Calculus I

First Semester

Lecturer 6

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2.6 Continuity

2.6.1 Continuity at a Point

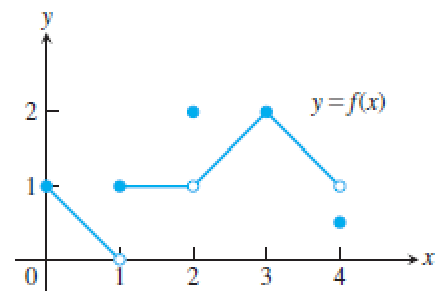
Definition:

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists. **(c lies in the domain of f).**
2. $\lim_{x \to c} f(x)$ exists. **(f has a limit as $x \to c$).**
3. $\lim_{x \to c} f(x) = f(c)$ **(The limit equals the function value).**

Example:

At which numbers does the function f in Figure appear to be not continuous? Explain why. What occurs at other numbers in the domain?



Solution:

First, we observe that the domain of the function is the closed interval [0, 4], so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$.

At the interior point x = 1, the function is not continuous.

Since the function fails to have a limit. It does have both a left-hand limit, $\lim_{x \to 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x \to 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$.However, the function value $f(1) = 1$ is equal to the limit from the right.

At the interior point x = 2, the function is not continuous.

At $x = 2$, the function does have a limit, $\lim_{x \to 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At the interior point x = 4, the function is not continuous.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x\to 4^-} f(x) = 1$,but again the value of the function $f(4) = 1/2$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

At $\boldsymbol{x = 3}$, the function has a limit, $\lim_{x\to 3} f(x) = 2$. Moreover, the limit is the same value as the function there, $f(3) = 2$. The function is continuous at $x = 3$.

At $\boldsymbol{x = 0}$, the function has a right-hand limit at this left endpoint, $\lim_{x\to 0^+} f(x) = 1$ and the value of the function is the same, $f(0) = 1$. The function is continuous from the right at $x = 0$. Because $x = 0$ is a left endpoint of the function's domain, we have that $\lim_{x\to 0} f(x) = 1$ and so f is continuous at $x = 0$.

At all other numbers $\boldsymbol{x = c}$ in the domain, the function has a limit equal to the value of the function, so $\lim_{x\to c} f(x) = f(c)$. For example, $\lim_{x\to 5/2} f(x) = f(5/2) = 5/2$. No breaks appear in the graph of the function at any of these numbers and the function is continuous at each of them.

The function f is continuous at every x in [0, 4] except $x = 1, 2$, and 4.

Remark:

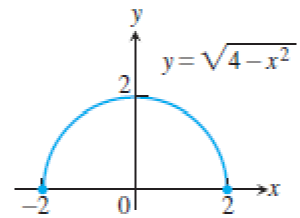
The following definitions capture the continuity ideas we observed in previous example.

Definition:

Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

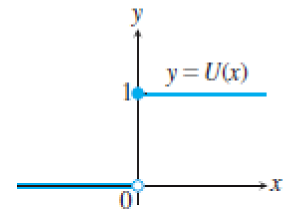
- The function f is ***continuous*** at c if $\lim_{x \to c} f(x) = f(c)$.
- The function f is ***right-continuous at c*** (or ***continuous from the right***) if $\lim_{x \to c^+} f(x) = f(c)$.
- The function f is ***left-continuous at c*** (or ***continuous from the left***) if $\lim_{x \to c^-} f(x) = f(c)$.

Example:



The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$. It is continuous at all points of this interval, including the endpoints $x = -2$ and $x = 2$.

Example:

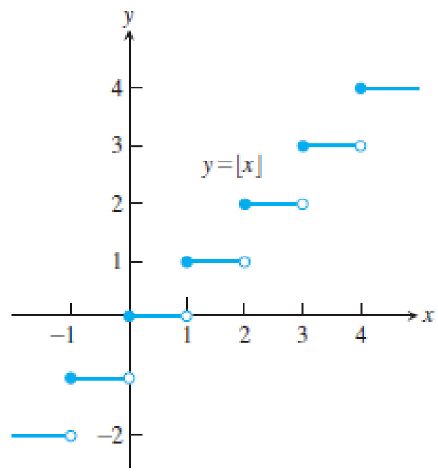


At which numbers does the function $U(x) = \begin{cases} 1 & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$ is not continuous?

Solution:

The unit step function $U(x)$ is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$.

Example:



At which numbers does the function $y = \lfloor x \rfloor$ is not continuous?

Solution:

The function $y = \lfloor x \rfloor$. Is discontinuous at every integer n, because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

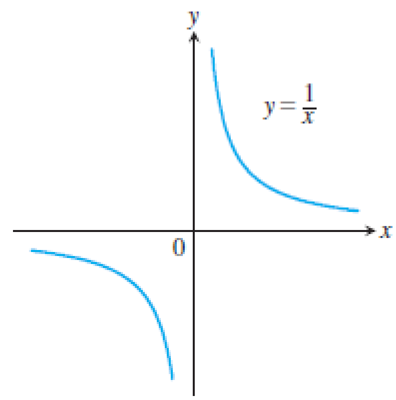
The greatest integer function is continuous at every real number other than the integers. For example, $\lim_{x \to 1.5} \lfloor x \rfloor = \lfloor 1.5 \rfloor = 1$.

In general, if $n - 1 < c < n, n$ an integer, then$\lim_{x \to c} \lfloor x \rfloor = \lfloor c \rfloor = n - 1$.

Remark:

We define a ***continuous function*** to be one that is continuous at every point in its domain. A function always has a specified domain, so if we change the domain, then we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a ***discontinuous*** function.

Example:



a) The function $f(x) = \frac{1}{x}$ is a continuous function because it is continuous at every point of its domain. The point $x = 0$ is not in the domain of the function f, so f is not continuous on any interval containing $x = 0$. Moreover, there is no way to extend f to a new function that is defined and continuous at $x = 0$. The function f does not have a removable discontinuity at $x = 0$.

b) The identity function $f(x) = x$ and constant functions are continuous everywhere.

Theorem (Properties of Continuous Functions):

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. ***Sums:*** $f + g$
2. ***Differences:*** $f - g$
3. ***Constant multiples:*** $k \cdot f$, **for any number k**
4. ***Products:*** $f \cdot g$
5. ***Quotients:*** f/g, **provided $g(c) \neq 0$**
6. ***Powers:*** f^n, **n a positive integer**
7. ***Roots:*** $\sqrt[n]{f}$, **provided it is defined on an interval containing c, where n is a positive integer**

Example:

a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x\to c} P(x) = P(c)$.

b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$).

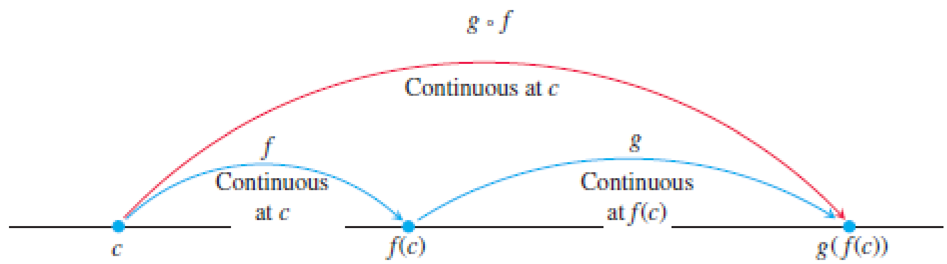
c) The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x\to 0}|x| = |0| = 0$.

d) The functions $y = sin\, x$ and $y = cos\, x$ are continuous at $x = 0$. Both functions are continuous everywhere.

e) All six trigonometric functions are continuous wherever they are defined. For example, $y = tan\, x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

Theorem (Compositions of Continuous Functions):

If f is continuous at c, and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.



Example:

Show that the following functions are continuous on their natural domains.

a) $y = \sqrt{x^2 - 2x - 5}$ **b)** $y = \frac{x^{2/3}}{1+x^4}$ **c)** $y = \left|\frac{x-2}{x^2-2}\right|$ **d)** $y = \left|\frac{x \sin x}{x^2+2}\right|$

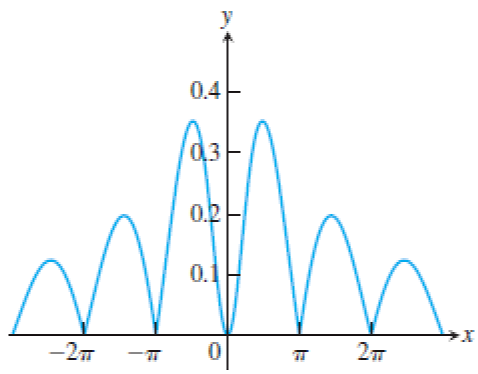
Solution:

a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$. The given function is then the composition of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.

b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

c) The quotient $x-2/x^2-2$ is continuous for all $x=\pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function.

d) Because the sine function is everywhere-continuous, the numerator term $x\ \sin x$ is the product of continuous functions, and the denominator term x^2+2 is an everywhere-positive polynomial. The given function is the composition of a quotient of continuous functions with the continuous absolute value function.



Theorem (Limits of Continuous Functions):

If $\lim_{x\to c} f(x)=b$ and g is continuous at the point b, then

$$\lim_{x\to c} g(f(x))=g(b).$$

Example:

As an application of Theorem Limits of Continuous Functions, we have the following calculations.

a) $\lim_{x\to\pi/2}\cos(2x+\sin(\frac{3\pi}{2}+x))=\cos(\lim_{x\to\pi/2}2x+\lim_{x\to\pi/2}(\sin(\frac{3\pi}{2}+x))$

$$=\cos(\pi+\sin 2\pi)=\cos\pi=-1.$$

b) $\lim_{x\to 1} sin^{-1}(\frac{1-x}{1-x^2})=sin^{-1}\left(\lim_{x\to 1}\frac{1-x}{1-x^2}\right)$ **Arcsine is continuous.**

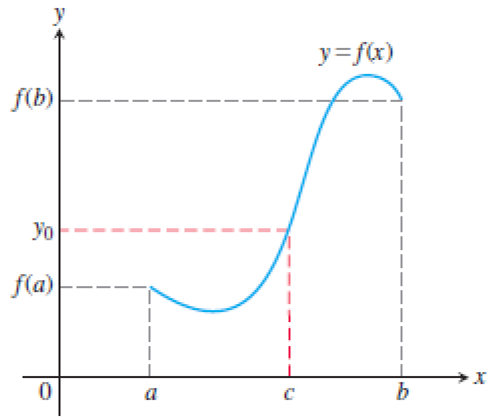
$=sin^{-1}\left(\lim_{x\to 1}\frac{1}{1+x}\right)$ **Cancel common factor $(1-x)$.**

$=sin^{-1}\frac{1}{2}=\frac{\pi}{6}.$

c) $\lim_{x\to 0}(\sqrt{x+1}\cdot e^{\tan x})=\lim_{x\to 0}\sqrt{x+1}\cdot\exp\,(\lim_{x\to 0}\tan x)$ **exp is continuous.**

$=1\cdot e^0=1.$

Theorem (The Intermediate Value Theorem for Continuous Functions):



If f is a continuous function on a closed interval $[\,a, b\,]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[\,a, b\,]$.

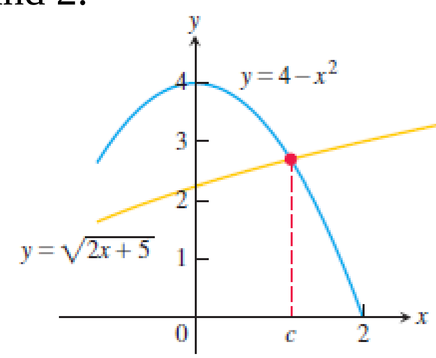
Example:

Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution:

Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2.

Example:



Use the Intermediate Value Theorem to prove that the equation $\sqrt{2x+5} = 4 - x^2$ has a solution.

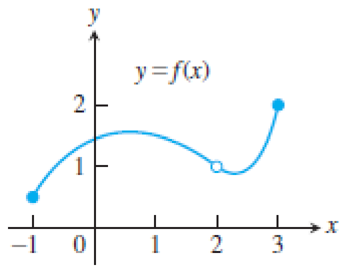
Solution:

We rewrite the equation as $\sqrt{2x+5} + x^2 - 4 = 0$, and set $f(x) = \sqrt{2x+5} + x^2 - 4$. Now $g(x) = \sqrt{2x+5}$ is continuous on the interval $[-5/2, \infty)$ since it is formed as the composition of two continuous functions, the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2 - 4$, and the quadratic function is continuous for all values of x. It follows that $f(x) = \sqrt{2x+5} + x^2 - 4$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} - 4 \approx -1.76$ and $f(2) = \sqrt{9} = 3$. Note that f is continuous on the finite closed interval $[0,2]$, which is a subset of the domain $[-5/2, \infty)$. Since the value $y_0 = 0$ is between the numbers $f(0) = -1.76$ and $f(2) = 3$, by the Intermediate Value Theorem there is a number $c \in [0,2]$ such that $f(c) = 0$. We have found a number c that solves the original equation.

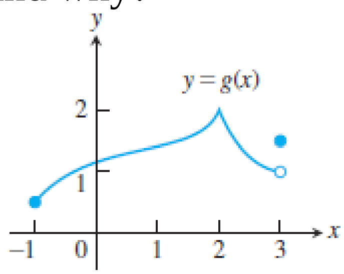
Exercises:

1. Say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

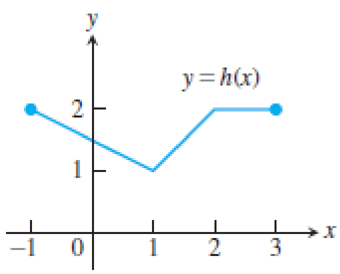
a)



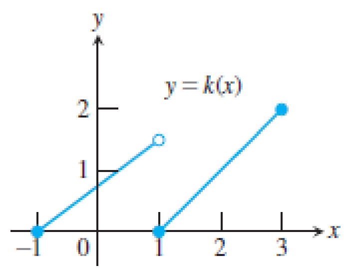
b)



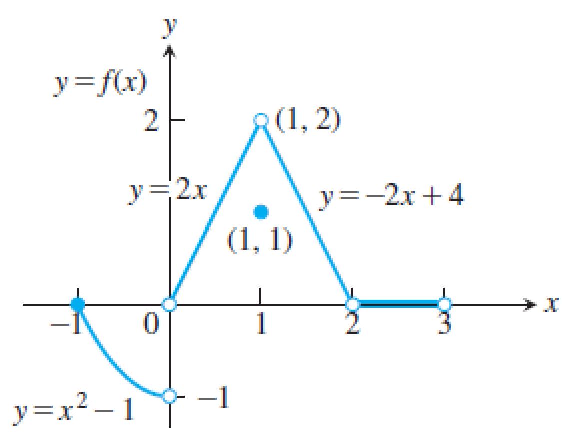
c)



d)



2. Let $f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0. & 2 < x < 3 \end{cases}$ graphed in the accompanying figure



a)
- **1)** Does $f(-1)$ exist?
- **2)** Does $\lim_{x \to -1^+} f(x)$ exist?
- **3)** Does $\lim_{x \to -1^+} f(x) = f(-1)$?
- **4)** Is f continuous at $x = -1$?

b)
- **1)** Does $f(1)$ exist?
- **2)** Does $\lim_{x \to 1} f(x)$ exist?
- **3)** Does $\lim_{x \to 1} f(x) = f(1)$?
- **4)** Is f continuous at $x = 1$?

c)
- **1)** Is f defined at $x = 2$? (Look at the definition of f.)
- **2)** Is f continuous at $x = 2$?

d) At what values of x is f continuous?

e) To what new value should $f(1)$ be changed to remove the discontinuity?

3. At what points are the following functions being continuous?

a) $y = \frac{1}{x-2} - 3x$ **b)** $y = \frac{1}{(x+2)^2} + 4$ **c)** $y = \frac{x+1}{x^2-4x+3}$

d) $y = \frac{x+3}{x^2-3x-10}$ **e)** $y = |x-1| + \sin x$ **f)** $y = \frac{1}{|x|+1} - \frac{x^2}{2}$

g) $y = \frac{\cos x}{x}$ **h)** $y = \frac{x+2}{\cos x}$ **i)** $y = \csc 2x$

j) $y = \tan\frac{\pi x}{2}$ **k)** $y = \frac{x\tan x}{x^2+1}$ **l)** $y = \frac{\sqrt{x^4+1}}{1+sin^2 x}$

m) $y = \sqrt{2x+3}$ **n)** $y = \sqrt[4]{3x-1}$ **o)** $y = (2x-1)^{1/3}$

p) $y = (2-x)^{1/5}$ **q)** $f(x) = \begin{cases} \frac{x^2-x-6}{x-3}, & x \neq 3 \\ 5. & x=3 \end{cases}$ **r)** $f(x) = \begin{cases} \frac{x^3-8}{x^2-4}, & x \neq 2, x \neq -2 \\ 3, & x = 2 \\ 4. & x = -2 \end{cases}$

s) $f(x) = \begin{cases} 1-x, & x<0 \\ e^x, & 0 \le x \le 1 \\ x^2+2. & x>1 \end{cases}$ **t)** $f(x) = \frac{x+3}{2-e^x}$

4. Find the limits in following. Are the functions continuous at the point being approached?

a) $\lim_{x\to\pi} \sin(x - \sin x)$ **b)** $\lim_{t\to 0} \sin\left(\frac{\pi}{2}\cos(\tan t)\right)$

c) $\lim_{y\to 1} \sec\left(y\, sec^2 y - tan^2 y - 1\right)$ **d)** $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos(\sin x^{1/3})\right)$

e) $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19-3\sec 2t}}\right)$ **f)** $\lim_{x\to\pi/6} \sqrt{csc^2 x + 5\sqrt{3}\tan x}$

g) $\lim_{x\to 0^+} \sin\left(\frac{\pi}{2} e^{\sqrt{x}}\right)$ **h)** $\lim_{x\to 1} cos^{-1}(\ln\sqrt{x})$

i) $\lim_{x\to 0} \sec\left(e^x + \pi\tan\frac{\pi}{4\sec x} - 1\right)$ **j)** $\lim_{x\to 0} \sin\left(\frac{\pi+\tan x}{\tan x - 2\sec x}\right)$

k) $\lim_{t\to 0} \tan\left(1 - \frac{\sin t}{t}\right)$ **l)** $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

5. For what value of a is $f(x) = \begin{cases} x^2-1, & x<3 \\ 2ax. & x \ge 3 \end{cases}$ continuous at every x?

6. For what value of b is $g(x) = \begin{cases} x, & x<-2 \\ bx^2. & x \ge -2 \end{cases}$ continuous at every x?

7. For what value of a is $f(x) = \begin{cases} a^2x-2a, & x \ge 2 \\ 12. & x<2 \end{cases}$ continuous at every x?

8. For what value of b is $g(x) = \begin{cases} \frac{x-b}{b+1}, & x<0 \\ x^2+b. & x>0 \end{cases}$ continuous at every x?

9. For what value of a and b is $f(x) = \begin{cases} -2, & x\leq -1 \\ ax-b, & -1<x<1 \\ 3. & x\geq 1 \end{cases}$ continuous at every x?

10. For what value of a and b is $g(x) = \begin{cases} ax+2b, & x\leq 0 \\ x^2+3a-b, & 0<x\leq 2 \\ 3x-5. & x>2 \end{cases}$ continuous at every x?

11. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval [-4, 4].

Calculus I

First Semester

Lecturer 7

Dr. Ban Jaffar AL-Taiy

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Chapter Three: Derivatives.

3.1 Tangent Lines and the Derivative at a Point:

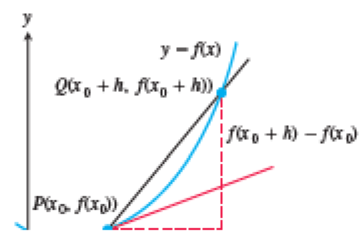
In this section we define the slope and tangent line to a curve at a point, and the derivative of a function at a point. The derivative gives a way to find both the slope of a graph and the instantaneous rate of change of a function.

3.1.1 Finding a Tangent Line to the Graph of a Function:

Definition:

The ***slope of the curve*** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \quad \text{(Provided the limit exists).}$$



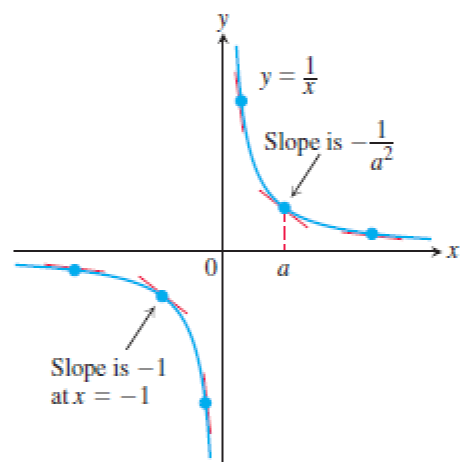
The ***tangent line*** to the curve at P is the line through P

Example:

a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

b) Where does the slope equal $-1/4$?

c) What happens to the tangent line to the curve at the point $(a, 1/a)$ as a change?

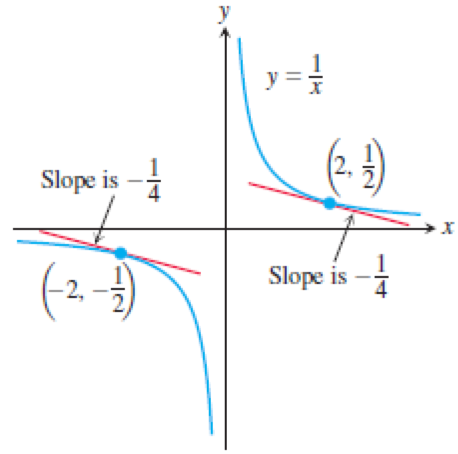


Solution:

a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to 0}\frac{\frac{a-a-h}{a(a+h)}}{h} = \lim_{h\to 0}\frac{\frac{-h}{a(a+h)}}{h} = \lim_{h\to 0}\frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim\limits_{h\to 0}$" before each fraction until the stage at which we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-\frac{1}{(-1)^2} = -1$.



b) The slope of $y = 1/x$ at the point where $x = a$ is $-\frac{1}{a^2}$. It will be $-\frac{1}{4}$, provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope $-\frac{1}{4}$ at the two points $(2, 1/2)$ and $(-2, -1/2)$.

c) The slope $-\frac{1}{a^2}$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent line becomes increasingly steep. We see this situation again as $a \to 0^-$. As a moves away from $x = 0$ in either direction, the slope approaches 0 and the tangent line levels off, becoming closer and closer to a horizontal line.

3.1.2 Derivative at a Point:

Definition:

The ***derivative of a function f at a point x_0***, denoted $f'(x_0)$, is

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

Provided the limit exists.

Remark:

All of the following are interpretations for the limit of the difference quotient

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$.

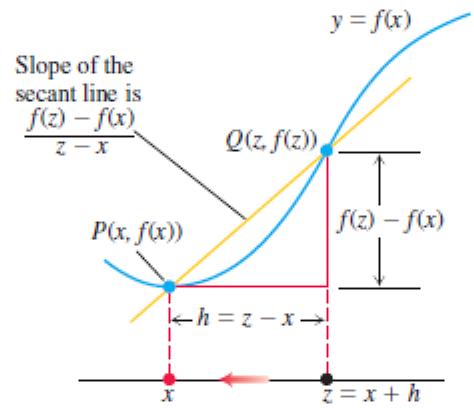
2. The slope of the tangent line to the curve $y = f(x)$ at $x = x_0$.

3. The rate of change of $f(x)$ with respect to x at $x = x_0$.

4. The derivative $f'(x_0)$ at $x = x_0$.

Remark:

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows. This formula is sometimes more convenient to use when finding a derivative function, and it focuses on the point z that approaches x.



$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}. \quad \text{(Alternative Formula for the Derivative)}$$

Remark:

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation $\frac{d}{dx} f(x)$ as another way to denote the derivative $f'(x)$.

Example:

Differentiate $f(x) = \frac{x}{x-1}$.

Solution:

We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have $f(x) = \frac{x}{x-1}$ and $f(x+h) = \frac{(x+h)}{(x+h)-1}$, so

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \textbf{Definition}$$

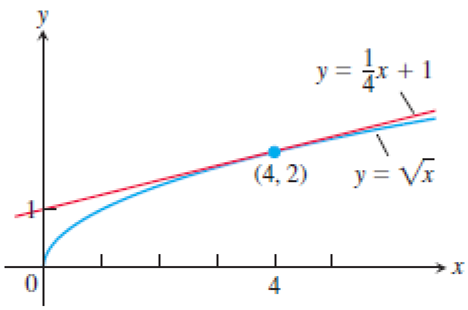
$$= \lim_{h \to 0} \frac{\frac{(x+h)}{(x+h)-1} - \frac{x}{x-1}}{h} \quad \textbf{Substitute.}$$

$$= \lim_{h \to 0} \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \quad \textbf{Simplify.}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. \quad \textbf{Cancel } h \neq 0 \textbf{ and evaluate.}$$

Example:



a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

b) Find the tangent line to the curve $f(x) = \sqrt{x}$ at $x = 4$.

Solution:

a) We use the alternative formula to calculate f':

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}$$

$$= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \qquad \frac{1}{a^2 - b^2} = \frac{1}{(a-b)(a+b)}$$

$$= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \qquad \text{Cancel and evaluate.}$$

b) The slope of the curve at $x = 4$ is $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

The tangent line is the line through the point (4,2) with slope $\frac{1}{4}$:

$$\frac{y-2}{x-4} = \frac{1}{4} \Rightarrow y = 2 + \frac{1}{4}(x - 4) \Rightarrow y = \frac{1}{4}x + 1.$$

Remark:

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' originate with Newton. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio; it simply denotes a derivative.

To indicate the value of a derivative at a specified number $x = a$, we use the notation $f'(a) = \frac{dy}{dx}\Big|_{x=a} = \frac{df}{dx}\Big|_{x=a} = \frac{d}{dx}f(x)\Big|_{x=a}$. For instance, in previous Example, $f'(4) = \frac{d}{dx}\sqrt{x}\Big|_{x=4} = \frac{1}{2\sqrt{x}}\Big|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

Exercises:

1. In following, find an equation for the tangent line to the curve at the given point. Then sketch the curve and tangent line together.

a) $y = 4 - x^2, (-1,3)$

b) $y = (x-1)^2 + 1, (1,1)$

c) $y = 2\sqrt{x}, (1,2)$

d) $y = \frac{1}{x^2}, (-1,1)$

e) $y = x^3, (-2,-8)$

f) $y = \frac{1}{x^3}, (-2, \frac{1}{8})$

2. In following, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

a) $f(x) = x^2 + 1, (2,5)$

b) $f(x) = x - 2x^2, (1,-1)$

c) $g(x) = \frac{x}{x-2}, (3,3)$

d) $g(x) = \frac{8}{x^2}, (2,2)$

e) $h(t) = t^3, (2,8)$

f) $h(t) = t^3 + 3t, (1,4)$

g) $f(x) = \sqrt{x}, (4,2)$

h) $f(x) = \sqrt{x+1}, (8,3)$

3. In following, find the slope of the curve at the point indicated.

a) $y = 5x - 3x^2, x = 1$

b) $y = x^3 - 2x + 7, x = -2$

c) $y = \frac{1}{x-1}, x = 3$

d) $y = \frac{x-1}{x+1}, x = 0$

4. Using the definition, calculate the derivatives of the functions in following. Then find the values of the derivatives as specified.

a) $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$

b) $F(x) = (x-1)^2 + 1; \quad F'(-1), F'(0), F'(2)$

c) $g(t) = \frac{1}{t^2}; \quad g'(-1), g'(2), g'(\sqrt{3})$

d) $k(z) = \frac{1-z}{2z}; \quad k'(-1), k'(1), k'(\sqrt{2})$

e) $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$

f) $r(s) = \sqrt{2s+1}; \quad r'(0), r'(1), r'(1/2)$

5. In following, find the indicated derivatives.

a) $\frac{dy}{dx}$ if $y = 2x^3$ **b)** $\frac{dr}{ds}$ if $r = s^3 - 2s^2 + 3$

c) $\frac{ds}{dt}$ if $s = \frac{t}{2t+1}$ **d)** $\frac{dv}{dt}$ if $v = t - \frac{1}{t}$

e) $\frac{dp}{dq}$ if $p = q^{3/2}$ **f)** $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{w^2-1}}$

6. In following, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

a) $f(x) = x + \frac{9}{x}, \; x = -3.$ **b)** $k(x) = \frac{1}{2+x}, \; x = 2.$

c) $s = t^3 - t^2, \; t = -1.$ **d)** $y = \frac{x+3}{1-x}, \; x = -2.$

7. In following, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

a) $y = f(x) = \frac{8}{\sqrt{x-2}}, (x,y) = (6,4)$ **b)** $w = g(z) = 1 + \sqrt{4-z}, (z,w) = (3,2)$

8. In following, find the values of the derivatives.

a) $\left.\frac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$ **b)** $\left.\frac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \frac{1}{x}$

c) $\left.\frac{dr}{d\theta}\right|_{\theta=0}$ if $r = \frac{2}{\sqrt{4-\theta}}$ **d)** $\left.\frac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

9. Use the alternative formula to find the derivative of the following functions

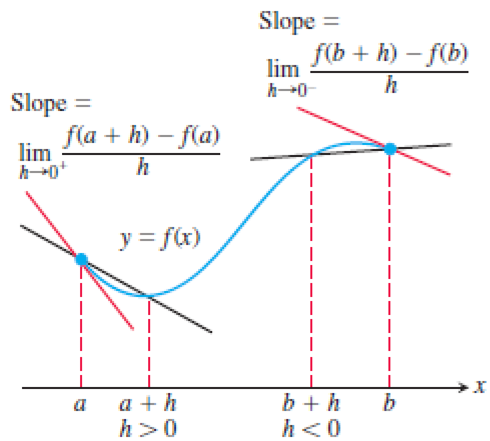
a) $f(x) = \frac{1}{x+2}$ **b)** $f(x) = x^2 - 3x + 4$

c) $g(x) = \frac{x}{x-1}$ **d)** $g(x) = 1 + \sqrt{x}$

3.2 Differentiability on an Interval; One-Sided Derivatives:

Definition:

A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits



$\lim_{h\to0^+} \frac{f(a+h)-f(a)}{h}$ **Right - hand derivative at a**

$\lim_{h\to0^-} \frac{f(b+h)-f(b)}{h}$ **Left - hand derivative at b,**

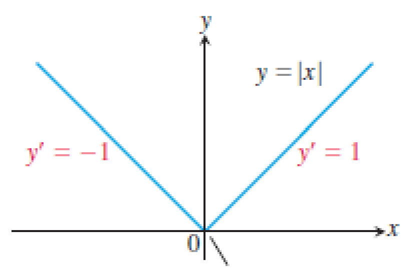
exist at the endpoints.

Remark:

Right-hand and left-hand derivatives may or may not be defined at any point of a function's domain. Because a function has a derivative at an interior point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

Example:

Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$ but has no derivative at $x = 0$.



Solution:

The graph of the function $y = mx + b$ is a straight line with slope m. Thus, to the right of the origin, when $x > 0$,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1.x) = 1. \quad |x| = x \text{ since } x > 0, \frac{d}{dx}(mx + b) = m$$

To the left, when $x < 0$,

$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1.x) = 1.$ **$|x| = -x$ since $x < 0$,**

The two branches of the graph come together at an angle at the origin, forming a non-smooth corner. There is no derivative at the origin because the one-sided derivatives differ there:

$$\text{Right-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0.$$

$$= \lim_{h\to 0^+} 1 = 1$$

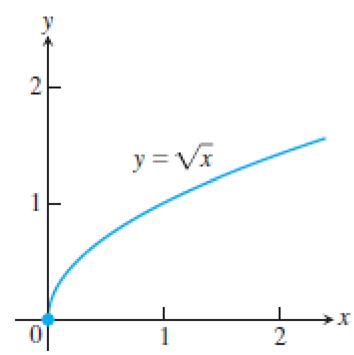
$$\text{Left-hand derivative of } |x| \text{ at zero} = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h}$$

$$= \lim_{h\to 0^-} \frac{h}{h} \qquad |h| = -h \text{ when } h > 0.$$

$$= \lim_{h\to 0^-} -1 = -1.$$

Example:

Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.



Solution:

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}.$$

We apply the definition to examine whether the derivative exists at $x = 0$:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h}-\sqrt{0}}{h} = \lim_{h\to 0^+} \frac{\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(h, \sqrt{h})$ on a graph of $y = \sqrt{x}$ approach, the graph has a vertical tangent line at the origin.

Theorem (Differentiable Implies Continuous):

If f has a derivative at $x = c$, then f is continuous at $x = c$.

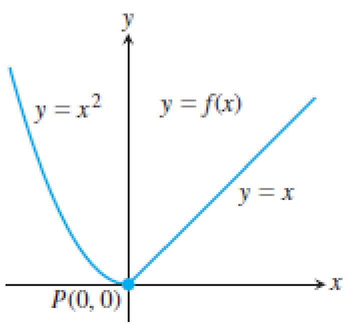
Remark:

The converse of Theorem (Differentiable Implies Continuous) is false. A function need not have a derivative at a point where it is continuous, for example the function $|x|$ is continuous at $x = 0$ not differentiable at $x = 0$.

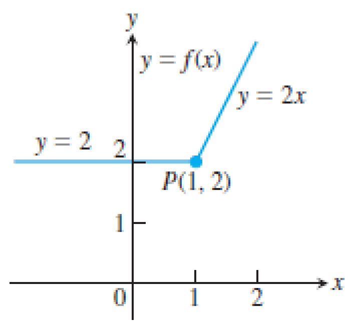
Exercises:

1. Compute the right-hand and left-hand derivatives as limits to show that the functions in following are not differentiable at the point *P*.

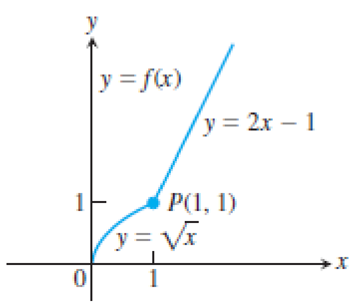
a)



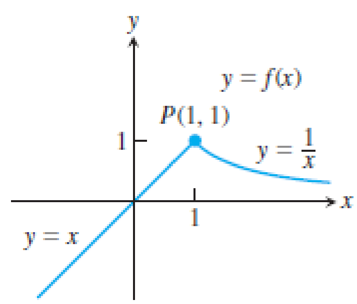
b)



c)



d)



2. In following, determine whether the piecewise-defined function is differentiable at $x = 0$.

a) $f(x) = \begin{cases} 2x - 1 & x \geq 0 \\ x^2 + 2x + 7 & x < 0 \end{cases}$ **b)** $g(x) = \begin{cases} x^{2/3} & x \geq 0 \\ x^{1/3} & x < 0 \end{cases}$

c) $f(x) = \begin{cases} 2x + \tan x & x \geq 0 \\ x^2, & x < 0 \end{cases}$ **d)** $f(x) = \begin{cases} 2x - x^3 - 1 & x \geq 0 \\ x - \frac{1}{x+1} & x < 0 \end{cases}$

3. Each figure in following shows the graph of a function over a closed interval *D*. At what domain points does the function appear to be

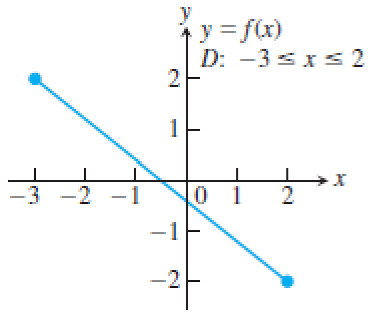
i. differentiable?

ii. continuous but not differentiable?

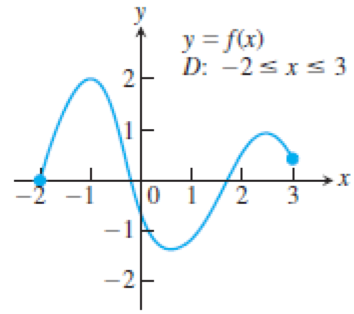
iii. neither continuous nor differentiable?

Give reasons for your answers.

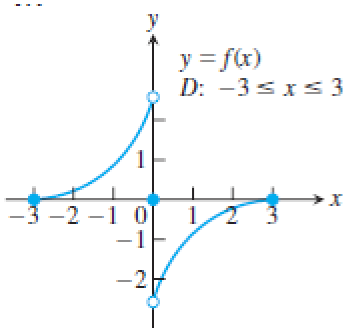
a)



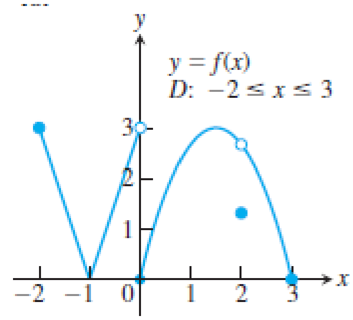
b)



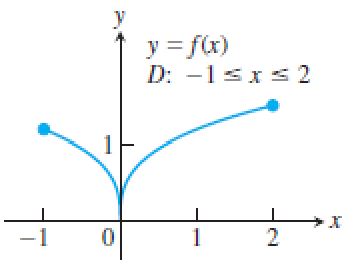
c)



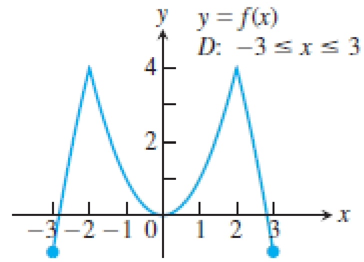
d)



e)



f)



Calculus I

First Semester

Lecturer 8

Dr. Ban Jaffar AL-Taiy

Taghreed Hussein Abed

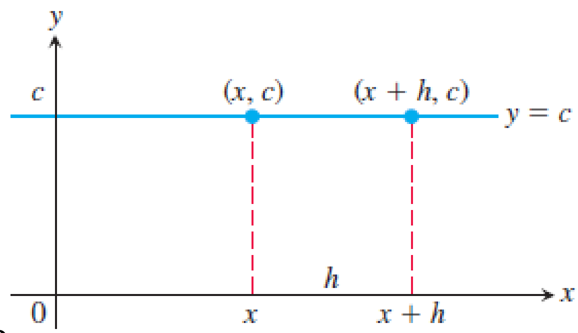
3.3 Differentiation Rules

This section introduces several rules that allow us to differentiate constant functions, power functions, polynomials, exponential functions, rational functions, and certain combinations of them, simply and directly, without having to take limits each time.

Theorem (Derivative of a Constant Function):

If f has the constant value $f(x) = c$, then $\frac{df}{dx} = \frac{d}{dx}(c) = 0$.

Proof:



We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c. At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0. \blacksquare$$

Theorem (Power Rule):

If n is any real number, then $\frac{d}{dx}x^n = nx^{n-1}$, for all x where the powers x^n and x^{n-1} are defined.

Example:

Differentiate the following powers of x.

a) x^3 b) $x^{2/3}$ c) $x^{\sqrt{2}}$ d) $\frac{1}{x^4}$ e) $x^{-4/3}$ f) $\sqrt{x^{2+\pi}}$

Solution:

a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$,

b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$,

c) $\frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$,

d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$,

e) $\frac{d}{dx}\left(x^{-4/3}\right) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$,

f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{\frac{2+\pi}{2}}\right) = \frac{d}{dx}\left(x^{1+\frac{\pi}{2}}\right) = (1+\frac{\pi}{2})x^{1+\frac{\pi}{2}-1} = (1+\frac{\pi}{2})x^{\frac{\pi}{2}}$,

Theorem (Derivative Constant Multiple Rule):

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

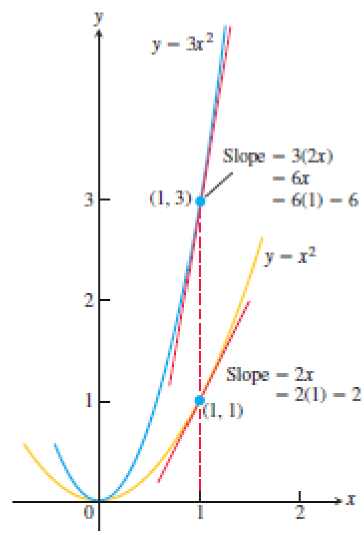
Proof:

$\frac{d}{dx}(cu) = \lim_{h\to 0} \frac{cu(x+h)-cu(x)}{h}$ **Derivative definition with $f(x) = cu(x)$.**

$= c\lim_{h\to 0} \frac{u(x+h)-u(x)}{h}$ **Constant Multiple Rule for Limits.**

$= c\frac{du}{dx}$. ■ **u is differentiable.**

Example:



a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x.$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3.

b) The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

Theorem (Derivative Sum Rule):

If u and are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Proof:

We apply the definition of the derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}(u(x)+v(x)) = \lim_{h\to 0} \frac{[u(x+h)+v(x+h)]-[u(x)+v(x)]}{h}$$

$$= \lim_{h\to 0} \left[\frac{u(x+h)-u(x)}{h} + \frac{v(x+h)-v(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}. \blacksquare$$

Remark:

1. Combining the Sum Rule with the Constant Multiple Rule gives the ***Difference Rule***, which says that the derivative of a difference of differentiable functions is the difference of their derivatives:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}\,. \blacksquare$$

2. The Sum Rule also extends to finite sums of more than two functions. If $u_1\,,\, u_2\,, \dots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$ and

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}\,.$$

Example:

Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

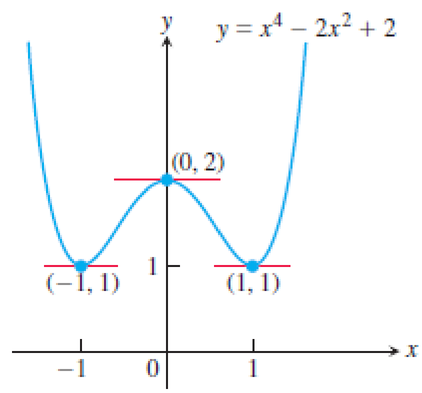
Solution:

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\tfrac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \quad \textbf{Sum and Difference Rules}$$

$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5.$$

Example:

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent lines? If so, where?



Solution:

The horizontal tangent lines, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2\,) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x:

$$4x^3 - 4x = 0 \Rightarrow 4x(x^2 - 1) = 0 \Rightarrow x = 0, 1, -1.$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangent lines at $x = 0, 1$, and -1. The corresponding points on the curve are (0, 2), (1,1), and (−1,1).

Definition:

For any numbers $a > 0$ and x, ***the exponential function with base a*** is

$$a^x = e^{x \ln a}.$$

Remark:

The derivative of $f(x) = a^x$, is

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \qquad \textbf{Derivative definition}$$

$$= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} \qquad a^{x+h} = a^x \cdot a^h$$

$$= \lim_{h\to 0} a^x \cdot \frac{a^h - 1}{h} \qquad \textbf{Factoring out } a^x$$

$$= a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h} \qquad a^x \textbf{ is constant as } h \to 0$$

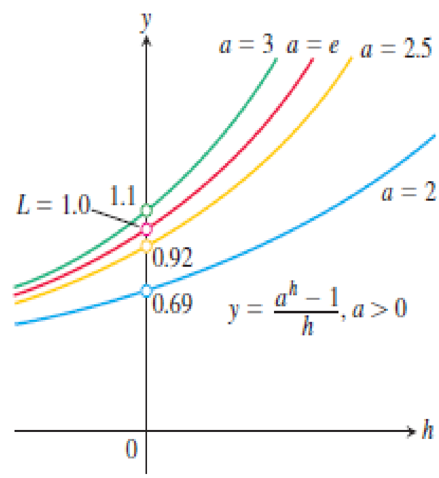
$$= \underbrace{\lim_{h\to 0} \frac{a^h - 1}{h}}_{\text{a fixed number } L} \cdot a^x.$$

Thus, we see that the derivative of a^x is a constant multiple L of a^x. The constant L is a limit we have not encountered before. Note, however, that it equals the derivative of $f(x) = a^x$ at $x = 0$:

$$f'(0) = \lim_{h\to 0} \frac{a^{0+h} - a^0}{h} = \lim_{h\to 0} \frac{a^h - 1}{h} = L.$$

The limit L is therefore the slope of the graph of $f(x) = a^x$ where it crosses the y-axis. Now we investigate values of L by graphing the function $y = (a^h - 1)/h$ and studying its behavior as h approaches 0.

The Figure shows the graphs of $y = (a^h - 1)/h$ for four different values of a. The limit L is approximately 0.69 if $a = 2$, about 0.92 if $a = 2.5$, and about 1.1 if $a = 3$. It appears that the value of L is 1 at some number a chosen between 2.5 and 3. That number is given by $a = e \approx 2.718281828$. With this choice of base, we obtain the natural exponential function $f(x) = e^x$, and see that it satisfies the property $f'(0) = \lim_{h\to 0} \frac{e^h - 1}{h} = 1$, because it is the exponential function whose graph has slope 1 when it crosses the y-axis.



That the limit is 1 implies an important relationship between the natural exponential function e^x and its derivative:

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\frac{e^h - 1}{h}\cdot e^x \qquad \textbf{Since } \frac{d}{dx}(a^x) = \lim_{h\to 0}\frac{a^h-1}{h}\cdot a^x \textbf{ with } a = e.$$

$$= 1\cdot e^x = e^x. \qquad \textbf{Since } \lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

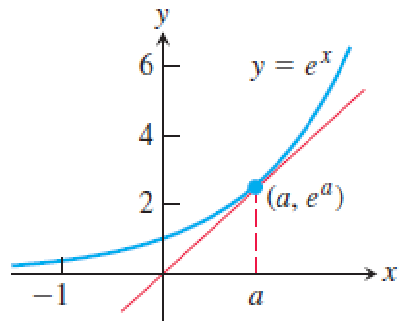
Therefore, the natural exponential function is its own derivative. Also, if $f(x) = c\cdot e^x$, c any constant then $\frac{d}{dx}(c\cdot e^x) = c\cdot\frac{d}{dx}(e^x) = c\cdot e^x$.

Example:

Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

Solution:

Since the line passes through the origin, its equation is of the form $y = mx$, where m is the slope. If it is tangent to the graph at the point (a, e^a), the slope is $m = (e^a - 0)/(a - 0)$. The slope of the natural exponential at $x = a$ is e^a. Because these slopes are the same, we then have that $e^a = e^a/a$. It follows that $a = 1$ and $m = e$. So, the equation of the tangent line is $y = xe$.



Theorem (Derivative Product Rule):

If u and v are differentiable at x, then so is their product u · v, and

$$\frac{d}{dx}(u\cdot v) = u\cdot\frac{dv}{dx} + \frac{du}{dx}\cdot v.$$

Proof:

$$\frac{d}{dx}(u\cdot v) = \lim_{h\to 0}\frac{[u(x+h)\cdot v(x+h)] - [u(x)\cdot v(x)]}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)\cdot v(x)$ in the numerator:

$$\frac{d}{dx}(u\cdot v) = \lim_{h\to 0}\frac{[u(x+h)\cdot v(x+h)] - u(x+h)\cdot v(x) + u(x+h)\cdot v(x) - [u(x)\cdot v(x)]}{h}$$

$$= \lim_{h\to 0} u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}$$

$$= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} + \lim_{h\to 0} v(x) + \lim_{h\to 0} \frac{u(x+h)-u(x)}{h}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. Therefore, $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + \frac{du}{dx} \cdot v$. ■

Example:

Find the derivative of **a)** $y = \frac{1}{x}(x^2 + e^x)$, **b)** $y = e^{2x}$.

Solution:

a) We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] = \frac{1}{x}(2x + e^x) + \left(-\frac{1}{x^2}\right)(x^2 + e^x)$$

$$\boldsymbol{\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + \frac{du}{dx} \cdot v}$$

$$\boldsymbol{\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}}$$

$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$$

$$= 1 + (x-1)\frac{e^x}{x^2}.$$

b) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + \frac{d}{dx}(e^x) \cdot e^x = 2\, e^x \cdot e^x = 2\, e^{2x}$.

Example:

Find the derivative of $y = (x^2+1)(x^3+3)$.

Solution:

From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}(x^2+1)(x^3+3) = (x^2+1)(3x^2) + (2x)(x^3+3)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x.$$

Theorem (Derivative Quotient Rule):

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and, and

$$\boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}\ .$$

Proof:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} = \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)} .$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $u(x)v(x)$ in the numerator. We then get

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x) + u(x)v(x) - u(x)v(x+h)}{hv(x+h)v(x)}$$

$$= \lim_{h\to 0} \frac{v(x)\frac{u(x+h)-u(x)}{h} - u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)} .$$

Taking the limits in the numerator and denominator now gives the Quotient Rule.

Example:

Find the derivative of $y = \frac{(t^2-1)}{(t^3+3)}$.

Solution:

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 3$:

$$\frac{dy}{dx} = \frac{(t^3+1)\cdot 2t - (t^2-1)\cdot 3t^2}{(t^3+3)^2} = \frac{2t^4+2t-3t^4+3t^2}{(t^3+3)^2} = \frac{-t^4+3t^2+2t}{(t^3+3)^2} .$$

Remark:

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

Example:

Find the derivative of $y = \frac{(x-1)(x^2-2x)}{x^4}$.

Solution:

Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression.

First expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum, Constant Multiple, and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} = -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4} .$$

3.3.1 Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the ***second derivative*** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

The symbol D^2 means that the operation of differentiation is performed twice.

If y'' is differentiable, its derivative, $y'' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$, is the ***third derivative*** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^n y,$$

denoting the **n*th derivative*** of y with respect to x for any positive integer n.

Example:

Find the fourth derivative of $y = x^3 - 3x^2 + 2$.

Solution:

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative:	$y' = 3x^2 - 6x$
Second derivative:	$y'' = 6x - 6$
Third derivative:	$y''' = 6$
Fourth derivative:	$y^{(4)} = 0.$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero.

Exercises:

1. In following, find the first and second derivatives.

a) $y = \frac{4x^3}{3} - x$ **b)** $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{2}$

c) $w = 3z^{-2} - \frac{1}{z}$ **d)** $s = -2t^{-1} + \frac{4}{t^2}$

e) $y = 6x^2 - 10x - 5x^{-2}$ **f)** $y = 4 - 2x - x^{-3}$

g) $r = \frac{1}{3s^2} - \frac{5}{2s}$ **h)** $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

2. In following, find y'

a) by applying the Product Rule and

b) by multiplying the factors to produce a sum of simpler terms to differentiate.

I. $y = (3 - x^2)(x^3 - x + 1)$ **II.** $y = (2x + 3)(5x^2 - 4x)$

III. $y = (x^2 + 1)(x + 5 + \frac{1}{x})$ **IV.** $y = (1 + x^2)(x^{3/4} - x^{-3})$

3. Find the derivatives of all orders of the functions in following

a) $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$ **b)** $y = \frac{x^5}{120}$

c) $y = (x - 1)(x + 2)(x + 3)$ **d)** $y = (4x^2 + 3)(2 - x)x$

e) $y = 2e^{-x} + e^{3x}$ **f)** $y = \frac{x^2 + 3e^x}{2e^x - x}$

g) $y = x^3 e^x$ **h)** $w = re^{-r}$

4. Suppose *u* and *v* are functions of *x* that are differentiable at $x = 0$ and that $u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2$. Find the values of the following derivatives at $x = 0$.

a) $\frac{d}{dx}(uv)$ **b)** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c)** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d)** $\frac{d}{dx}(7v - 2u)$

5. Suppose *u* and *v* are differentiable functions of *x* and that $u(1) = 2$, $u'(1) = 0, v(1) = 5, v'(1) = -1$. Find the values of the following derivatives at $x = 1$.

a) $\frac{d}{dx}(uv)$ **b)** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c)** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d)** $\frac{d}{dx}(7v - 2u)$

6. a) Normal line to a curve Find an equation for the line perpendicular to the tangent line to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

b) Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

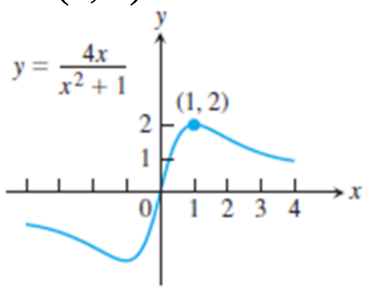
c) Tangent lines having specified slope Find equations for the tangent lines to the curve at the points where the slope of the curve is 8.

7. a) Horizontal tangent lines: Find equations for the horizontal tangent lines to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangent lines at the points of tangency.

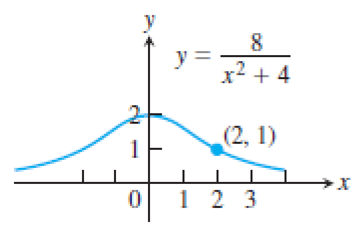
b) Smallest slope: What is the smallest slope on the curve? At what point

point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent line at this point.

8. Find the tangent lines to Newton's serpentine (graphed here) at the origin and the point (1, 2).



9. Find the tangent line to the Witch of Agnesi (graphed here) at the point (2, 1).



10. Quadratic tangent to identity function: The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $y = x$ at the origin. Find a, b, and c.

11. Quadratics having a common tangent: The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point (1, 0). Find a, b, and c.

12. Find all points (*x*, *y*) on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.

13. Find all points (*x*, *y*) on the graph of $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x2 + 1$ with tangent lines parallel to the line $8x - 2y = 1$.

14. Find all points (*x*, *y*) on the graph of $y = x/(x - 2)$ with tangent lines perpendicular to the line $y = 2x + 3$.

15. Find all points (*x*, *y*) on the graph of $f(x) = x^2$ with tangent lines passing through the point (3, 8).

16. Assume that functions *f* and *g* are differentiable with $f(1) = 2$, $f'(1) = -3$, $g(1) = 4$, and $g'(1) = -2$. Find the equation of the line tangent to the graph of $F(x) = f(x)g(x)$ at $x = 1$.

17. Assume that functions *f* and *g* are differentiable with $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, and $g'(2) = 1$. Find an equation of the line perpendicular to the graph of $F(x) = \frac{f(x)+3}{x-g(x)}$ at $x = 2$.

18. Find the value of a that makes the following function differentiable for all *x*-values.

$$g(x)=\begin{cases} ax & if\ x<0 \\ x^2-3x, & if\ x\geq 0\end{cases}.$$

19. Find the values of a and b that make the following function differentiable for all x-values.

$$f(x)=\begin{cases} ax+b & if\ x>-1 \\ bx^2-3, & if\ x\leq -1\end{cases}.$$

$y=x^3e^x$

$y'=x^3e^x+e^x3x^2=(x^3+3x^2)e^x$

$y''=(x^3+3x^2)e^x+e^x(3x^2+6x)=(x^3+6x^2+6x)e^x$

$y'''=(x^3+6x^2+6x)e^x+e^x(3x^2+12x+6)$

$=(x^3+9x^2+18x+6)e^x$

$y^{(4)}=(x^3+9x^2+18x+6)e^x+e^x(3x^2+18x+18)$

$=(x^3+12x^2+32x+18)e^x$

$y^{(n)}=P(x)e^x, P(x)$ is a polynomial of degree 3

$w=re^{-r}$

$w'=-re^{-r}+e^{-r}=(1-r)e^{-r}$

$w''=-(1-r)e^{-r}+e^{-r}(-1)=-(2-r)e^{-r}$

$w'''=(2-r)e^{-r}+e^{-r}=(3-r)e^{-r}$

$w''''=-(3-r)e^{-r}+e^{-r}(-1)=-(4-r)e^{-r}$

$w^{(n)}=(-1)^{n+1}(n-r)e^{x}$

Calculus I

First Semester

Lecturer 9

Dr. Ban Jaffar AL-Taiy

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3.4 Derivatives of Trigonometric Functions

This section shows how to differentiate the six basic trigonometric functions.

Theorem (Derivative of the Sine Function):

$$\frac{d}{dx}(\sin x) = \cos x.$$

Proof:

Let $f(x) = \sin x$ then

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$ **Derivative definition**

$= \lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h}$ $\{\sin(x+h) = \sin x\cos h + \cos x\sin h\}$

$= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$

$= \lim_{h\to 0}\left(\sin x\cdot\frac{(\cos h-1)}{h}\right) + \lim_{h\to 0}\left(\cos x\,.\frac{\sin h}{h}\right)$

$= \sin x\cdot\underbrace{\lim_{h\to 0}\frac{\cos h-1}{h}}_{\text{limit 0}} + \cos x\ .\underbrace{\lim_{h\to 0}\left(\frac{\sin h}{h}\right)}_{\text{limit 1}}$

$= \sin x\cdot 0 + \cos x\ .1 = \cos x.$

Example:

Differentiate the following.

a) $y = x^2 - \sin x$ **b)** $y = x^2\sin x$ **c)** $y = \frac{\sin x}{x}$

Solution:

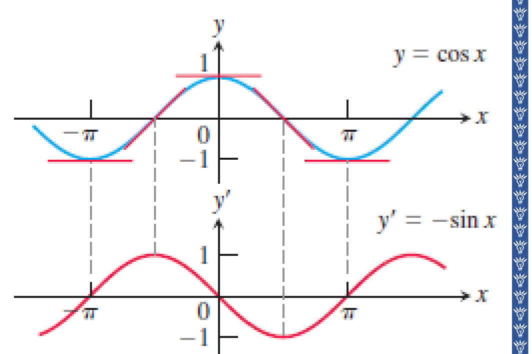
a) $y = x^2 - \sin x \Rightarrow \frac{dy}{dx} = 2x - \cos x.$

b) $y = x^2\sin x \Rightarrow \frac{dy}{dx} = x^2\cos x + 2x\sin x.$

c) $y = \frac{\sin x}{x} \Rightarrow \frac{dy}{dx} = \frac{x\cos x-\sin x\cdot 1}{x^2}.$

Theorem (Derivative of the Cosine Function):

$$\frac{d}{dx}(\cos x) = -\sin x.$$



Proof:

Let $f(x) = \cos x$ then

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$ **Derivative definition**

$$= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \qquad \{\cos(x+h) = \cos x \cos h + \cos x \cos h\}$$

$$= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h\to 0}\left(\cos x \cdot \frac{(\cos h - 1)}{h}\right) - \lim_{h\to 0}\left(\sin x \,.\frac{\sin h}{h}\right)$$

$$= \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} - \sin x \;.\underbrace{\lim_{h\to 0}\left(\frac{\sin h}{h}\right)}_{\text{limit } 1}$$

$$= \cos x \cdot 0 - \sin x \;.1 = -\sin x.$$

Example:

Differentiate the following.

a) $y = 5x + \cos x$ **b)** $y = \sin x \cos x$ **c)** $y = \frac{\cos x}{1 - \sin x}$

Solution:

a) $y = 5x + \cos x \Rightarrow \frac{dy}{dx} = 5 - \sin x.$

b) $y = \sin x \cos x \Rightarrow \frac{dy}{dx} = \sin x\,(-\sin x) + \cos x \cos x.$

$\Rightarrow \quad = -\sin^2 x + \cos^2 x = \cos^2 x - \sin^2 x.$

c) $y = \frac{\cos x}{1 - \sin x} \quad \Rightarrow \frac{dy}{dx} = \frac{(1 - \sin x)\,(-\sin x) - \cos x \cdot (-\cos x)}{(1 - \sin x)^2}.$

$\Rightarrow \quad = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}.$

$\Rightarrow \quad = \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \{\sin^2 x + \cos^2 x = 1\}$

$\Rightarrow \quad = \frac{1}{1 - \sin x}$

Remark (The derivatives of the other trigonometric functions):

$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$

Proof:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cdot \frac{d}{dx}\sin x - \sin x .\frac{d}{dx}\cos x}{\cos^2 x} = \frac{\cos x \cdot \cos x - \sin x .(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x \cdot \frac{d}{dx}\cos x - \cos x .\frac{d}{dx}\sin x}{\sin^2 x} = \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-(\cos^2 x + \sin^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x\cdot\frac{d}{dx}1 - 1.\frac{d}{dx}\cos x}{\cos^2 x} = \frac{\cos x\cdot 0 - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \sec x\cdot\tan x.$$

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\frac{1}{\sin x} = \frac{\cos x\cdot\frac{d}{dx}1 - 1.\frac{d}{dx}\sin x}{\sin^2 x} = \frac{0 - \cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x\cdot\cot x.$$

Exercises:

1. In following, find $\frac{dy}{dx}$

a) $y = \frac{3}{x} + 5\sin x$

b) $y = x^2\cos x$

c) $y = \sqrt{x}\sec x + 3$

d) $y = x^2\cot x - \frac{1}{x^2}$

e) $y = \sin x\tan x$

f) $y = \frac{\cos x}{x} + \frac{x}{\cos x}$

g) $y = \frac{\cot x}{1+\cot x}$

h) $y = x^2\cos x - 2x\sin x - 2\cos x$

i) $y = (\sec x + \tan x)(\sec x - \tan x)$

j) $y = x^3\cos x\sin x$

2. In following, find $\frac{ds}{dt}$

a) $s = \tan t - t$

b) $s = t^2 - \sec t + 1$

c) $s = \frac{1+\csc t}{1-\csc t}$

d) $s = \frac{\sin t}{1-\cos t}$

3. In following, find $\frac{dr}{d\theta}$

a) $r = 4 - \theta^2\sin\theta$

b) $r = \theta\sin\theta + \cos\theta$

c) $r = \sec\theta\csc\theta$

d) $r = (1 + \sec\theta)\sin\theta$

4. In following, find $\frac{dp}{dq}$

a) $p = 5 + \frac{1}{\cot q}$

b) $p = (1 + \csc q)\cos q$

c) $p = \frac{\sin q+\cos q}{\cos q}$

d) $p = \frac{\tan q}{1+\tan q}$

e) $p = \frac{q\sin q}{q^2-1}$

f) $p = \frac{3q+\tan q}{q\sec q}$

5. Find y'' if a) $y = \csc x$. b) $y = \sec x$.

6. Find $y^{(4)} = d^4y/dx^4$ if a) $y = -2\sin x$. b) $y = 9\cos x$.

3.5 The Chain Rule

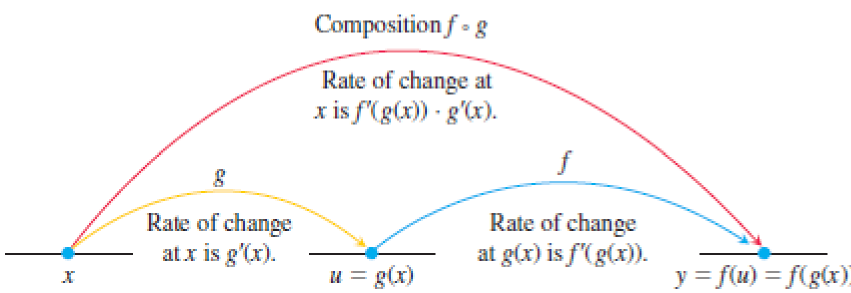
The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule.

Theorem(The Chain Rule):

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, where $\frac{dy}{du}$ is evaluated at $u = g(x)$.



Example:

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution:

We know that the velocity is dx/dt. In this instance, x is a composition of two functions: $x = cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -sin(u) \qquad \boldsymbol{x = cos(u)}$$

$$\frac{du}{dt} = 2t \qquad \boldsymbol{u = t^2 + 1}$$

By the Chain Rule,

$$\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt} = -\sin(u) \cdot 2t = -\sin(t^2 + 1) \cdot 2t = -2t \cdot \sin(t^2 + 1).$$

Remark:

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the Chain Rule are supposed to be evaluated. So, it sometimes helps to write the Chain Rule using functional notation. If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x) \cdot g'(x)$.

In words, differentiate the "***outside***" function f and evaluate it at the "***inside***" function $g(x)$ left alone; then multiply by the derivative of the "***inside function***".

Example:

Differentiate $sin\ (x^2 + \ x)$ with respect to x.

Solution:

We apply the Chain Rule directly and find

$$\frac{d}{dx} sin\ \underbrace{(x^2 + \ x)}_{\text{inside}} = \cos(\underbrace{x^2 + \ x}_{\substack{\text{inside} \\ \text{left alone}}}) \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of} \\ \text{the inside}}} .$$

Remark:

We sometimes have to use the Chain Rule two or more times to find a derivative.

Example:

Find the derivative of $g(t) = tan\ (5 - sin\ 2t)$.

Solution:

Notice here that the tangent is a function of $5 - sin\ 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$g'(t) = \frac{d}{dt} tan\ (5 - sin\ 2t)$$

$$= sec^2\ (5 - sin\ 2t) \cdot \frac{d}{dt}(5 - \sin 2t) \qquad \textbf{Derivative of tan u with } \boldsymbol{u = 5 - sin\ 2t}$$

$$= sec^2\ (5 - sin\ 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \qquad \textbf{Derivative of } \boldsymbol{5 - sin\ u} \textbf{ with } \boldsymbol{u = 2t}$$

$$= sec^2\ (5 - sin\ 2t) \cdot \left((-\cos 2t) \cdot 2\right)$$

$$= -2(-\cos 2t) sec^2\ (5 - sin\ 2t) \cdot$$

Remark:

If n is any real number and f is a power function, $f(u) = u^n$, the Power Rule tells us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the ***Power Chain Rule***:

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}. \qquad \boldsymbol{\frac{d}{dx}(u^n) = nu^{n-1}}$$

Example:

The Power Chain Rule simplifies computing the derivative of a power of an expression.

a) $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6\frac{d}{dx}(5x^3 - x^4)$ **Power Chain Rule with $u = 5x^3 - x^4, n = 7$**

$= 7(5x^3 - x^4)^6(15x^2 - 4x^3).$

b) $\frac{d}{dx}\left(\frac{1}{3x-2}\right) = \frac{d}{dx}(3x-2)^{-1}$

$= -1(3x-2)^{-2}\frac{d}{dx}(3x-2)$ **Power Chain Rule with $u = 3x - 2, n = -1$**

$= -1(3x-2)^{-2}(3)$

$= \frac{3}{(3x-2)^2}.$

In part (b) we could also find the derivative with the Quotient Rule.

c) $\frac{d}{dx}(\sin^5 x) = 5\sin^4 x\frac{d}{dx}\sin x$ **Power Chain Rule with $u = \sin x, n = 5$, because $\sin^n x$ means $(\sin x)^n, n \neq -1$.**

$= 5\sin^4 x\cos x.$

Example:

we saw that the derivative of absolute value function

$$\frac{d}{dx}(|x|) = \frac{x}{|x|},\ x \neq 0,$$

$$= \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}.$$

is not differentiable at $x = 0$. However, the function is differentiable at all other real numbers, as we now show. Since $|x| = \sqrt{x^2}$, we can derive the following formula, which gives an alternative to the more direct analysis seen before.

$$\frac{d}{dx}|x| = \frac{d}{dx}\sqrt{x^2} = \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2)$$

Power Chain Rule with

$\boldsymbol{u = x^2, n = 1/2, x \neq 0}$

$\boldsymbol{\sqrt{x^2} = |x|}$

$$= \frac{1}{2|x|} \cdot 2x$$

$$= \frac{x}{|x|}, x \neq 0.$$

Example:

Show that the slope of every line tangent to the curve $y = 1/(1-2x)^3$ is positive.

Solution:

We find the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}(1-2x)^{-3}$$

$$= -3(1-2x)^{-4} \cdot \frac{d}{dx}(1-2x)$$ **Power Chain Rule with $\boldsymbol{u = (1-2x), n = -3,}$**

$$= -3(1-2x)^{-4} \cdot (-2)$$

$$= \frac{6}{(1-2x)^4}.$$

At any point (x, y) on the curve, the denominator is nonzero, and the slope of the tangent line is $\frac{dy}{dx} = \frac{6}{(1-2x)^4}$, which is the quotient of two positive numbers.

Exercises:

1. In following, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'\big(g(x)\big)g'(x)$.

a) $y = 6u - 9, u = (1/2)x^4$

b) $y = 2u^3, u = 8x - 1$

c) $y = \sin u, u = 3x + 1$

d) $y = \cos u, u = -\frac{x}{3}$

e) $y = \sqrt{u}, u = \sin x$

f) $y = \sin u, u = x - \cos x$

g) $y = \tan u, u = \pi x$

h) $y = -\sec u, u = \frac{1}{x} + 7x$

2. In following, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

a) $y = (2x+1)^5$
b) $y = (4-3x)^9$
c) $y = \left(1 - \frac{x}{7}\right)^{-7}$
d) $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$
e) $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$
f) $y = \sqrt{3x^2 - 4x + 6}$
g) $y = sec\ (tan\ x)$
h) $y = \cot\left(\pi - \frac{1}{x}\right)$
i) $y = \tan^3 x$
j) $y = 5\cos^{-4} x$

3. In following, find dy/dt

a) $y = \left(t^{-3/4} \sin t\right)^{4/3}$
b) $y = \left(\frac{t^2}{t^3 - 4t}\right)^3$
c) $y = \left(\frac{3t-4}{5t+2}\right)^{-5}$
d) $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$
e) $y = \sqrt{1 + \cos\,(t^2)}$
f) $y = 4\sin\left(\sqrt{1 + \sqrt{t}}\right)$
g) $y = \tan^2\,(\sin^3 t)$
h) $y = \sqrt{3t + \sqrt{2 + \sqrt{1-t}}}$

4. Find y'' in following.

a) $y = \left(1 + \frac{1}{x}\right)^3$
b) $y = \left(1 - \sqrt{x}\right)^{-1}$
c) $y = \frac{1}{9}\cot(3x - 1)$
d) $y = 9\tan\left(\frac{x}{3}\right)$
e) $y = x(2x+1)^4$
f) $y = x^2(x^3 - 1)^5$

5. For each of the following functions, solve both $f'^{(x)} = 0$ and $f''(x) = 0$ for x.

a) $f(x) = x(x-4)^3$
b) $f(x) = \sec^2 x - 2\tan x$ for $0 \le x \le 2\pi$

6. In following, find the value of $(f \circ g)'$ at the given value of x.

a) $f(u) = u^5 + 1,\ u = g(x) = \sqrt{x}, x = 1.$
b) $f(u) = 1 - \frac{1}{u},\ u = g(x) = \frac{1}{1-x}, x = -1.$
c) $f(u) = cot\frac{\pi u}{10},\ u = g(x) = 5\sqrt{x}, x = 1.$
d) $f(u) = u + \frac{1}{\cos^2 u},\ u = g(x) = \pi x, x = 1/4.$
e) $f(u) = \frac{2u}{u^2+1},\ u = g(x) = 10x^2 + x + 1, x = 0.$
f) $f(u) = \left(\frac{u-1}{u+1}\right)^2,\ u = g(x) = \frac{1}{x^2} - 1, x = -1.$

7. Assume that $f'(3) = -1, g'(2) = 5, g(2) = 3$, and $y = f(g(x))$. What is y at $x = 2$?

8. If $r = sin\,(f(t)), f(0) = \pi/3$, and $f'(0) = 4$, then what is $\frac{dr}{dt}$ at $t = 0$?

9. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	$1/3$	-3
3	3	-4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a) $2f(x), \quad x = 2$
b) $f(x) + g(x), \quad x = 3$
c) $f(x) \cdot g(x), \quad x = 3$
d) $f(x)/g(x), \quad x = 2$
e) $f(g(x)), \quad x = 2$
f) $\sqrt{f(x)}, \quad x = 2$
g) $1/g^2(x), \quad x = 3$
h) $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

10. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$1/3$
1	3	-4	$-1/3$	$-8/3$

Find the derivatives with respect to x of the following combinations at the given value of x.

a) $5f(x) - g(x), \quad x = 1$
b) $f(x)g^3(x), \quad x = 0$
c) $\frac{f(x)}{g(x)+1}, \quad x = 1$
d) $f(g(x)), \quad x = 0$
e) $g(f(x)), \quad x = 0$
f) $(x^{11} + f(x))^{-2}, \quad x = 1$
g) $f(x + g(x)), \quad x = 0$

11. Find ds/dt when $\theta = 3\pi/2$ if $s = cos\,\theta$ and $d\theta/dt = 5$.

12. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

13. Find dy/dt if $y = x$ using the Chain Rule with y as a composition of

a) $y = (u/5) + 7\,, u = 5x - 35$ **b)** $y = 1 + (1/u)\,, u = 1/(x - 1)$

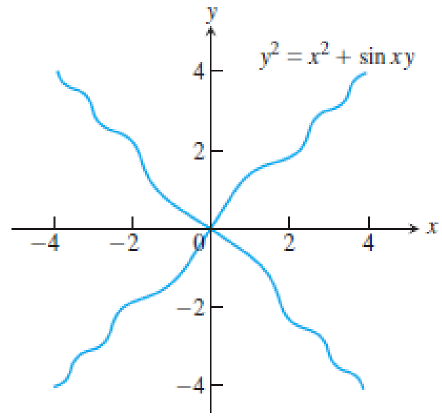
14. Find dy/dt if $y = x^{3/2}$ using the Chain Rule with y as a composition of

a) $y = u^3 + 7\,, u = \sqrt{x}$ **b)** $y = \sqrt{u}\,, u = x^3$

3.6 Implicit Differentiation

To calculate the derivatives of implicitly defined functions, we

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.



Example:

Find dy/dx if $y^{2} = x^2 + sin\ xy$.

Solution:

We differentiate the equation implicitly.

$$y^{2} = x^2 + sin\ xy$$

$$\frac{d}{dx}(y^{2}) = \frac{d}{dx}(x^2) + \frac{d}{dx}(sin\ xy)$$ **Differentiate both sides with respect to x ⋯**

$$2y\frac{dy}{dx} = 2x + \cos xy\frac{d}{dx}(xy)$$ **⋯treating y as a function of x and using the Chain Rule**

$$2y\frac{dy}{dx} = 2x + \cos xy\left(y + x\frac{dy}{dx}\right)$$ **Treat xy as a product.**

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ **Collect terms with dy/dx.**

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ **Solve for dy/dx.**

Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves both variables x and y, not just the independent variable x.

Implicit differentiation can also be used to find higher derivatives.

Example:

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution:

To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \textbf{Treat } y \textbf{ as a function of } x.$$

$$y' = \frac{x^2}{y} \quad \text{when } y \neq 0 \qquad \textbf{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

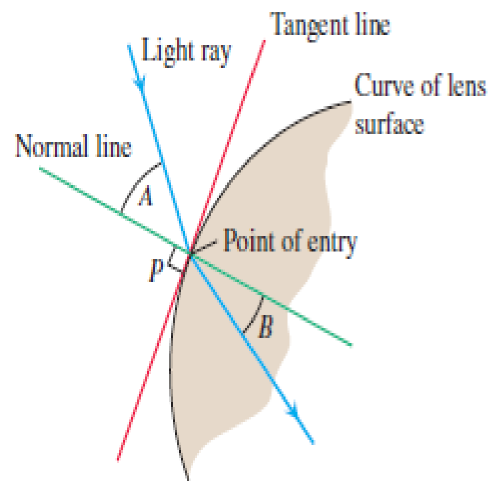
$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y\prime}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = \frac{x^2}{y}$ to express y'' in terms of *x* and *y*.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \cdot \frac{x^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3} \quad \text{when } y \neq 0.$$

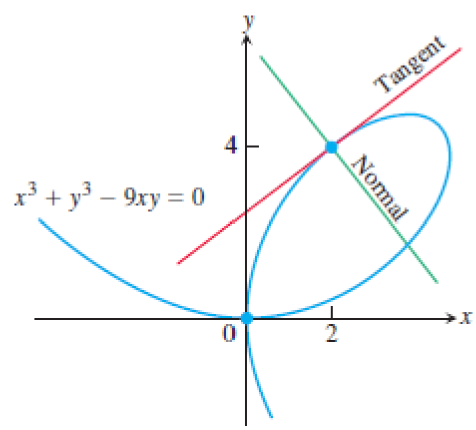
Remark:

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B). This line is called the normal line to the surface at the point of entry. In a profile view of a lens like the one in Figure, the ***normal line*** is the line perpendicular (also said to be ***orthogonal***) to the tangent line of the profile curve at the point of entry.



Example:

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there.



Solution:

The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0.$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$ …

$$3x^2 + 3y^2\frac{dy}{dx} - 9(x\frac{dy}{dx} + y\frac{dx}{dx}) = 0$$ **Differentiate both sides with respect to x.**

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$ **Treat xy as a product and y as a function of x.**

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$ **Solve for dy/dx.**

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y-x^2}{y^2-3x}\right|_{(2,4)} = \frac{3(4)-2^2}{4^2-3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope 4/5:

$$y = 4 + \frac{4}{5}(x - 2) \Rightarrow y = \frac{4}{5}x + \frac{12}{2}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope -5/4:

$$y = 4 - \frac{5}{4}(x - 2) \Rightarrow y = -\frac{5}{4}x + \frac{13}{2}.$$

Exercises:

1. Use implicit differentiation to find dy/dx in following.

a) $x^2y + xy^2 = 6$ **b)** $x^3 + y^3 = 18xy$

c) $y^2 = \frac{x-1}{x+1}$ **d)** $x^3 = \frac{2x-y}{x+y}$

e) $x = \sec y$ **f)** $xy = \cot(xy)$

g) $x + \tan(xy) = 0$ **h)** $x^4 + \sin y = x^3y^2$

i) $y\sin\left(\frac{1}{y}\right) = 1 - xy$ **j)** $x\cos(2x + 3y) = y\sin x$

2. Find $dr/d\theta$ in following

a) $\theta^{1/2} + r^{1/2} = 1$ **b)** $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{3/2} + \frac{4}{3}\theta^{3/4}$

c) $\sin(r\theta) = \frac{1}{2}$ **d)** $\cos r + \cos\theta = r$

3. In following, use implicit differentiation to find dy/dx and then d^2y/dx^2. Write the solutions in terms of x and y only.

a) $x^2 + y^2 = 1$ **b)** $x^{2/3} + y^{2/3} = 1$

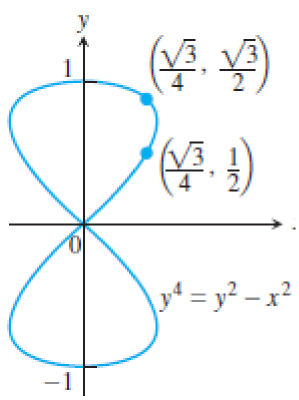
c) $y^2 = x^2 + 2x$ **d)** $y^2 - 2x = 1 - 2y$

e) $2\sqrt{y} = x - y$ **f)** $xy + y^2 = 1$

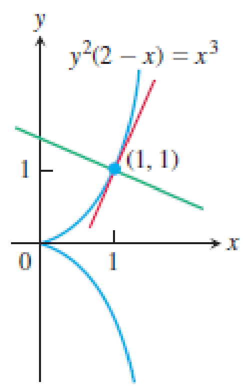
g) $3 + \sin y = y - x^3$ **h)** $\sin y = x\cos y - 2$

4. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point (2, 2).
5. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point (0, -1).
6. In following, find the slope of the curve at the given points.
a) $y^2 + x^2 = y^4 - 2x$ at (-2, 1) and (-2, -1).
b) $(x^2 + y^2)^2 = (x - y)^2$ at (-2, 1) and (-2, -1).
7. In following, verify that the given point is on the curve and find the lines that are **I)** tangent and **II)** normal to the curve at the given point.
a) $x^2 + xy - y^2 = 1, (2,3)$
b) $x^2 + y^2 = 25, (3, -4)$
c) $x^2y^2 = 9, (-1,3)$
d) $y^2 - 2x - 4y - 1 = 0, (-2,1)$
e) $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, (-1,0)$
f) $x^2 - \sqrt{3}xy + 2y^2 = 5, (\sqrt{3}, 2)$
g) $2xy + \pi \sin y = 2\pi, (1, \pi/2)$
h) $x \sin 2y = y \cos 2x, (\pi/4, \pi/2)$
i) $y = 2\sin(\pi x - y), (1,0)$
j) $x^2 cos^2 y - \sin y = 0, (0, \pi)$
8. Parallel tangents Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the *x*-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?
9. Normal parallel to a line Find the normal to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

10. The eight curve. Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.

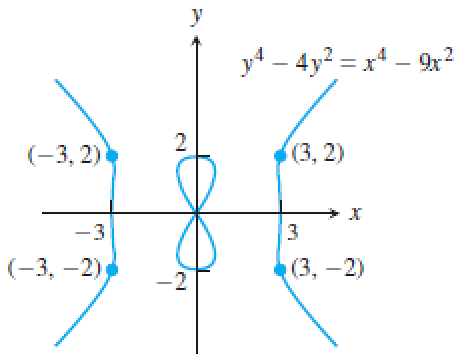


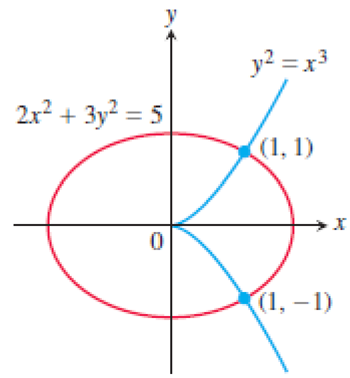
11. Find equations for the tangent and normal to the cissoid of Diocles $y^2(2 - x) = x^3$ at (1, 1).



12. Find the slopes of the curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.

13. Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 5$ at the points $(1, \pm 1)$? Give reasons for your answer.





14. For The folium of Descartes (See Figure)

a) Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

b) At what point other than the origin does the folium have a horizontal tangent?

c) Find the coordinates of the point A, where the folium has a vertical tangent.