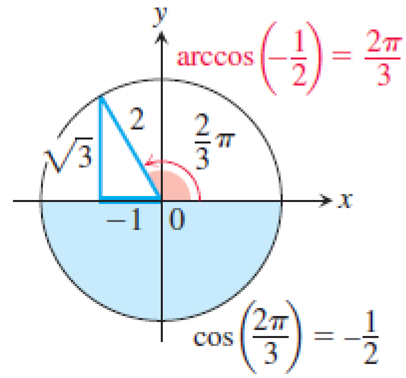
(b) We see that $\arccos(-\frac{1}{2}) = \frac{2\pi}{3}$ because $cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\frac{2\pi}{3}$ belongs to the range$[\ 0, \pi\ ]$ of the arcsine function.

Remark:

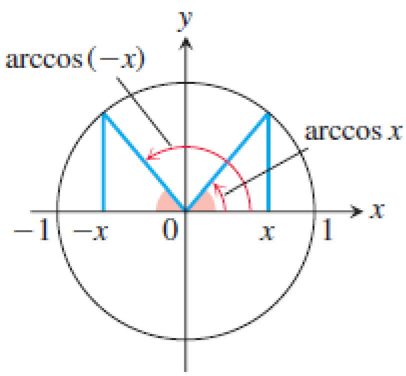
Using the same procedure illustrated in previous example, we can create the following table of common values for the arcsine and arccosine functions.

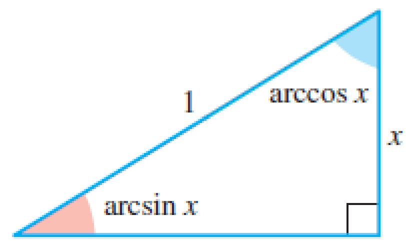
x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$





Remark:





the arccosine of x satisfies the identity

$$arccos\ x + arccos\ (-x) = \pi\ ,$$

Or

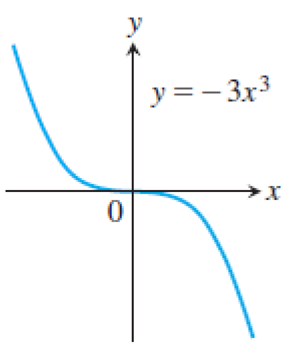
$$arccos\ (-x) = \pi - arccos\ x.$$

Also, we can see that for $x > 0$, $arcsin\ x + arccos\ x = \pi/2$.

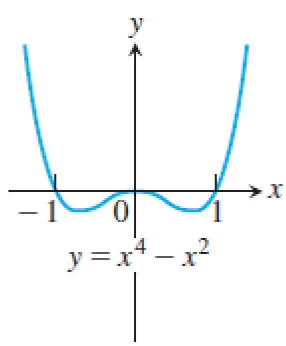
Exercises:

1. Which of the functions graphed in following are one-to-one, and which are not?

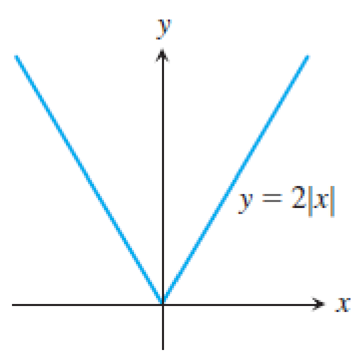
a)



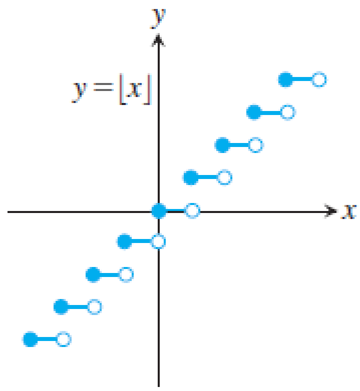
b)



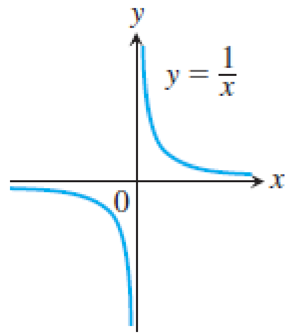
c)



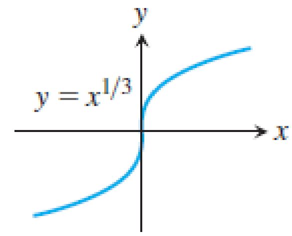
d)



e)



f)



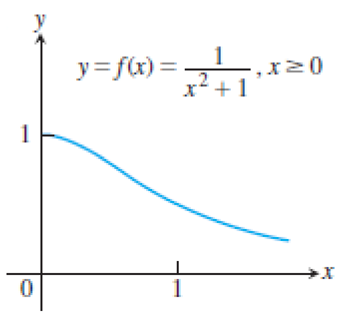
2. In following, determine from its graph whether the function is one-to-one.

a) $f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$ **b)** $f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$

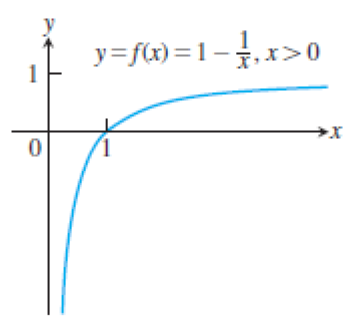
c) $f(x) = \begin{cases} 1 - \frac{x}{2}, & x \leq 0 \\ \frac{x}{x+2}, & x > 0 \end{cases}$ **d)** $f(x) = \begin{cases} 2 - x^2, & x \leq 1 \\ x^2, & x > 1 \end{cases}$

3. Each of following shows the graph of a function $y = f(x)$. Copy the graph and draw in the line $y = x$. Then use reflection with respect to the line $y = x$ to add the graph of f^{-1} to your sketch. (It is not necessary to find a formula for f^{-1}.) Identify the domain and range of f^{-1}.

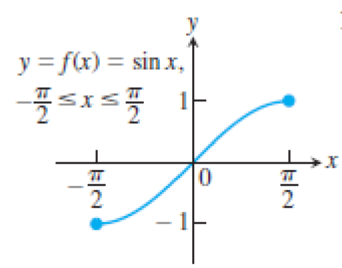
a)



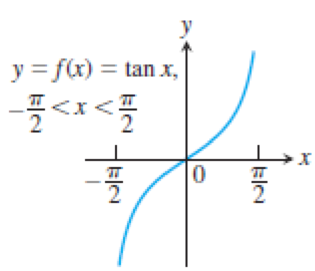
b)



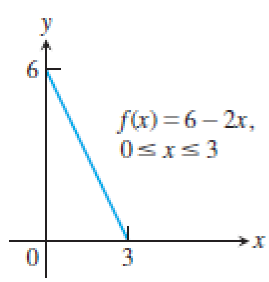
c)



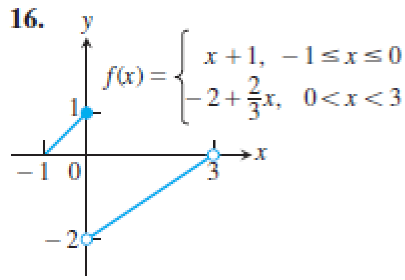
d)



e)



f) 16.



4. a) Graph the function $f(x) = \sqrt{1 - x^2}, 0 \leq x \leq 1$. What symmetry does the graph have?

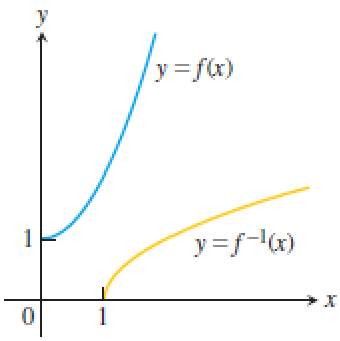
b) Show that f is its own inverse. (Remember that $\sqrt{x^2} = x$ if $x \geq 0$.

5. **a)** Graph the function $f(x) = 1/x$. What symmetry does the graph have?

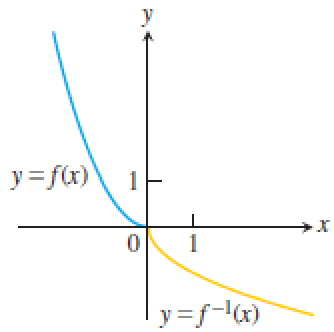
b) Show that f is its own inverse.

6. Each of following gives a formula for a function $y = f(x)$ and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

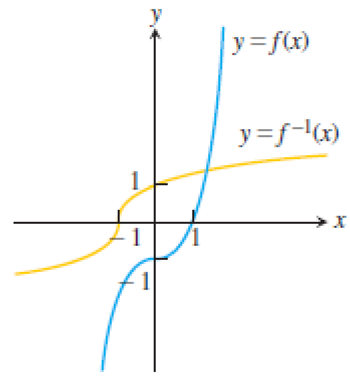
a) $f(x) = x^2 + 1, x \geq 0$



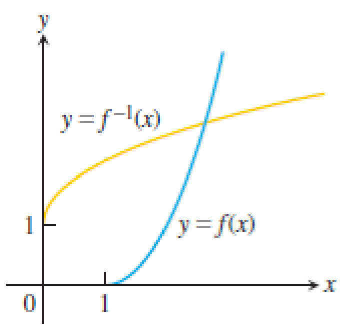
b) $f(x) = x^2, x \leq 0$



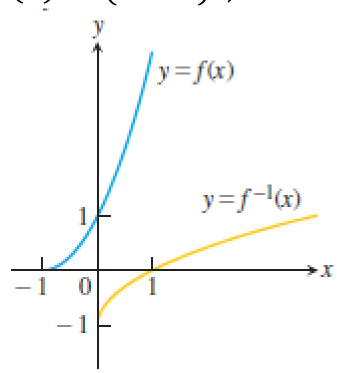
c) $f(x) = x^3 - 1$



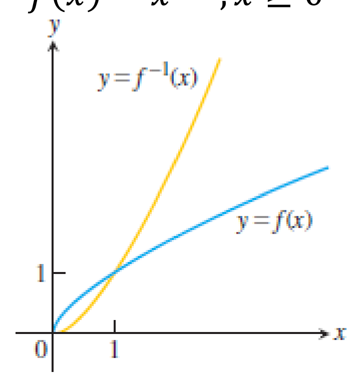
d) $f(x) = x^2 - 2x + 1, x \geq 1$



e) $f(x) = (x+1)^2, x \geq -1$



f) $f(x) = x^{2/3}, x \geq 0$



8. Each of following gives a formula for a function $y = f\ (x)$. In each case, find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f\ (x)\right) = x$.

a) $f(x) = x^5$ **b)** $f(x) = x^4, x \geq 0$ **c)** $f(x) = x^3 + 1$

d) $f(x) = (1/2)x - 7/2$ **e)** $f(x) = 1/x^2, x > 0$ **f)** $f(x) = 1/x^3, x \neq 0$

g) $f(x) = \frac{x+3}{x-2}$ **h)** $f(x) = \frac{\sqrt{x}}{\sqrt{x}-3}$

9. Express the following logarithms in terms of ln 2 and ln 3.

a) $ln\ 0.75$ **b)** $ln(4/9)$ **c)** $ln(1/\ 2)$ **d)** $ln\sqrt[3]{9}$ **e)** $ln3\sqrt{2}$ **f)** $ln\sqrt{13.5}$

10. Use the properties of logarithms to write the expressions in following as a single term.

a) $\ln\sin\theta - \ln\left(\frac{\sin\theta}{5}\right)$ **b)** $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$ **c)** $\frac{1}{2}\ln(4t^4) - \ln 2$

d) $\ln\sec\theta + \ln\cos\theta$ **e)** $\ln(8x + 4) - \ln 2^2$ **f)** $3\ln\sqrt[3]{t^2 - 1} - \ln(t + 1)$

11. Find simpler expressions for the quantities in following:

a) $e^{\ln 7.2}$ **b)** $e^{-\ln x^2}$ **c)** $e^{\ln x - \ln y}$ **d)** $e^{\ln(x^2+y^2)}$

e) $e^{-\ln 0.3}$ **f)** $e^{\ln \pi x - \ln 2}$ **g)** $2\ln\sqrt{e}$ **h)** $\ln(\ln e^e)$

i) $\ln(e^{-x^2-y^2})$ **j)** $\ln(e^{\sec\theta})$ **k)** $\ln(e^{(e^x)})$ **l)** $\ln(e^{2\ln x})$

12. In following, solve for *y* in terms of *t* or *x*, as appropriate.

a) $\ln y = 2t + 4$ **b)** $\ln y = -t + 5$

c) $\ln(y - b) = 5t$ **d)** $\ln(c - 2y) = t$

e) $\ln(y - 1) - \ln 2 = x + \ln x$ **f)** $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

13. Express the ratios in following as ratios of natural logarithms and simplify.

a) $\frac{\log_2 x}{\log_3 x}$ **b)** $\frac{\log_2 x}{\log_8 x}$ **c)** $\frac{\log_x a}{\log_{x^2} a}$ **d)** $\frac{\log_9 x}{\log_3 x}$ **e)** $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x}$ **f)** $\frac{\log_a b}{\log_b a}$

14. In following, find the exact value of each expression.

a) $sin^{-1}\left(\frac{-1}{2}\right)$ **b)** $sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **c)** $sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$ **d)** $cos^{-1}\left(\frac{1}{2}\right)$ **e)** $cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

f) $cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **g)** $\arccos(-1)$ **h)** $\arccos(0)$ **i)** $\arcsin(-1)$ **j)** $\arcsin\left(\frac{-1}{\sqrt{2}}\right)$

Calculus I

First Semester

Lecturer 4

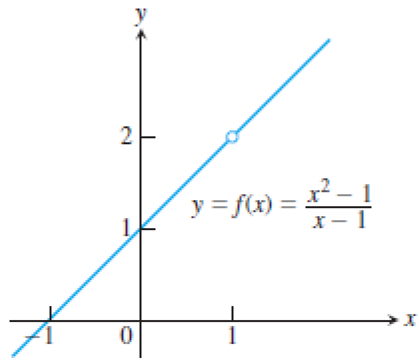
Dr. Ban Jaffar AL-Taiy

Taghreed Hussein Abed

Chapter Two: Limits and Continuity.

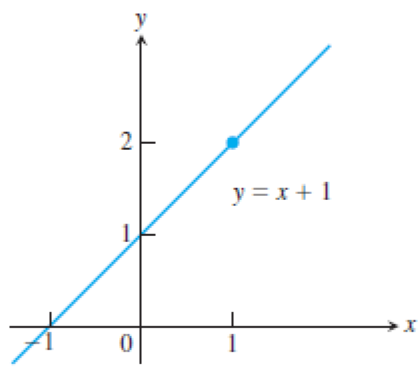
2.1 Limits of Function Values

In this section we hall study the function's $y = f(x)$ behavior near a particular point c, but not at c. For instance, how does the behave of the function $f(x) = \frac{x^2-1}{x-1}$ near $x = 1$.



The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for } x \neq 1.$$



The graph of f is the line $y = x + 1$ with the point (1, 2) removed. This removed point is shown as a "hole". Even though $f(1)$ is not defined, it is clear that we can make the value of f(x) as close as we want to 2 by choosing x close enough to 1.

x		$f(x) = \frac{x^2-1}{x-1}$
0.9		1.9
0.99		1.99
0.999		1.999
0.999999		1.999999
	1.1	2.1
	1.01	2.01
	1.001	2.001
	1.000001	2.000001

suppose $f(x)$ is defined on an open interval about c, except possibly at c itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, we say that f approaches the limit L as x approaches c, and write

$$\lim_{x\to c} f(x) = L,$$

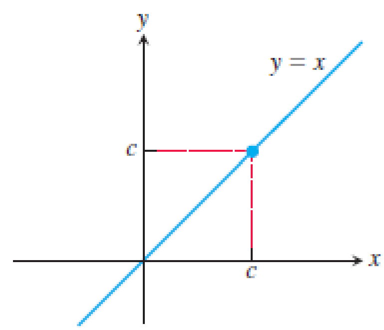
which is read "the limit of $f(x)$ as x approaches c is L." For instance, in Example we would say that $f(x) = \frac{x^2-1}{x-1}$ approaches the limit 2 as x approaches 1, and write

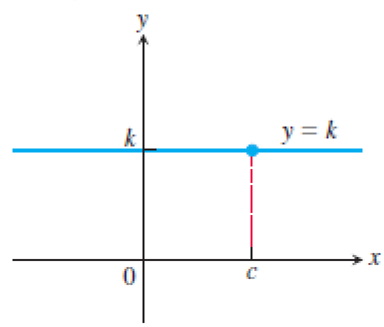
$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

Example:

a) If f is the identity function $f(x) = x$, then for any value of c

$$\lim_{x\to c} f(x) = \lim_{x\to c} x = c.$$





b) If f is the constant function $f(x) = k$ (function with the constant value k), then for any value of c

$$\lim_{x\to c} f(x) = \lim_{x\to c} k = k.$$

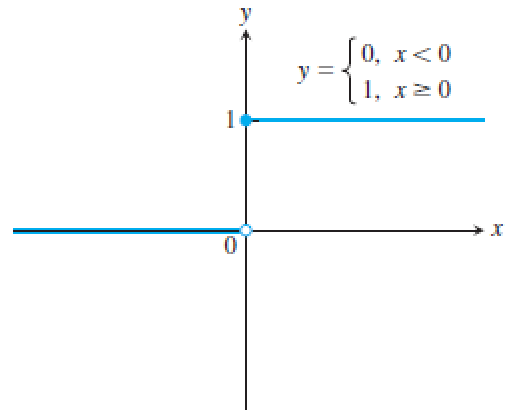
Example:

Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

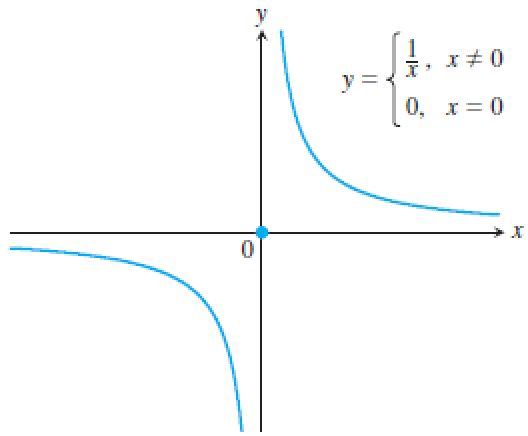
a) $U(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$, **b)** $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, **c)** $f(x) = \begin{cases} 0 & x \leq 0 \\ \sin\frac{1}{x} & x > 0 \end{cases}$.

Solution:

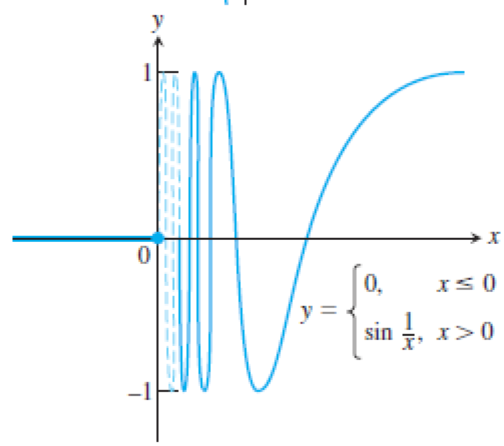
a) It jumps: The unit step function $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no single value L approached by $U(x)$ as $x \to 0$.



b) It grows too "large" to have a limit: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to any fixed real number. We say the function is not bounded.

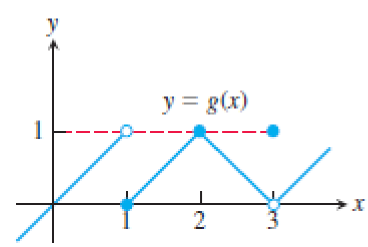


c) It oscillates too much to have a limit: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between +1 and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$.



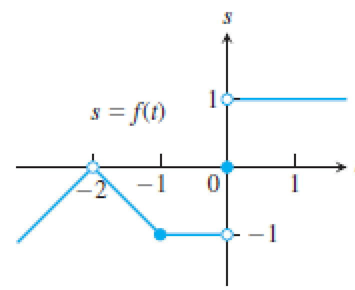
Exercises:

1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.



a) $\lim_{x \to 1} g(x)$, **b)** $\lim_{x \to 2} g(x)$, **c)** $\lim_{x \to 3} g(x)$, **d)** $\lim_{x \to 2.5} g(x)$.

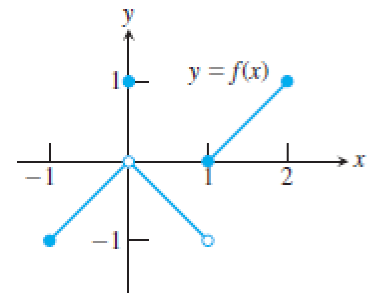
2. For the function f(t) graphed here, find the following limits or explain why they do not exist.



a) $\lim_{t\to -2} f(t)$ **b)** $\lim_{t\to -1} f(t)$

c) $\lim_{t\to 0} f(t)$ **d)** $\lim_{t\to 0.5} f(t)$

3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



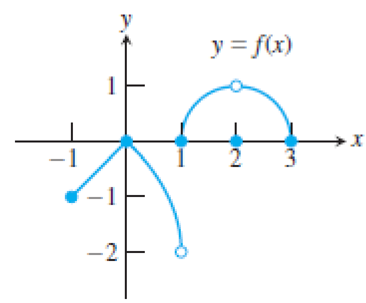
a) $\lim_{x\to 0} f(x)$ exists. **b)** $\lim_{x\to 0} f(x) = 0$

c) $\lim_{x\to 0} f(x) = 1$ **d)** $\lim_{x\to 1} f(x) = 1$

e) $\lim_{x\to 1} f(x) = 0$ **f)** $\lim_{x\to c} f(x)$ exists at every point c in (-1,1).

g) $\lim_{x\to 1} f(x)$ does not exist.

4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a) $\lim_{x\to 2} f(x)$ does not exist. **b)** $\lim_{x\to 2} f(x) = 2.$

c) $\lim_{x\to 1} f(x)$ does not exist. **d)** $\lim_{x\to c} f(x)$ exists at every point c in (-1,1).

e) $\lim_{x\to c} f(x)$ exists at every point c in (1,3).

5. Find the limits in following

a) $\lim_{x\to -3}(x^2 - 13)$ **b)** $\lim_{x\to 2}(-x^2 + 5x - 2)$

c) $\lim_{t\to 6} 8(t-5)(t-7)$ **d)** $\lim_{x\to -2}(x^3 - 2x^2 + 4x + 8)$

e) $\lim_{x\to 2} \frac{2x+5}{11-x^3}$ **f)** $\lim_{s\to 2/3}(8-3s)(2s-1)$

g) $\lim_{s\to -1/2} 4x(3x+4)^2$ **h)** $\lim_{y\to 2} \frac{y+2}{y^2+5y+6}$

i) $\lim_{y\to -3}(5-y)^{4/3}$ **j)** $\lim_{z\to 4} \sqrt{z^2 - 10}$

k) $\lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1}$ **l)** $\lim_{x\to -3} \frac{x+3}{x^2+4x+3}$

m) $\lim_{x\to 5} \frac{x-5}{x^2-25}$ **n)** $\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$

o) $\lim_{x\to -5} \frac{x^2+3x-10}{x+5}$ **p)** $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2}$

q) $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$

r) $\lim_{y\to 0} \frac{y^3+8y^2}{3y^4-16y^2}$

s) $\lim_{x\to -2} \frac{-2x-4}{x^3+2x^2}$

t) $\lim_{x\to 1} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

u) $\lim_{x\to 1} \frac{x^{-1}-1}{x-1}$

v) $\lim_{u\to 2} \frac{v^3-8}{v^4-16}$

w) $\lim_{u\to 1} \frac{u^4-1}{u^3-1}$

x) $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}}$

y) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$

z) $\lim_{x\to \pi/4} \sin^2 x$

aa) $\lim_{x\to 0}(2\sin x - 1)$

bb) $\lim_{x\to \pi/3} \tan x$

cc) $\lim_{x\to 0} \sec x$

dd) $\lim_{x\to 0}(x^2-1)(2-\cos x)$

ee) $\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x}$

ff) $\lim_{x\to 0} \sqrt{7+\sec^2 x}$

gg) $\lim_{x\to -\pi} \sqrt{x+4}\cos(x+\pi)$

2.2 The Limit Laws

To calculate limits of functions that are arithmetic combinations of functions having known limits, we can use several fundamental rules.

Theorem (Limit Laws):

If L, M, c, and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ then

1. **Sum Rule:** $\lim_{x\to c}(f(x)+g(x)) = L+M$
2. **Difference Rule:** $\lim_{x\to c}(f(x)-g(x)) = L-M$
3. **Constant Multiple Rule:** $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
4. **Product Rule:** $\lim_{x\to c}(f(x)\cdot g(x)) = L.M$
5. **Quotient Rule:** $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M},\quad M\neq 0.$
6. **Power Rule:** $\lim_{x\to c}[f(x)]^n = L^n$, **$n$ a positive integer.**
7. **Root Rule:** $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, **$n$ a positive integer.**

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

Example:

Use the fundamental rules of limits to find the following limits.

a) $\lim_{x \to c}(x^3 + 4x^2 - 3)$ b) $\lim_{x \to c} \frac{x^4 - x^2 - 1}{x^2 + 5}$ c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution:

a) $\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ **Sum and Difference Rules**

$= c^3 + 4c^2 - 3$ **Power and Multiple Rules**

b) $\lim_{x \to c} \frac{x^4 - x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 - x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$ **Quotient Rule**

$= \frac{\lim_{x \to c} x^4 - \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ **Sum and Difference Rules**

$= \frac{c^4 - c^2 - 1}{c^2 + 5}$ **Power or Product Rule**

c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)}$ **Root Rule with $n = 2$**

$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$ **Difference Rule**

$= \sqrt{4(-2)^2 - 3}$ **Product and Multiple Rules**

$= \sqrt{16 - 3}$

$= \sqrt{13}$

Theorem (Limits of Polynomials):

If $P(x) = a_n x^n + a_{n-1} x^n + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(x) = a_n c^n + a_{n-1} c^n + \cdots + a_0.$$

Theorem (Limits of Rational Functions):

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0.$$

Example:

Evaluate $\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$

Solution:

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$= \lim_{x\to 0}\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$= \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$ **Common factor x^2**

$$= \lim_{x\to 0}\frac{1}{\left(\sqrt{x^2+100}+10\right)}$$ **Cancel x^2 for $x \neq 0$.**

$$= \frac{1}{\left(\sqrt{0^2+100}+10\right)}$$ **Denominator not 0 at $x = 0$; substitute**

$$= \frac{1}{20} = 0.05.$$

Theorem (The Sandwich Theorem):

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

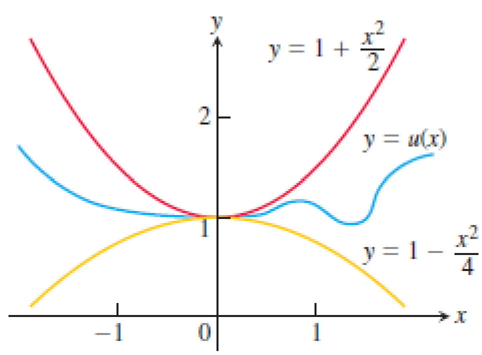
Then $\lim_{x\to c} f(x) = L$.

Example:

Given that

$$1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.



Solution:

Since $\lim_{x\to 0}\left(1-\frac{x^2}{4}\right) = 1$ and $\lim_{x\to 0}\left(1-\frac{x^2}{2}\right) = 1$,by the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$.

Example:

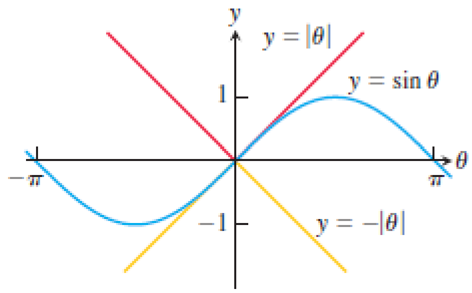
The Sandwich Theorem helps us establish several important limit rules:

a) $\lim_{\theta \to 0} \sin\theta = 0$ **b)** $\lim_{\theta \to 0} \cos\theta = 1$

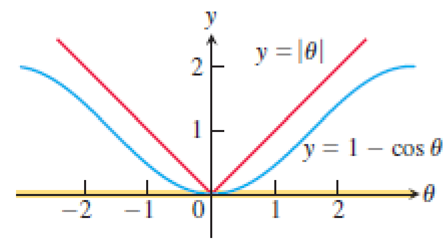
c) For any function f, $\lim_{x \to c} |f(x)| = 0$ implies $\lim_{x \to c} f(x) = 0$.

Solution:

a) Since $-|\theta| \leq \sin\theta \leq |\theta|$, for all θ and since $\lim_{\theta \to 0} -|\theta| \lim_{\theta \to 0} |\theta| = 0$, we have $\lim_{\theta \to 0} \sin\theta = 0$.



b) Since $-|\theta| \leq 1 - \cos\theta \leq |\theta|$, for all θ and since $\lim_{\theta \to 0} -|\theta| \lim_{\theta \to 0} |\theta| = 0$, we have $\lim_{\theta \to 0} 1 - \cos\theta = 0$ or $\lim_{\theta \to 0} \cos\theta = 1$.



c) Since $-|f(x)| \leq f(x) \leq |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x \to c} f(x) = 0$.

Theorem:

If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

Exercises:

1. If $\sqrt{5 - 2x^2} \leq f(x) \leq \sqrt{5 - x^2}$ for $-1 \leq x \leq 1$, find $\lim_{x \to 0} f(x)$.

2. If $2 - x^2 \leq g(x) \leq 2$ for all x, find $\lim_{x \to 0} g(x)$.

3. It can be shown that the inequalities $1 - \frac{x^2}{6} \leq \frac{x \sin x}{2 - 2\cos x} < 1$ hold for all values of x close to zero. What, if anything, does this tell you about $\lim_{x \to 0} \frac{x \sin x}{2 - 2\cos x}$? Give reasons for your answer.

4. Suppose that the inequalities $\frac{1}{2} - \frac{x^2}{24} \leq \frac{1-\cos x}{x^2} < \frac{1}{2}$ hold for values of x close to zero. What, if anything, does this tell you about $\lim_{x\to 0} \frac{1-\cos x}{x^2}$? Give reasons for your answer.

5. Find the limits in following

a) $\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h}$

b) $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

c) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$

d) $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3}$

e) $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2}$

f) $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}}$

g) $\lim_{x\to -3} \frac{-2-\sqrt{x^2-5}}{x+3}$

2.3 The Precise Definition of a Limit

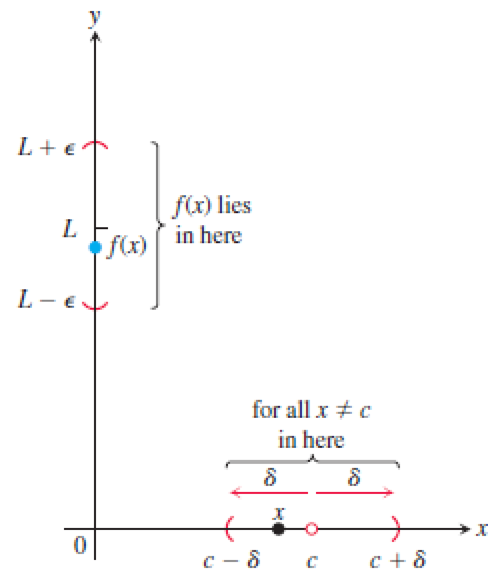
Definition:

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number L, and write

$$\lim_{x\to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$



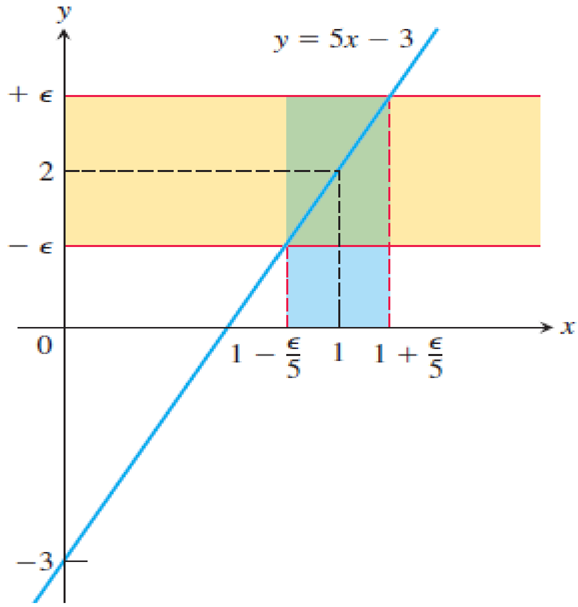
Examples: Testing the Definition

Example:

Show that $\lim_{x\to 1}(5x-3) = 2$.

Solution:

Set c = 1, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of c = 1, that is, whenever $0 < |x - 1| < \delta$, it is true that $f(x)$ is within distance ϵ of $L = 2$, so $|f(x) - 2| < \epsilon$.



We find δ by working backward from the ϵ -inequality:

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < \epsilon \Rightarrow |x - 1| < \epsilon/5.$$

Thus, we can take $\delta = \epsilon/5$. If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5\tfrac{\epsilon}{5} = \epsilon.$$

Which proves that $\lim_{x \to 1}(5x - 3) = 2$.

Example:

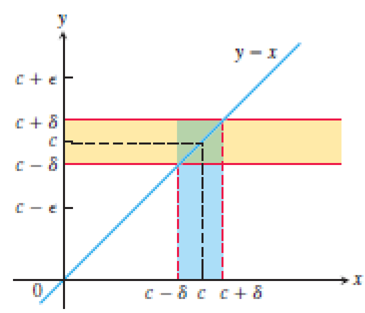
Prove the following results

a) $\lim_{x \to c} x = c$

b) $\lim_{x \to c} k = k$. (k constant).

Solution:

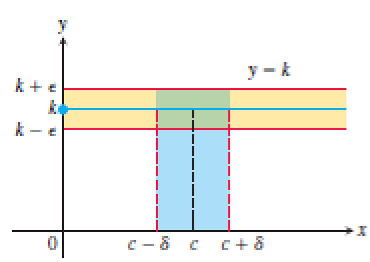
a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x



$$0 < |x - c| < \delta \Rightarrow |x - c| < \epsilon.$$

The implication will hold if δ equals ϵ or any smaller positive number. This proves that $\lim_{x \to c} x = c$.

b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x



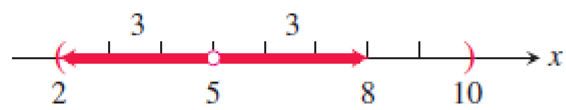
$$0 < |x - c| < \delta \Rightarrow |k - k| < \epsilon.$$

Since k - k = 0, we can use any positive number for δ and the implication will hold. This proves that $\lim_{x \to c} k = k$.

Example:

For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

$$0 < |x-5| < \delta \Rightarrow \left|\sqrt{x-1} - 2\right| < 1.$$



Solution:

We organize the search into two steps.

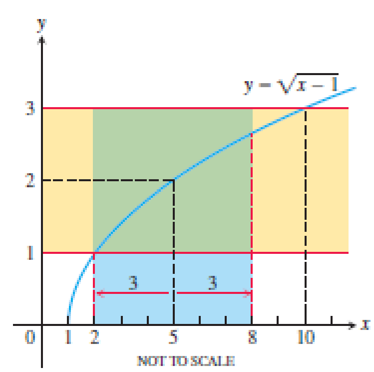
1. Solve the inequality $\left|\sqrt{x-1} - 2\right| < 1$ to find an interval containing $x = 5$ on which the inequality holds for all $x \neq 5$.

$$\left|\sqrt{x-1} - 2\right| < 1$$
$$-1 < \sqrt{x-1} - 2 < 1$$
$$1 < \sqrt{x-1} < 3$$
$$1 < x-1 < 9$$
$$2 < x < 10.$$

The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.

2. Find a value of $\delta > 0$ to place the centered interval $5 - \delta < x < 5 + \delta$ (centered at $x = 5$) inside the interval (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3. If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x-5| < \delta$ will automatically place x between 2 and 10 to make $\left|\sqrt{x-1} - 2\right| < 1$:

$$0 < |x-5| < \delta \Rightarrow \left|\sqrt{x-1} - 2\right| < 1.$$



Example:

Prove that $\lim_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$.

Solution:

Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

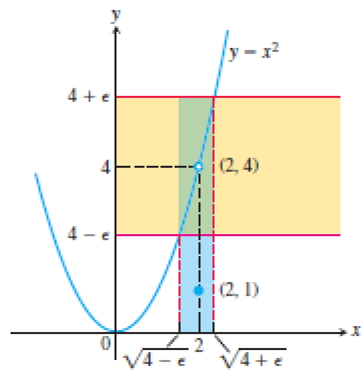
$$0 < |x-2| < \delta \Rightarrow |f(x)-4| < \epsilon.$$

1. Solve the inequality $|f(x)-4| < \epsilon$ to find an interval containing $x = 2$ on which the inequality holds for all $x \neq 2$.

For $x \neq c = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$|x^2-4| < \epsilon$$
$$-\epsilon < x^2 - 4 < \epsilon$$
$$4-\epsilon < x^2 < 4+\epsilon$$
$$\sqrt{4-\epsilon} < |x| < \sqrt{4+\epsilon} \qquad \{\text{Assumes } \epsilon < 4\}$$
$$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon} \qquad \text{An open interval about } x = 2 \text{ that solves the inequality}$$

The inequality $|f(x)-4| < \epsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$.



2. Find a value of $\delta > 0$ that places the centered interval $(2-\delta, 2+\delta)$ inside the interval $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$. Take δ to be the distance from $x = 2$ to the nearer endpoint of $(\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$. In other words, take $\delta = min\{2-\sqrt{4-\epsilon},\ \sqrt{4+\epsilon}-2\}$, the minimum (the smaller) of the two numbers $2-\sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon}-2$. If δ has this or any smaller positive value, the inequality $0 < |x-2| < \delta$ will automatically place x between $\sqrt{4-\epsilon}$ and $\sqrt{4+\epsilon}$ to make $|f(x)-4| < \epsilon$. For all x,

$$0 < |x-2| < \delta \Rightarrow |f(x)-4| < \epsilon.$$

If $\epsilon \geq 4$, then we take δ to be the distance from $x = 2$ to the nearer endpoint of the interval $(0, \sqrt{4+\epsilon})$. In other words, take $\delta = min\{2,\ \sqrt{4+\epsilon}-2\}$.

Exercises:

1. In following sketch, the interval (a, b) on the x-axis with the point c inside. Then find a value of $\delta > 0$ such that for all x, $0 < |x - c| < \delta \Rightarrow a < x < b$.

a) $a = 1, b = 7, c = 5$ **b)** $a = 1, b = 7, c = 2$

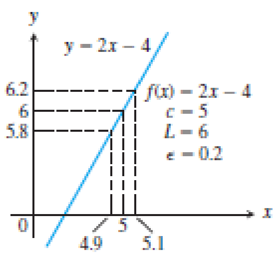
c) $a = -7/2\,, b = -1/2\,, c = -3$ **d)** $a = -7/2\,, b = -1/2\,, c = -3/2$

e) $a = 4/9\,, b = 4/7\,, c = 1/2$

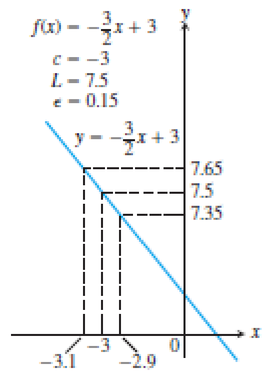
2. In Exercises 7–14, use the graphs to find a $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

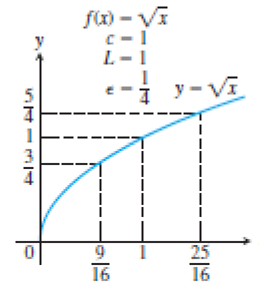
a)



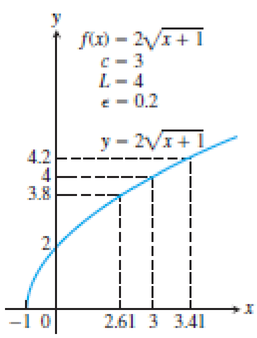
b)



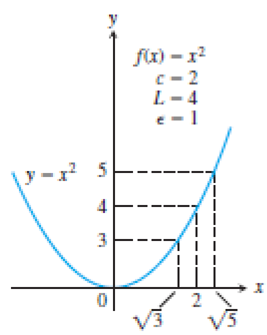
c)



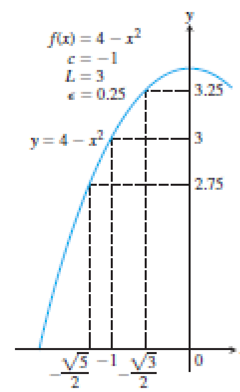
d)



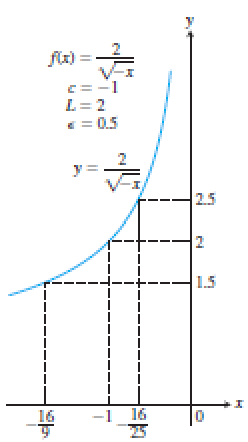
e)



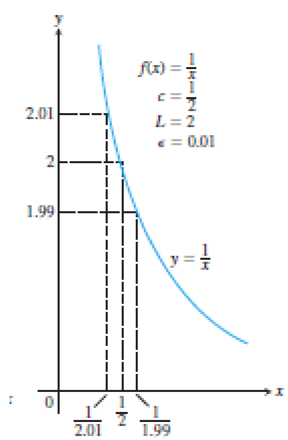
f)



g)



h)



3. Each of following gives a function $f(\text{x})$ and numbers L, c, and $\epsilon > 0$. In each case, find an open interval about c on which the inequality $|f(x) - L| < \epsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \epsilon$.

a) $f(x) = x + 1, L = 5, c = 4, \epsilon = 0.01$
b) $f(x) = 2x - 2, L = -6, c = -2, \epsilon = 0.02$
c) $f(x) = \sqrt{x+1}, L = 1, c = 0, \epsilon = 0.1$
d) $f(x) = \sqrt{x}, L = 1/2, c = 1/4, \epsilon = 0.1$
e) $f(x) = \sqrt{19 - x}, L = 3, c = 10, \epsilon = 1$
f) $f(x) = \sqrt{x - 7}, L = 4, c = 23, \epsilon = 1$
g) $f(x) = 1/x, L = 1/4, c = 4, \epsilon = 0.05$
h) $f(x) = x^2 + 1, L = 3, c = \sqrt{3}, \epsilon = 0.1$
i) $f(x) = x^2, L = 4, c = -2, \epsilon = 0.5$
j) $f(x) = 1/x, L = -1, c = -1, \epsilon = 0.1$

4. Each of following gives a function $f(x)$, a point c, and a positive number ϵ. Find $L = \lim_{x \to c} f(x)$.Then find a number $\delta > 0$ such that for all x

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

a) $f(x) = 3 - 2x, c = 3, \epsilon = 0.02$
b) $f(x) = -3x - 2, c = -1, \epsilon = 0.03$
c) $f(x) = \frac{x^2-4}{x-2}, c = 2, \epsilon = 0.05$
d) $f(x) = \frac{x^2+6x+5}{x+5}, c = -5, \epsilon = 0.05$
e) $f(x) = \sqrt{1 - 5x}, c = -3, \epsilon = 0.5$
f) $f(x) = 4/x, c = 4, \epsilon = 0.4$

5. Prove the limit statements of following

a) $\lim_{x \to 4}(9 - x) = 5$
b) $\lim_{x \to 3}(3x - 7) = 2$
c) $\lim_{x \to 9}\sqrt{x - 5} = 2$
d) $\lim_{x \to 0}\sqrt{4 - x} = 2$
e) $\lim_{x \to 1} f(x) = 1 \; if \; f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$
f) $\lim_{x \to -2} f(x) = 4 \; if \; f(x) = \begin{cases} x^2 & x \neq -2 \\ 1 & x = -2 \end{cases}$
g) $\lim_{x \to 1}(1/x) = 1$
h) $\lim_{x \to \sqrt{3}}(1/x^2) = 1/3$
i) $\lim_{x \to -3}\frac{x^2-9}{x+3} = -6$
j) $\lim_{x \to 1}\frac{x^2-1}{x-1} = 2$
k) $\lim_{x \to 1} f(x) = 2 \; if \; f(x) = \begin{cases} 4 - 2x & x < 1 \\ 6x - 4 & x \geq 1 \end{cases}$
l) $\lim_{x \to 0} f(x) = 0 \; if \; f(x) = \begin{cases} 2x & x < 0 \\ x/2 & x \geq 0 \end{cases}$
m) $\lim_{x \to 0} x \sin 1/x = 0$
n) $\lim_{x \to 0} x^2 \sin 1/x = 0$

Calculus I

First Semester

Lecturer 5

Dr. Ban Jaffar AL-Taiy

Taghreed Hussein Abed

2.4 One-Sided Limits

Definition:

We say that $f(x)$ has ***right-hand limit L at c***, and write

$$\lim_{x\to c^+} f(x) = L,$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

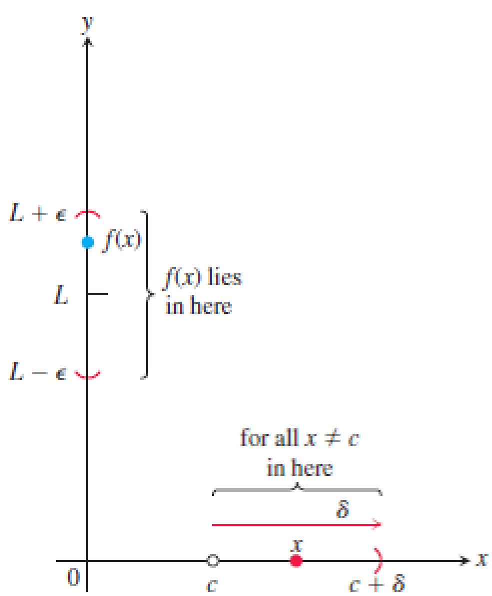
$$c < x < c + \delta \Rightarrow |f(x) - L| < \epsilon.$$

We say that f has ***left-hand limit L at c***, and write

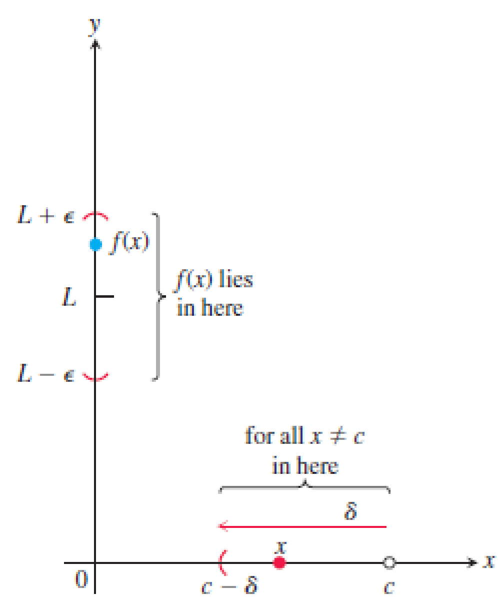
$$\lim_{x\to c^-} f(x) = L,$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \Rightarrow |f(x) - L| < \epsilon.$$



Intervals associated with
the definition of right-hand limit.



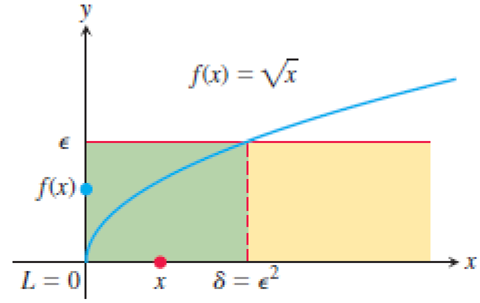
Intervals associated with
the definition of left-hand limit.

Example

Prove that $\lim_{x \to 0} \sqrt{x} = 0$.

Solution:

Let $\epsilon > 0$ be given. Here c = 0 and L = 0, so we want to find a $\delta > 0$ such that for all *x*



$$0 < x < \delta \Rightarrow \left|\sqrt{x} - 0\right| < \epsilon,$$

$$0 < x < \delta \Rightarrow \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives $x < \epsilon^2$ if $0 < x < \delta$. If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \ \Rightarrow \sqrt{x} < \epsilon \qquad \text{or} \qquad 0 < x < \epsilon^2 \ \Rightarrow \left|\sqrt{x} - 0\right| < \epsilon.$$

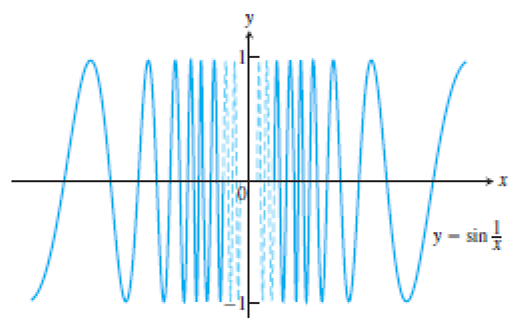
According to the definition, this shows that $\lim_{x \to 0} \sqrt{x} = 0$.

Example:

Show that $y = sin\left(\frac{1}{x}\right)$ has no limit as *x* approaches zero from either side.

Solution:

As x approaches zero, its reciprocal, $\frac{1}{x}$, grows without bound and the values of $sin\left(\frac{1}{x}\right)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero.



This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$.

Theorem (Limit of the Ratio $(sin\,\theta)/\theta$ as $\theta \to 0$):

$$\lim_{\theta \to 0} \frac{sin\,\theta}{\theta} = 1 \quad (\theta \text{ in radians}) \qquad (1)$$

Example:

Show that **a)** $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and **b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution:

a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h}$$

$$= -\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \sin\theta \qquad \text{Let } \theta = h/2.$$

$$= -(1)(0).$$

b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\lim_{x \to 0} \frac{\sin 2x}{5x} = \lim_{x \to 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} = \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} = \frac{2}{5}(1) = \frac{2}{5}$$

Now, Eq. (1) applies with $\theta = 2x$.

Example:

Find $\lim_{t \to 0} \frac{\tan t \cdot \sec 2t}{3t}$

Solution:

From the definition of $\tan t$ and $\sec 2t$, we have

$$\lim_{t \to 0} \frac{\tan t \cdot \sec 2t}{3t} = \lim_{t \to 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t}$$

$$= \frac{1}{3} \lim_{t \to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t}$$

$$= \frac{1}{3}(1)(1)(1).$$

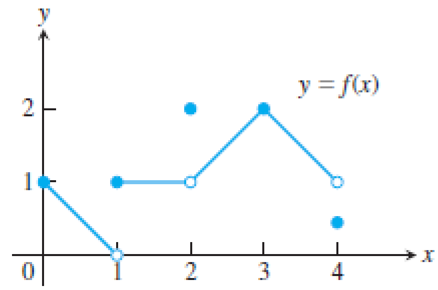
Theorem:

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Example:

For the function graphed in following Figure



At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$,

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$ $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$ $\lim_{x\to 2^-} f(x) = 1$,

$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$,

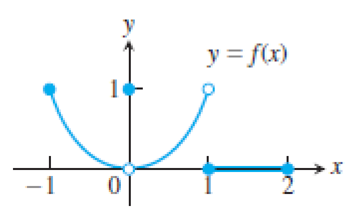
At $x = 4$ $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$.

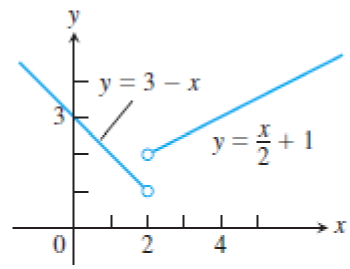
Exercises:

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



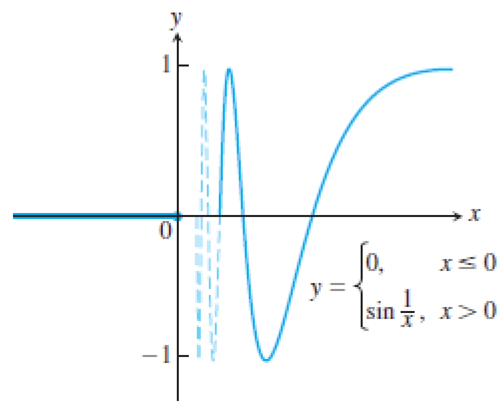
a) $\lim_{x\to -1^+} f(x) = 1$ **b)** $\lim_{x\to 0^-} f(x) = 0$ **c)** $\lim_{x\to 0^-} f(x) = 1$
d) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ **e)** $\lim_{x\to 0} f(x)$ exists. **f)** $\lim_{x\to 0} f(x) = 0$
g) $\lim_{x\to 0} f(x) = 1$ **h)** $\lim_{x\to 1} f(x) = 1$ **i)** $\lim_{x\to 1} f(x) = 0$
j) $\lim_{x\to 2^-} f(x) = 2$ **k)** $\lim_{x\to -1^-} f(x)$ does not exist. **l)** $\lim_{x\to 2^+} f(x) = 0$

2. Let $f(x) = \begin{cases} 3 - x & x < 2 \\ \frac{x}{2}+1 & x > 2. \end{cases}$



a) Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.
b) Does $\text{l}\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?
c) Find $\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4^-} f(x)$.
d) Does $\text{l}\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

3. Let $f(x) = \begin{cases} 0 & x \le 0 \\ \sin\frac{1}{x} & x > 0. \end{cases}$



a) Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?
b) Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?
c) Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

4. Graph the following functions. Then answer these questions
a) What are the domain and range of f?
b) At what points c, if any, does $\lim_{x\to c} f(x)$ exist?
c) At what points does only the left-hand limit exist?
d) At what points does only the right-hand limit exist?

1. $f(x) = \begin{cases} \sqrt{1-x^2} & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ 2 & x = 2 \end{cases}$

2. $f(x) = \begin{cases} x & -1 \le x < 0 \text{ or } 0 < x \le 1 \\ 1 & x = 0 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$

5. Find the limits in following

a) $\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}}$ **b)** $\lim_{h\to 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

c) $\lim_{h\to 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}$ **d)** $\lim_{x\to 1^-} \left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)$

e) $\lim\limits_{x\to 2^-}(x+3)^{\frac{|x+2|}{x+2}}$

f) $\lim\limits_{x\to 2^+}(x+3)^{\frac{|x+2|}{x+2}}$

g) $\lim\limits_{x\to 1^-}\frac{\sqrt{2x}(x-1)}{|x-1|}$

h) $\lim\limits_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}$

i) $\lim\limits_{\theta\to 3^+}\frac{\lfloor\theta\rfloor}{\theta}$

j) $\lim\limits_{\theta\to 3^-}\frac{\lfloor\theta\rfloor}{\theta}$

k) $\lim\limits_{t\to 4^+}(t-\lfloor t\rfloor)$

l) $\lim\limits_{t\to 4^-}(t-\lfloor t\rfloor)$

m) $\lim\limits_{t\to 0}\frac{\sin kt}{t}$ (k constant)

n) $\lim\limits_{h\to 0^-}\frac{h}{\sin 3h}$

o) $\lim\limits_{x\to 0}\frac{\tan 2x}{x}$

p) $\lim\limits_{x\to 0}\frac{x\csc 2x}{\cos 5x}$

q) $\lim\limits_{x\to 0}6x^2(\cot x)(\csc 2x)$

r) $\lim\limits_{x\to 0}\frac{x+x\cos x}{\sin x\cos x}$

s) $\lim\limits_{x\to 0}\frac{x^2-x+\sin x}{2x}$

t) $\lim\limits_{\theta\to 0}\frac{1-\cos\theta}{\sin 2\theta}$

u) $\lim\limits_{x\to 0}\frac{x-x\cos x}{\sin^2 3x}$

v) $\lim\limits_{y\to 0}\frac{\sin 3y\cot 5y}{y\cot 4y}$

w) $\lim\limits_{\theta\to 0}\frac{\tan\theta}{\theta^2\cot 3\theta}$

x) $\lim\limits_{\theta\to 0}\frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

2.5 Limits Involving Infinity, Infinite Limits.

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x\to\pm\infty$. We further extend the concept of limit to infinite limits. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these ideas to analyze the graphs of functions having horizontal or vertical asymptotes.

2.5.1 Finite Limits as $x\to\pm\infty$:

Definition:

1. We say that f(x) has the ***limit L as x approaches infinity*** and write

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \epsilon \text{ whenever } x > M.$$

2. We say that f(x) has the ***limit L as x approaches negative infinity*** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

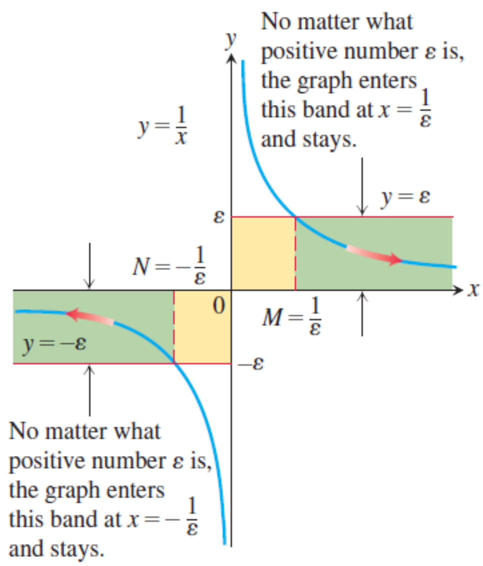
$$|f(x) - L| < \epsilon \text{ whenever } x < N.$$

Example(*):

Show that

a) $\lim_{x \to \infty} \frac{1}{x} = 0$ **b)** $\lim_{x \to -\infty} \frac{1}{x} = 0$

Solution:



a) Let $\epsilon > 0$ be given. We must find a number M

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon \text{ whenever } x > M.$$

The implication will hold if $M = 1/\epsilon$ or any larger positive number. This proves $\lim_{x \to \infty} \frac{1}{x} = 0$.

b) Let $\epsilon > 0$ be given. We must find a number N

such that

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon \text{ whenever } x < N.$$

The implication will hold if $N = -1/\epsilon$ or any number less than $-1/\epsilon$. This proves $\lim_{x \to \infty} \frac{1}{x} = 0$.

Remark:

Limits at infinity have properties similar to those of finite limits.

Theorem:

All the Limit Laws are true when we replace $\lim_{x \to c}$ ***by*** $\lim_{x \to \infty}$ ***or*** $\lim_{x \to -\infty}$***.***
That is, the variable x may approach a finite number c or $\pm\infty$***.***

Example:

Used the properties of Limit Laws to calculate limits in the same way as when x approaches a finite number c.

a) $\lim\limits_{x\to\infty}(5+\frac{1}{x}) = \lim\limits_{x\to\infty} 5 + \lim\limits_{x\to\infty}\frac{1}{x}$ **Sum Rule**

$= 5 + 0 = 5.$ **Known limits**

b) $\lim\limits_{x\to-\infty}\frac{\pi\sqrt{3}}{x^2} = \lim\limits_{x\to-\infty}\pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$ **Product Rule**

$= \lim\limits_{x\to-\infty}\pi\sqrt{3}\cdot\lim\limits_{x\to-\infty}\frac{1}{x}\cdot\lim\limits_{x\to-\infty}\frac{1}{x}$ **Known limits**

$= \pi\sqrt{3}\cdot 0\cdot 0 = 0.$

2.5.2 Limits at Infinity of Rational Functions:

Remark:

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

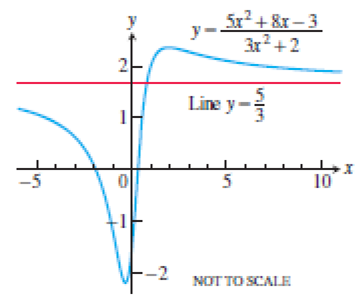
The following examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

Examples:

a) $\lim\limits_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} = \lim\limits_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}$

Divide numerator and denominator by x^2.

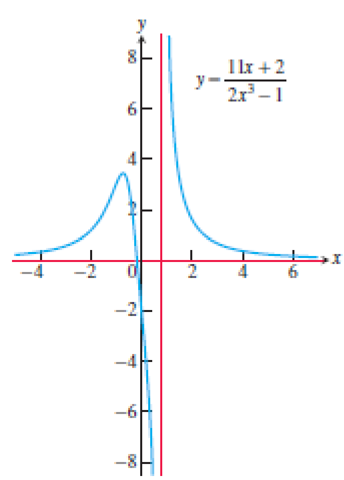
$= \frac{5+0-0}{3+0} = \frac{5}{3}.$



b) $\lim\limits_{x\to-\infty}\frac{11x+2}{2x^3-1} = \lim\limits_{x\to\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}$

Divide numerator and denominator by x^3.

$= \frac{0+0}{2-0} = 0.$



2.5.3 Horizontal Asymptotes:

Remark:

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an ***asymptote*** of the graph.

Looking at $f(x) = 1/x$, we observe that the x-axis is an asymptote of the curve on the right because $\lim_{x\to\infty} \frac{1}{x} = 0$ and on the left because $\lim_{x\to-\infty} \frac{1}{x} = 0$. We say that the x-axis is a **horizontal asymptote** of the graph of $f(x) = 1/x$.

Definition:

A line $y = b$ is **a horizontal asymptote** of the graph of a function

$y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

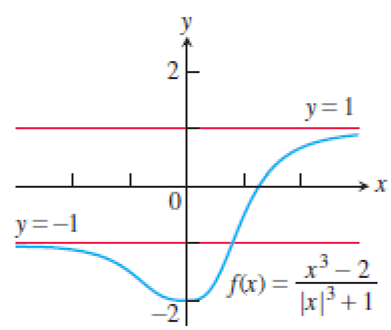
Remark:

The graph of a function can has zero, one, or two horizontal asymptotes, depending on whether the function has limits as $x \to \infty$ and as $x \to -\infty$. For example, the function $f(x) = \frac{5x^2+8x-3}{3x^2+2}$ has the line $y = \frac{5}{3}$ as a horizontal asymptote on both the right and the left because $\lim_{x\to\infty} f(x) = \frac{5}{3}$ and $\lim_{x\to\infty} f(x) = \frac{5}{3}$.

Example:

Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3-2}{|x|^3+1}.$$



Solution:

We calculate the limits as $x \to \pm\infty$.

For $x \geq 0$: $\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty} \frac{x^3-2}{x^3+1} = \lim_{x\to\infty} \frac{1-(2/x^3)}{1+(1/x^3)} = 1.$

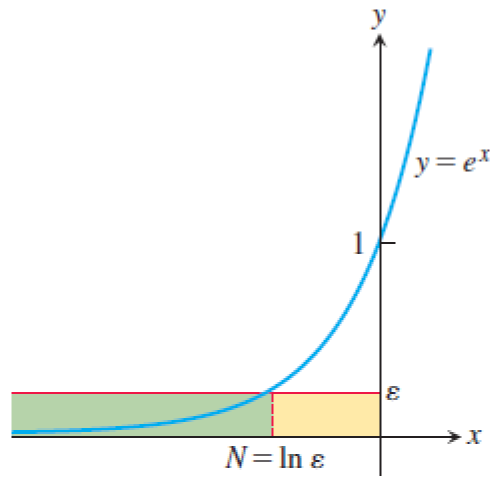
For $x < 0$: $\lim_{x\to-\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to-\infty} \frac{x^3-2}{(-x)^3+1} = \lim_{x\to-\infty} \frac{1-(2/x^3)}{-1+(1/x^3)} = -1.$

The horizontal asymptotes are $y = -1$ and $y = 1$. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x.

Example:

Find the horizontal asymptotes of the graph of

$$f(x) = e^x.$$



Solution:

The x-axis (the line $y = 0$) is a horizontal asymptote of the graph of $f(x) = e^x$ because $\lim_{x\to-\infty} e^x = 0$. To see this, we use the definition of a limit as x approaches $-\infty$. So let $\epsilon > 0$ be given, but arbitrary. We must find a constant N such that $|e^x - 0| < \epsilon$ whenever $x < N$.

Now $|e^x - 0| = e^x$, so the condition that needs to be satisfied whenever $x < N$ is $e^x < \epsilon$.

Let $x = N$ be the number where $e^x = \epsilon$. Since e^x is an increasing function, if $x < N$, then $e^x < \epsilon$. We find N by taking the natural logarithm of both sides of the equation $e^N = \epsilon$, so $N = ln\epsilon$. With this value of N the condition is satisfied, and we conclude that $\lim_{x\to-\infty} e^x = 0$.

Remark:

Sometimes it is helpful to transform a limit in which x approaches ∞ to a new limit by setting $t = 1/x$ and seeing what happens as t approaches 0.

Example:

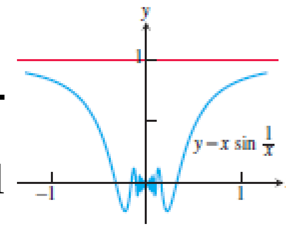
Find **a)** $\lim_{x\to\infty} \sin\frac{1}{x}$ **b)** $\lim_{x\to\pm\infty} x\sin\frac{1}{x}$

Solution:

a) We introduce the new variable $t = 1/x$. From Example (*), we know that $t \to 0^+$ as $x \to \infty$. Therefore, $\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0^+} \sin t = 0$.

b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0^+}\frac{\sin t}{t} = 1$ and $\lim_{x\to-\infty} x\sin\frac{1}{x} = \lim_{t\to 0^-}\frac{\sin t}{t} = 1$.



The graph of $x\sin\frac{1}{x}$, show that the line y = 1 is a horizontal asymptote.

Remark:

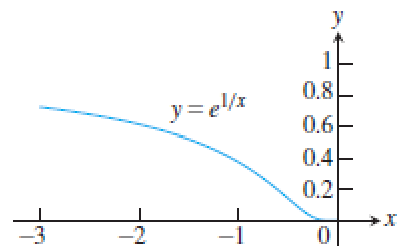
Similarly, we can investigate the behavior of $y = f\left(\frac{1}{x}\right)$ as $x \to 0$ by investigating $y = f(t)$ as $t \to \pm\infty$, where $t = 1/x$.

Example:

Find $\lim_{x\to 0^-} e^{1/x}$.

Solution:

We let $t = 1/x$ so that $\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^{t} = 0$.

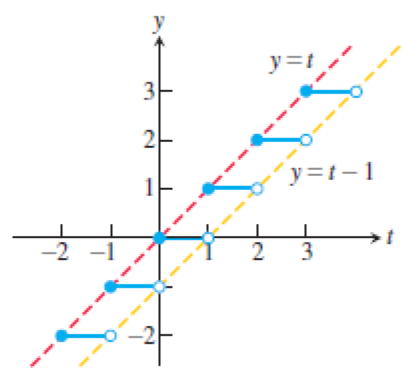


Example:

Find $\lim_{x\to 0^+} x\left\lfloor\frac{1}{x}\right\rfloor$.

Solution:

We let $t = 1/x$ so that $\lim_{x\to 0^+} x\left\lfloor\frac{1}{x}\right\rfloor = \lim_{t\to\infty}\frac{1}{t}\lfloor t\rfloor$.



From the graph of $x\left\lfloor\frac{1}{x}\right\rfloor$, we see that $t - 1 \le \lfloor t\rfloor \le t$, which gives

$$1 - \frac{1}{t} \le \frac{1}{t}\lfloor t\rfloor \le 1$$

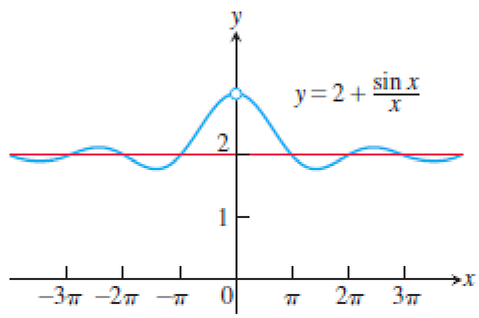
Multiply inequalities by $\frac{1}{t} > 0$.

It follows from the Sandwich Theorem that $\lim_{t\to\infty}\frac{1}{t}\lfloor t\rfloor = 1$, so 1 is the value of the limit we seek.

Remark:

The Sandwich Theorem also holds for limits as $x \to \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions at very large values of x in magnitude consistent with whether $x \to \infty$ or $x \to -\infty$.

Example:



Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution:

We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x\to\pm\infty}\left|\frac{1}{x}\right| = 0$, we have $\lim_{x\to\pm\infty}\frac{\sin x}{x} = 0$ by the Sandwich Theorem. Hence, $\lim_{x\to\pm\infty}\left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2$, and the line $y = 2$ is a horizontal asymptote of the curve on both left and right. This example illustrates that a curve may cross one of its horizontal asymptotes many times.

Example:

Find $\lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right)$.

Solution:

Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \to \infty$, so what happens to the difference in the limit is unclear (we cannot subtract from because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic expression:

$$\lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right) = \lim_{x\to\infty}\left(x - \sqrt{x^2 + 16}\right) \cdot \frac{x+\sqrt{x^2+16}}{x+\sqrt{x^2+16}}$$ **Multiply and divide**

by the conjugate.

$$= \lim_{x\to\infty} \frac{x^2-(x^2+16)}{x+\sqrt{x^2+16}} = \lim_{x\to\infty} \frac{-16}{x+\sqrt{x^2+16}}$$

As $x \to \infty$, the denominator in this last expression becomes arbitrarily large, while the numerator remains constant, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x\to\infty} \frac{-16}{x+\sqrt{x^2+16}} = \lim_{x\to\infty} \frac{\frac{-16}{x}}{1+\sqrt{\frac{x^2}{x^2}+\frac{16}{x^2}}} = \frac{0}{1+\sqrt{1+0}} = 0.$$

2.5.4 Oblique Asymptotes:

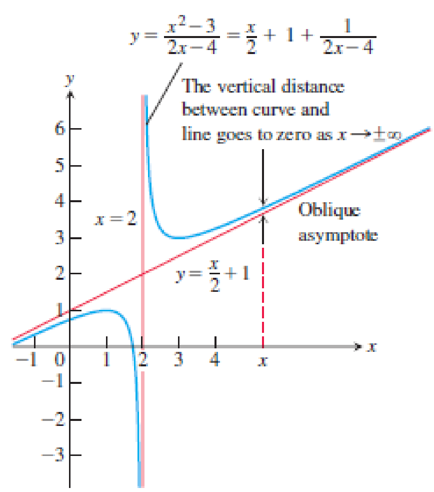
Remark:

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

Example:

Find the oblique asymptote of the graph of $f(x) = \frac{x^2-3}{2x-4}$.

Solution:



$$f(x) = \frac{x^2-3}{2x-4} = \frac{2\cdot(x^2-3)}{2.(2x-4)} = \frac{2x^2-6}{2(2x-4)} = \frac{2x^2-8+2}{2(2x-4)}$$

$$= \frac{(2x^2-8)+2}{2(2x-4)} = \frac{(2x^2-8)}{2(2x-4)} + \frac{2}{2(2x-4)}$$

$$= \frac{(2x^2-4x+4x-8)}{2(2x-4)} + \frac{(2x^2-4x+4x-8)+2}{2(2x-4)}$$

$$f(x) = \frac{x^2-3}{2x-4} = \frac{x^2-4+1}{(2x-4)} = \frac{x^2-4}{(2x-4)} + \frac{1}{(2x-4)}$$

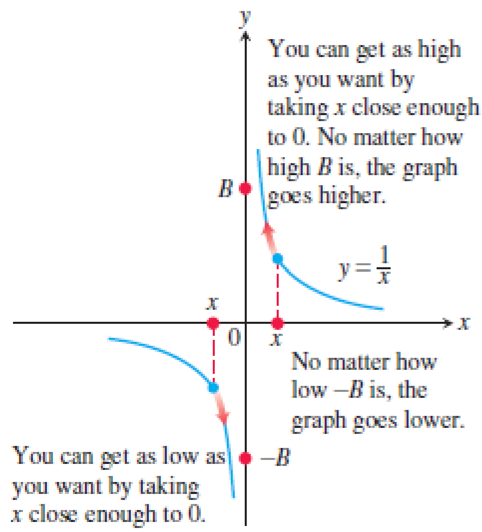
$$= \frac{(x-2)(x+2)}{2(x-2)} + \frac{1}{(2x-4)} = \frac{(x+2)}{2} + \frac{1}{(2x-4)} = \underbrace{(\frac{x}{2}+1)}_{linear\ g(x)} + \frac{1}{(2x-4)}$$

As $x \to \pm\infty$, the part $\frac{1}{(2x-4)}$, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line $g(x) = \frac{x}{2} + 1$ an asymptote of the graph of f. The line $y = g(x)$ is an asymptote both to the right and to the left.

Notice in Example

2.5.5 Infinite Limits:

Remark:



Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still.

Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that f(x) approaches ∞ as $x \to 0^+$. We write $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty$.

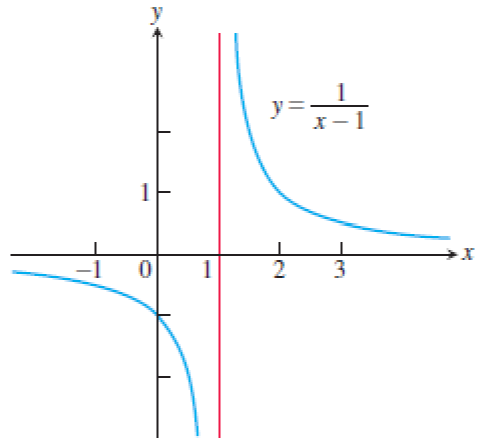
In writing this equation, we are not saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, this expression is just a concise way of saying that $\lim_{x\to 0^+} \frac{1}{x}$ does not exist because $\frac{1}{x}$ becomes arbitrarily large and positive as $x \to 0^+$.

Example:

Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$.

Geometric Solution:

The graph of $y = \frac{1}{x-1}$ is the graph of $y = \frac{1}{x}$ shifted 1 unit to the right. Therefore, $y = \frac{1}{x-1}$ behaves near 1 exactly the way $y = \frac{1}{x}$ behaves near 0:



$$\lim_{x\to 1^+}\frac{1}{x-1} = \infty \quad \text{and} \quad \lim_{x\to 1^-}\frac{1}{x-1} = -\infty.$$

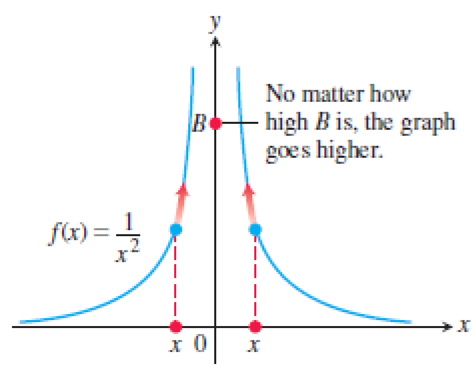
Analytic Solution:

Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $\frac{1}{x-1} \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $\frac{1}{x-1} \to -\infty$.

Example:

Discuss the behavior of $f(x) = \frac{1}{x^2}$ as $x \to 0$.

Solution:



As x approaches zero from either side, the values of $\frac{1}{x^2}$ are positive and become arbitrarily large. This means that $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{x^2} = \infty$.

The function $y = \frac{1}{x}$ shows no consistent behavior as $x \to 0$. We have $\frac{1}{x} \to \infty$ if $x \to 0^+$, but $\frac{1}{x} \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0}\frac{1}{x}$ is that it does not exist. The function $y = \frac{1}{x^2}$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0}\frac{1}{x^2} = \infty$.

Example:

The following examples illustrate that rational functions can behave in various ways near zeros of the denominator

a) $\lim_{x\to 2}\frac{(x-2)^2}{x^2-4} = \lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{(x-2)}{(x+2)} = 0.$ **Can substitute 2 for x after algebraic manipulation eliminates division by 0.**

b) $\lim_{x\to 2}\frac{(x-2)}{x^2-4} = \lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{(x+2)} = \frac{1}{4}.$ **Again substitute 2 for x after algebraic manipulation eliminates division by 0.**

c) $\lim_{x\to 2^+}\frac{(x-3)}{x^2-4} = \lim_{x\to 2^+}\frac{(x-3)}{(x-2)(x+2)} = -\infty.$ **The values are negative for $x > 2$ near 2.**

d) $\lim_{x\to 2^-}\frac{(x-3)}{x^2-4} = \lim_{x\to 2^-}\frac{(x-3)}{(x-2)(x+2)} = \infty.$ **The values are positive for $x < 2$ near 2.**

e) $\lim_{x\to 2}\frac{(x-3)}{x^2-4} = \lim_{x\to 2}\frac{(x-3)}{(x-2)(x+2)}$ does not exist. **Limits from left and from right differ.**

f) $\lim_{x\to 2}\frac{(2-x)}{(x-2)^3} = \lim_{x\to 2}\frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2}\frac{-1}{(x-2)^2} = -\infty.$ **Denominator is positive, so values are negative near $x = 2$.**

In parts (a) and (b), the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus, a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator.

Example:

Find $\lim_{x\to -\infty}\frac{2x^5-6x^4+1}{3x^2+x-7}$.

Solution:

We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$\lim_{x\to -\infty}\frac{2x^5-6x^4+1}{3x^2+x-7} = \lim_{x\to -\infty}\frac{2x^3-6x^2+x^{-2}}{3+x^{-1}-7x^{-2}} = \lim_{x\to -\infty}\frac{2x^2(x-3)+x^{-2}}{3+x^{-1}-7x^{-2}} = -\infty,$$ **$x^{-2} \to 0, (x-3) \to -\infty$**

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$.

2.5.6 Precise Definitions of Infinite Limits:

Definition:

1. We say that $f(x)$ **approaches infinity as x approaches c**, and write

$$\lim_{x\to c} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that

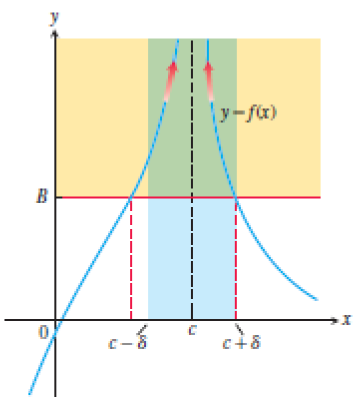
$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that $f(x)$ **approaches negative infinity as x approaches c**, and write

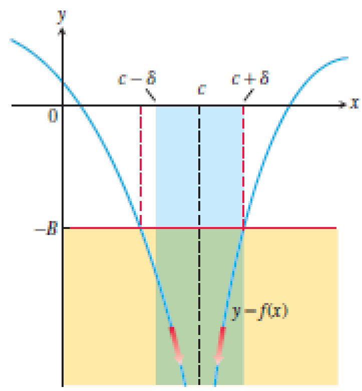
$$\lim_{x\to c} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that

$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$



For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.



For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

Example:

Prove that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Solution:

Given B > 0, we want to find $\delta > 0$ such that

$$\frac{1}{x^2} > B \quad \text{whenever} \quad 0 < |x - 0| < \delta.$$

Now, $\frac{1}{x^2} > B$ if and only if $x^2 < \frac{1}{B}$ or, equivalently, $|x| < \frac{1}{\sqrt{B}}$. Thus, choosing $\delta = \frac{1}{\sqrt{B}}$ (or any smaller positive number), we see that if $0 < |x| < \delta$ then $\frac{1}{x^2} > \frac{1}{\delta^2} \geq B$. Therefore, by definition, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

2.5.7 Vertical Asymptotes:

Definition:

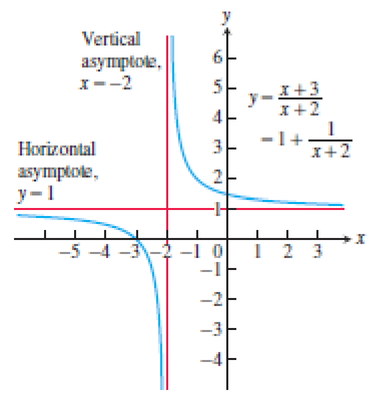
A line $x = a$ is a ***vertical asymptote*** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a^-} f(x) = \pm\infty \, .$$

Example:

Find the horizontal and vertical asymptotes of the curve $y = \frac{x+3}{x+2}$.

Solution:



We are interested in the behavior as $x \to \pm\infty$, and the behavior as $x \to -2$, where the denominator is zero.

$$y = \frac{x+3}{x+2} = \frac{x+2+1}{x+2} = \frac{x+2}{x+2} + \frac{1}{x+2} = 1 + \frac{1}{x+2} \, .$$

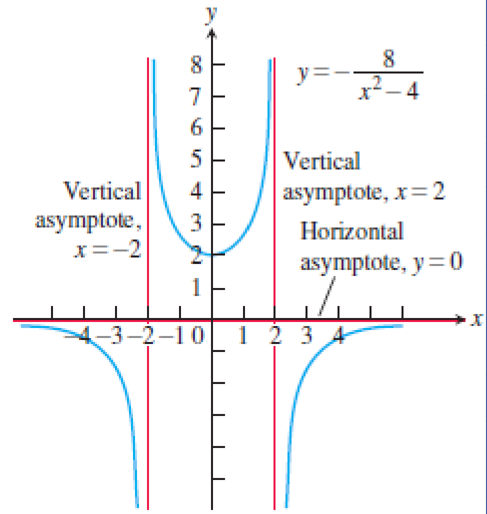
We see that the curve in question is the graph of $f(x) = \frac{1}{x}$ shifted 1 unit up and 2 units left. The asymptotes, instead of being the coordinate axes, are

now the lines y = 1 and $x = -2$. As $x \to \pm\infty$., the curve approaches the horizontal asymptote y = 1; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$.

Example:

Find the horizontal and vertical asymptotes of the curve $f(x) = \frac{-8}{x^2-4}$.

Solution:



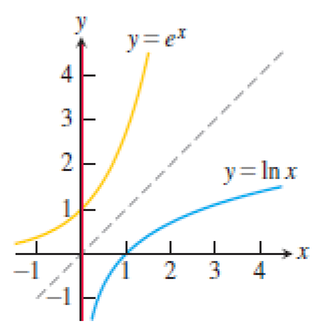
We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$., where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

a) The behavior as $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line y = 0 is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well. Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

b) The behavior as $x \to \pm 2$. Since $\lim_{x\to 2^+} f(x) = -\infty$ and $\lim_{x\to 2^-} f(x) = \infty$,the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points.

Example:



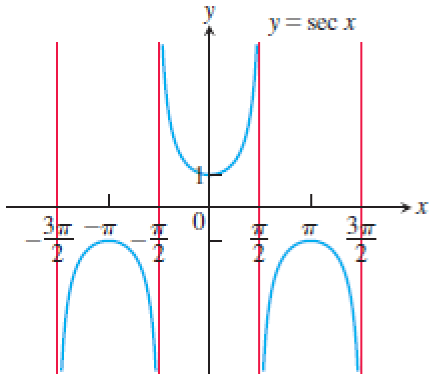
The graph of the natural logarithm function has the y-axis (the line $x = 0$) as a vertical asymptote. We see this from the graph (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x-axis is a horizontal asymptote of $y = e^x$. Thus,

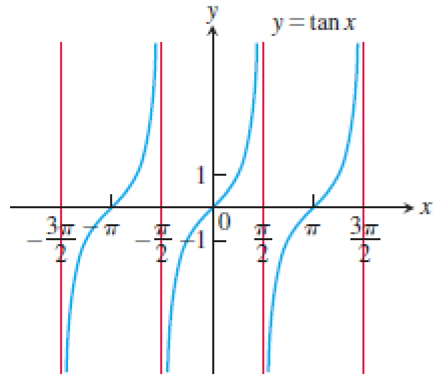
$$\lim_{x\to 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$.

Example:

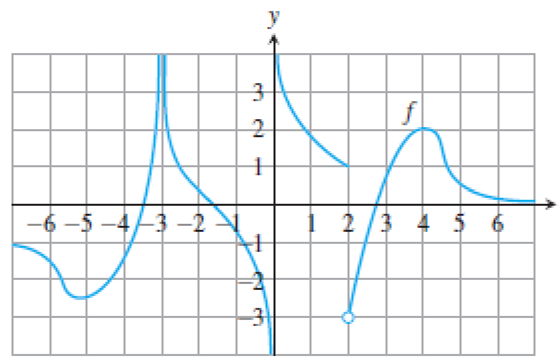
The curves $y = \sec x = \frac{1}{\cos x}$ and $y = \tan x = \frac{\sin x}{\cos x}$ both have vertical asymptotes at odd-integer multiples of $\pi/2$, which are the points where $cos\ x = 0$.





Exercises:

1. For the function f whose graph is given, determine the following limits. Write ∞ or $-\infty$ where appropriate



a) $\lim_{x\to 4} f(x)$ **b)** $\lim_{x\to 2^+} f(x)$ **c)** $\lim_{x\to 2^-} f(x)$

d) $\lim_{x\to 2} f(x)$ **e)** $\lim_{x\to 3^+} f(x)$ **f)** $\lim_{x\to 3^-} f(x)$

g) $\lim_{x\to -3} f(x)$ **h)** $\lim_{x\to 0^+} f(x)$ **i)** $\lim_{x\to 0^-} f(x)$

j) $\lim_{x\to 0} f(x)$ **k)** $\lim_{x\to \infty} f(x)$ **l)** $\lim_{x\to -\infty} f(x)$

2. In following, find the limit of each function

a) as $x \to \infty$ and

b) as $x \to -\infty$.

i. $f(x) = \frac{2}{x} - 3$ ii. $f(x) = \pi - \frac{2}{x^2}$ iii. $g(x) = \frac{1}{2+(1/x)}$

iv. $g(x) = \frac{1}{8-(5/x^2)}$ v. $h(x) = \frac{-5+(7/x)}{3-(1/x^2)}$ vi. $h(x) = \frac{3-(2/x)}{4+(\sqrt{2}/x^2)}$

3. Find the limits in following

a) $\lim_{x\to\infty} \frac{\sin 2x}{x}$ **b)** $\lim_{\theta\to -\infty} \frac{\cos\theta}{3\theta}$

c) $\lim_{t\to -\infty} \frac{2-t+\sin t}{t+\cos t}$ **d)** $\lim_{r\to\infty} \frac{r+\sin r}{2r+7-5\sin r}$

4. In following, find the limit of each rational function

a) as $x \to \infty$ and

b) as $x \to -\infty$.

Write ∞ or $-\infty$ where appropriate.

i) $f(x) = \frac{2x+3}{5x+7}$

ii) $f(x) = \frac{2x^3+7}{x^3-x^2+x+7}$

iii) $h(x) = \frac{7x^3}{x^3-3x^2+6x}$

iv) $g(x) = \frac{10x^5+x^4+31}{x^6}$

v) $f(x) = \frac{3x^7+5x^2-1}{6x^3-7x+3}$

vi) $h(x) = \frac{5x^8-2x^3+9}{3+x-4x^5}$

5. The process by which we determine limits of rational functions applies equally well to ratios containing non-integer or negative powers of *x*. Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in following. Write ∞ or $-\infty$ where appropriate.

a) $\lim\limits_{x\to\infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$

b) $\lim\limits_{x\to-\infty} \left(\frac{x^2+x-1}{8x^2-3}\right)^{1/3}$

c) $\lim\limits_{x\to-\infty} \left(\frac{1-x^3}{x^2+7x}\right)^5$

d) $\lim\limits_{x\to\infty} \frac{2+\sqrt{x}}{2-\sqrt{x}}$

e) $\lim\limits_{x\to\infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$

f) $\lim\limits_{x\to-\infty} \frac{\sqrt[3]{x}-5x+3}{2x+x^{2/3}-4}$

g) $\lim\limits_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$

h) $\lim\limits_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$

6. Find the limits in following. Write ∞ or $-\infty$ where appropriate.

a) $\lim\limits_{x\to 0^+} \frac{1}{3x}$

b) $\lim\limits_{x\to 0^-} \frac{5}{2x}$

c) $\lim\limits_{x\to 5^-} \frac{3x}{2x+10}$

d) $\lim\limits_{x\to 7} \frac{4}{(x-7)^2}$

e) $\lim\limits_{x\to 0} \frac{-1}{x^2(x+1)}$

f) $\lim\limits_{x\to 0} \frac{-1}{x^{2/3}}$

g) $\lim\limits_{x\to(\pi/2)^-} \tan x$

h) $\lim\limits_{x\to(-\pi/2)^+} \sec x$

i) $\lim\limits_{\theta\to 0^-} (1+\csc\theta)$

j) $\lim\limits_{\theta\to 0} (2-\cot\theta)$

7. Find the limits in following. Write ¬∞ or -∞ where appropriate.

1) lim $\frac{1}{x^2-4}$ as

a) $x \to 2^+$ b) $x \to 2^-$

c) $x \to -2^+$ d) $x \to -2^-$

2) lim $\frac{x}{x^2-1}$ as

a) $x \to 1^+$ b) $x \to 1^-$

c) $x \to -1^+$ d) $x \to -1^-$

3) lim $\left(\frac{x^2}{2}-\frac{1}{x}\right)$ as

a) $x \to 0^+$ b) $x \to 0^-$

c) $x \to \sqrt[3]{2}$ d) $x \to -1$

4) lim $\frac{x^2-1}{2x+4}$ as

a) $x \to -2^+$ b) $x \to -2^-$

c) $x \to 1^+$ d) $x \to 0^-$

5) lim $\frac{x^2-3x+2}{x^3-2x^2}$ as

a) $x \to 0^+$ b) $x \to 2^+$

6) lim $\frac{x^2-3x+2}{x^3-4x}$ as

a) $x \to 2^+$ b) $x \to -2^+$

c) $x \to 2^-$ **d)** $x \to 2$
e) What, if anything, can be said about the limit as $x \to 0$?

7) **lim** $\left(2-\frac{3}{x^{1/3}}\right)$ **as**

a) $x \to 0^+$ **b)** $x \to 0^-$

9) **lim** $\left(\frac{1}{x^{2/3}}+\frac{2}{(x-1)^{2/3}}\right)$ **as**

a) $x \to 0^+$ **b)** $x \to 0^-$
c) $x \to 1^+$ **d)** $x \to 1^-$

c) $x \to 0^-$ **d)** $x \to 1^+$
e) What, if anything, can be said about the limit as $x \to 0$?

8) **lim** $\left(\frac{1}{x^{3/5}}+7\right)$ **as**

a) $x \to 0^+$ **b)** $x \to 0^-$

10) **lim** $\left(\frac{1}{x^{1/3}}+\frac{2}{(x-1)^{4/3}}\right)$ **as**

a) $x \to 0^+$ **b)** $x \to 0^-$
c) $x \to 1^+$ **d)** $x \to 1^-$

8. Graph the rational functions in following. Include the graphs and equations of the asymptotes and dominant terms.

a) $y=\frac{1}{x-1}$ **b)** $y=\frac{1}{x+1}$ **c)** $y=\frac{1}{2x+4}$
d) $y=\frac{-3}{x-3}$ **e)** $y=\frac{x+3}{x+2}$ **f)** $y=\frac{2x}{x+1}$

9. Determine the domain of each function in following. Then use various limits to find the asymptotes.

a) $y=4+\frac{3x^2}{x^2+1}$ **b)** $y=\frac{2x}{x^2-1}$ **c)** $y=\frac{\sqrt{x^2+4}}{x}$ **d)** $y=\frac{x^3}{x^3-8}$

10. Find the limits in following

a) $\lim_{x\to\infty}\left(\sqrt{x+9}-\sqrt{x+4}\right)$ **b)** $\lim_{x\to\infty}\left(\sqrt{x^2+25}-\sqrt{x^2-1}\right)$ **c)** $\lim_{x\to-\infty}\left(\sqrt{x^2+3}+x\right)$
d) $\lim_{x\to-\infty}\left(2x+\sqrt{4x^2+3x-2}\right)$ **e)** $\lim_{x\to\infty}\left(\sqrt{9x^2-x}-3x\right)$ **f)** $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$

11. Graph the rational functions in following. Include the graphs and equations of the asymptotes.

a) $y=\frac{x^2}{x-1}$ **b)** $y=\frac{x^2+1}{x-1}$ **c)** $y=\frac{x^2-4}{x-1}$
d) $y=\frac{x^2-1}{2x+4}$ **e)** $y=\frac{x^2-1}{x}$ **f)** $y=\frac{x^3+1}{x^2}$