Calculus II

Second Semester

Lecturer 4

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5.3 Trigonometric Integrals:

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral $\int sec^2 x\, dx = \tan x + C$. The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

5.3.1 Products of Powers of Sines and Cosines:

We begin with integrals of the form

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If ***m*** is **odd**, we write m as $2k+1$ and use the identity $sin^2 x = 1 - cos^2 x$ to obtain

$$sin^m x = sin^{2k+1} x = (sin^2 x)^k \, sin\, x = (1 - cos^2 x)^k \, sin\, x. \qquad \textbf{(1)}$$

Then we combine the single $sin\, x$ with dx in the integral and set $sin\, x\, dx$ equal to $-d(cos\, x)$.

Case 2 If ***n*** is **odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $cos^2 x = 1 - sin^2 x$ to obtain

$$cos^n x = cos^{2k+1} x = (cos^2 x)^k \, cos\, x = (1 - sin^2 x)^k \, cos\, x.$$

We then combine the single $cos\, x$ with dx and set $cos\, x\, dx$ equal to $d(sin\, x)$.

Case 3 If both **m** and **n** are **even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$sin^2 x = \frac{1-\cos 2x}{2}, \quad cos^2 x = \frac{1+\cos 2x}{2} \qquad \textbf{(2)}$$

to reduce the integrand to one in lower powers of $cos\, 2x$.

Here are some examples illustrating each case.

Example:

Evaluate $\int \sin^3 x \cos^2 x\, dx$.

Solution:

This is an example of Case 1.

$$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x \sin x\, dx \quad \textbf{m is odd.}$$
$$= \int (1 - cos^2 x)(\cos^2 x)(-d(\cos x)) \quad \sin x\, dx = -d(\cos x)$$
$$= \int (1 - u^2)(u^2)(-du) \quad u = \cos x$$
$$= \int (u^4 - u^2)(-du) \quad \textbf{Multiply terms.}$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{cos^5 x}{5} - \frac{cos^3 x}{3} + C. \blacksquare$$

Example:

Evaluate $\int \cos^5 x\, dx$.

Solution:

This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx$$
$$= \int (1 - sin^2 x)^2 d(\sin x) \quad \cos x\, dx = d(\sin x)$$
$$= \int (1 - u^2)^2 du \quad u = \sin x$$
$$= \int (1 - 2u^2 + u^4) du \quad \textbf{Square } 1 - u^2.$$
$$= u - \tfrac{2}{3}u^3 + \tfrac{1}{5}u^5 + C = \sin x - \tfrac{2}{3}sin^3 x + \tfrac{1}{5}sin^5 x + C. \blacksquare$$

Example:

Evaluate $\int \sin^2 x \cos^4 x\, dx$.

Solution:

This is an example of Case 3.

$$\int \sin^2 x \cos^4 x\, dx = \int (\frac{1 - cos\,2x}{2})(\frac{1 + cos\,2x}{2})^2 dx \quad \textbf{m and n both even}$$
$$= \tfrac{1}{8} \int (1 - cos\, 2x)(1 + 2\cos 2x + \cos^2 2x) dx$$
$$= \tfrac{1}{8} \int (1 + cos\, 2x - \cos^2 2x - \cos^3 2x) dx$$
$$= \tfrac{1}{8}[x + \tfrac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x) dx]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x\, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$
$$= \frac{1}{2}(x + \frac{1}{4}\sin 4x)$$

Omit constant of integration until final result.

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\, dx = \int (1 - \sin^2 2x) \cos 2x\, dx$$

$\boldsymbol{u = \sin 2x, du = 2\cos 2x\, dx}$

$$= \frac{1}{2}(1 - u^2)du = \frac{1}{2}(\sin 2x - \frac{1}{3}\sin^3 2x$$

Again omit C.

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C. \blacksquare$$

5.3.2 Eliminating Square Roots:

In the next example, we use the identity $cos^2\, \theta = \frac{1+\cos 2\theta}{2}$ to eliminate a square root.

Example:

Evaluate $\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx$.

Solution:

To eliminate the square root, we use the identity

$$cos^2\, \theta = \frac{1+\cos 2\theta}{2} \quad \text{or} \quad 1 + \cos 2\theta = 2cos^2\, \theta$$

With $\theta = 2x$, this becomes $1 + \cos 4x = 2cos^2\, 2x$. Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2cos^2 2\, x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{cos^2 2\, x}\, dx$$
$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x| dx$$
$$= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \boldsymbol{\cos 2x \geq 0 \text{ on } [0, \pi/4]}$$
$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}. \blacksquare$$

5.3.3 Integrals of Powers of tan *x* and sec *x*:

We know how to integrate the tangent and secant functions and their squares. To integrate higher powers, we use the identities $tan^2\, x = sec^2\, x - 1$ and $sec^2\, x = tan^2\, x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

Example:

Evaluate $\int \tan^4 x\, dx$.

Solution:

$$\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1) dx$$
$$= \int \tan^2 x \cdot \sec^2 x\, dx - \int \tan^2 x\, dx$$
$$= \int \tan^2 x \cdot \sec^2 x\, dx - \int (\sec^2 x - 1) dx$$
$$= \int \tan^2 x \cdot \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx$$

In the first integral, we let $u = tan\, x$, $du = \sec^2 x\ dx$ and have

$$\int \mathrm{u}^2 du = \tfrac{1}{3}\mathrm{u}^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \tfrac{1}{3}\tan^3 x - \tan x + x + C. \blacksquare$$

Example:

Evaluate $\int \sec^3 x\, dx$.

Solution:

We integrate by parts using $u = sec\ x, dv = sec^2\ x\ dx, v = tan\ x,$ $du = \ sec\ x\ tan\ x\ dx$ .Then

$$\int \sec^3 x\, dx = \sec x \cdot \tan x - \int (\tan x)(\sec x \tan x\, dx)$$
$$= \sec x \cdot \tan x - \int (\sec^2 x - 1) \sec x\, dx \quad \boldsymbol{tan^2\, x = sec^2\, x - 1}$$
$$= \sec x \cdot \tan x + \int \sec x\ dx - \int \sec^3 x\, dx\ .$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\, dx = \sec x \cdot \tan x + \int \sec x\ dx,$$

and $\int \sec^3 x\, dx = \frac{1}{2}\sec x \cdot \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C. \blacksquare$

Example:

Evaluate $\int \tan^4 x \cdot \sec^4 x\, dx$.

Solution:

$$\int \tan^4 x \cdot \sec^4 x\, dx = \int \tan^4 x\,(1 + tan^2\ x\,) \sec^2 x\, dx \quad \boldsymbol{\sec^2 x = tan^2\ x + 1}$$
$$= \int (\tan^4 x + tan^6\ x\,) \sec^2 x\, dx$$
$$= \int (\tan^4 x) \sec^2 x\, dx + \int (tan^6\ x\,) \sec^2 x\, dx$$
$$= \int u^4 du + \int u^6 du = \frac{u^5}{5} + \frac{u^7}{7} + C \quad \boldsymbol{u = tan\ x, du = sec^2\ x\ dx}$$
$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C. \blacksquare$$

5.3.4 Products of Sines and Cosines:

The integrals

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \int \cos mx \cos nx \, dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]. \qquad \textbf{(3)}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x]. \qquad \textbf{(4)}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \qquad \textbf{(5)}$$

Example:

Evaluate $\int \sin 3x \cdot \cos 5x \, dx$.

Solution:

From Equation (4) with $m = 3$ and $n = 5$, we get

$$\int \sin 3x \cdot \cos 5x \, dx = \frac{1}{2}\int[\sin(-2x) + \sin 8x]dx$$

$$= \frac{1}{2}\int[\sin 8x - \sin 2x]dx$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \blacksquare$$

Exercises:

1. Evaluate the integrals in following Exercises:

1) $\int \cos 2x \, dx$
2) $\int_0^{\pi} 3\sin\frac{x}{3}dx$
3) $\int \cos^3 x \sin x \, dx$
4) $\int \sin^4 2x \cos 2x \, dx$
5) $\int \sin^3 x \, dx$
6) $\int \cos^3 4x \, dx$
7) $\int \sin^5 x \, dx$
8) $\int_0^{\pi} \sin^5 \frac{x}{2}dx$
9) $\int \cos^3 x \, dx$
10) $\int_0^{\pi/6} 3\cos^5 3x dx$
11) $\int \sin^3 x \cos^3 x \, dx$
12) $\int \cos^3 2x \sin^5 2x \, dx$
13) $\int \cos^2 x \, dx$
14) $\int_0^{\pi/2} \sin^2 x dx$
15) $\int_0^{\pi/2} \sin^7 y dy$
16) $\int 7\cos^7 t \, dt$
17) $\int_0^{\pi} 8\sin^4 x dx$
18) $\int 8\cos^4 2\pi x \, dx$
19) $\int 16\sin^2 x \cos^2 x \, dx$
20) $\int_0^{\pi} 8\sin^4 y \cos^2 y \, dy$
21) $\int 8\cos^3 2\theta \sin 2\theta \, d\theta$
22) $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$
23) $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}dx$
24) $\int_0^{\pi} \sqrt{1-\cos 2x} \, dx$

25) $\int_0^{\pi} \sqrt{1-\sin^2 t}\,dt$

26) $\int_0^{\pi} \sqrt{1-\cos^2 \theta}\,d\theta$

27) $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}} dx$

28) $\int_0^{\pi/6} \sqrt{1+\sin x}\,dx$

29) $\int_{5\pi/6}^{\pi} \frac{\cos^4 x}{\sqrt{1-\sin x}} dx$

30) $\int_{\pi/2}^{3\pi/4} \sqrt{1-\sin 2x}\,dx$

31) $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta$

32) $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2} dt$

33) $\int \sec^2 x \tan x\,dx$

34) $\int \sec x \tan^2 x\,dx$

35) $\int \sec^3 x \tan x\,dx$

36) $\int \sec^3 x \tan^3 x\,dx$

37) $\int \sec^2 x \tan^2 x\,dx$

38) $\int \sec^4 x \tan^2 x\,dx$

39) $\int_{-\pi/3}^{0} 2\sec^3 x\,dx$

40) $\int e^x \sec^3 x\,dx$

41) $\int \sec^4 \theta\,d\theta$

42) $\int 3\sec^4 3x\,dx$

43) $\int_{\pi/4}^{\pi/2} \csc^4 \theta\,d\theta$

44) $\int \sec^6 x\,dx$

45) $\int 4\tan^3 x\,dx$

46) $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\,dx$

47) $\int \tan^5 x\,dx$

48) $\int \cot^6 2x\,dx$

49) $\int_{\pi/6}^{\pi/3} \cot^3 x\,dx$

50) $\int 8\cot^4 t\,dt$

51) $\int sin\,3x \cos 2x\,dx$

52) $\int sin\,2x \cos 3x\,dx$

53) $\int_{-\pi}^{\pi} \sin 3x \sin 3x\,dx$

54) $\int_0^{\pi/2} \sin x \cos x\,dx$

55) $\int cos\,3x \cos 4x\,dx$

56) $\int_{-\pi/2}^{\pi/2} cos\,x \cos 7x\,dx$

2. Following exercises require the use of various trigonometric identities before you evaluate the integrals.

a) $\int \sin^2 \theta \cos 3\theta\,d\theta$

b) $\int \cos^2 2\theta \sin \theta\,d\theta$

c) $\int \cos^3 \theta \sin 2\theta\,d\theta$

d) $\int \sin^3 \theta \cos 2\theta\,d\theta$

e) $\int sin\,\theta \cos \theta \cos 3\theta\,d\theta$

f) $\int sin\,\theta \sin 2\theta \sin 3\theta\,d\theta$

3. Use any method to evaluate the integrals in following:

a) $\int \frac{\sec^3 x}{\tan x} dx$

b) $\int \frac{\sin^3 x}{\cos^4 x} dx$

c) $\int \frac{\tan^2 x}{\csc x} dx$

d) $\int \frac{cot\,x}{\cos^2 x} dx$

e) $\int x \sin^2 x\,dx$

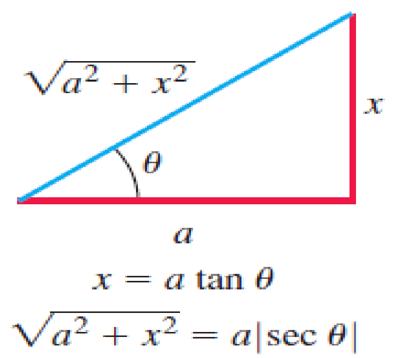
f) $\int x \cos^3 x\,dx$

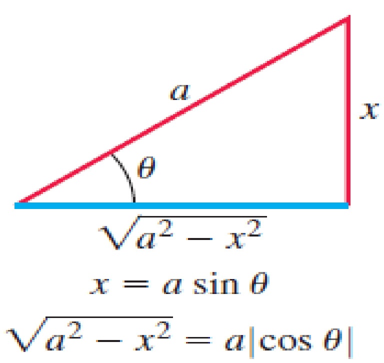
5.4 Trigonometric Substitutions:

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are $x = a\,tan\,u, x = a\,sin\,u$, and $x = a\,sec\,u$. These substitutions are effective in transforming integrals involving $\sqrt{a^2+x^2}, \sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$ into integrals we can evaluate directly since they come from the reference right triangles in Figure

With $\boldsymbol{x = a\,tan\,\theta}$,

$$\boldsymbol{a^2+x^2} = a^2 + a^2\tan^2\theta$$
$$= a^2(1+\tan^2\theta)$$
$$= \boldsymbol{a^2\sec^2\theta}.$$





With $\boldsymbol{x = a\,sin\,\theta}$,

$$\boldsymbol{a^2-x^2} = a^2 - a^2\sin^2\theta$$
$$= a^2(1-\sin^2\theta)$$
$$= \boldsymbol{a^2\cos^2\theta}.$$

With $\boldsymbol{x = a\,sec\,\theta}$,

$$\boldsymbol{x^2-a^2} = a^2\sec^2\theta - a^2$$
$$= a^2(\sec^2\theta - 1)$$
$$= \boldsymbol{a^2\tan^2\theta}.$$

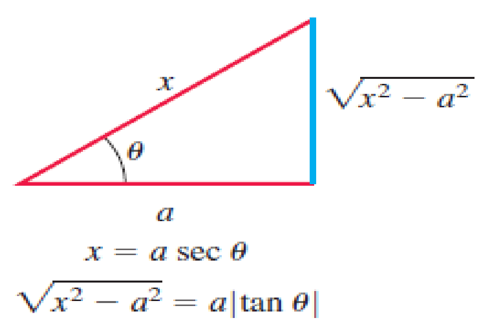


FIGURE: Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

Remark:

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\,tan\,\theta$, we want to be able to set $\theta = tan^{-1}(x/a)$ after the integration takes place. If $x = a\,sin\,\theta$, we want to be able to set $\theta = sin^{-1}(x/a)$ when we're done, and similarly for $x = a\,sec\,\theta$.

As we know from Section 3.8, the functions in these substitutions have inverses only for selected values of θ. For reversibility,

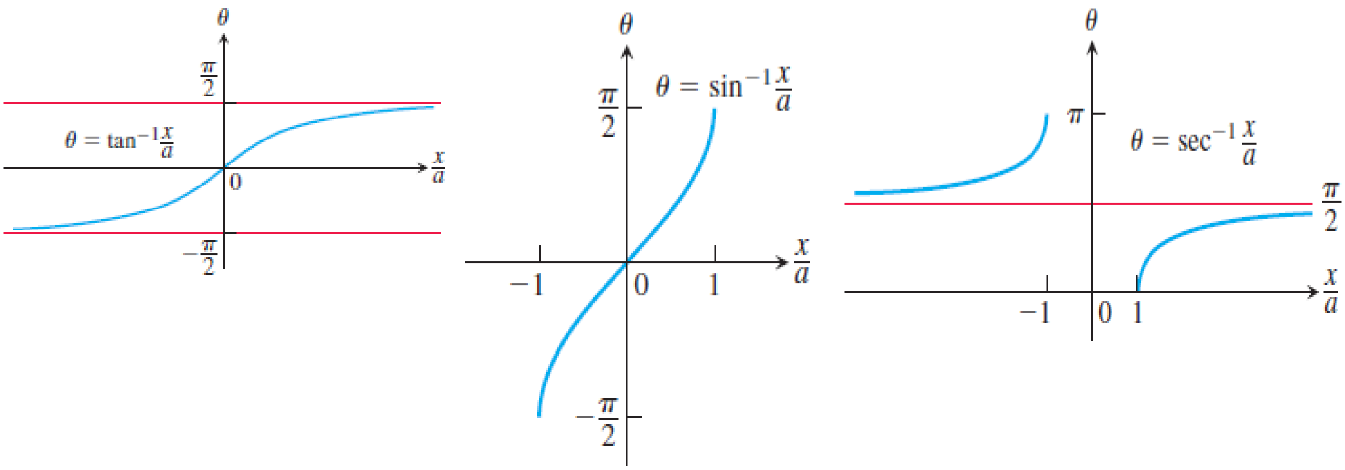


FIGURE: The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

$$x = a\,tan\,\theta \;\text{ requires }\; \theta = tan^{-1}\left(\tfrac{x}{a}\right) \;\text{ with }\; -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2},$$

$$x = a\,sin\,\theta \;\text{ requires }\; \theta = sin^{-1}\left(\tfrac{x}{a}\right) \;\text{ with }\; -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2},$$

$$x = a\,sec\,\theta \;\text{ requires }\; \theta = sec^{-1}\left(\tfrac{x}{a}\right) \;\text{ with }\; \begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1, \\ \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le 1. \end{cases}$$

To simplify calculations with the substitution $x = a\,sec\,\theta$, we will restrict its use to integrals in which $\frac{x}{a} \ge 1$. This will place θ in $[0, \frac{\pi}{2})$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

Procedure for a Trigonometric Substitution:

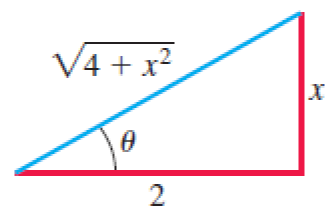
1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4. Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

Example:

Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$.

Solution:

We set



$$x = 2\tan\theta\,,\ dx = 2\sec^2\theta\, d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\,,$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$$

Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta d\theta}{|\sec\theta|}$$
$$= \int \sec\theta\, d\theta$$
$$= ln|\sec\theta + \tan\theta| + C$$
$$= ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

$\sqrt{\sec^2\theta} = |\sec\theta|$

$\sec\theta > 0$ **for** $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Notice how we expressed $ln|\sec\theta + \tan\theta|$in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ and read the ratios from the triangle. ■

Example:

Here we find an expression for the inverse hyperbolic sine function in terms of the natural logarithm. Following the same procedure as in the previous Example, we find that

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \sec\theta \, d\theta \qquad \boldsymbol{x = a\tan\theta\,, dx = \sec\theta d\theta}$$

$$= ln|\sec\theta + \tan\theta| + C$$

$$= ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C.$$

Since $sinh^{-1}(x/a)$ is also an antiderivative of $\frac{1}{\sqrt{a^2+x^2}}$, so the two antiderivatives differ by a constant, giving

$$sinh^{-1}\left(\frac{x}{a}\right) = ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + C.$$

Setting $x = 0$ in this last equation, we find $0 = ln\,|1| + C$, so $C = 0$.
Since $\sqrt{a^2+x^2} > |x|$, we conclude that

$$\boldsymbol{sinh^{-1}\left(\frac{x}{a}\right) = ln\,\left(\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right).}$$

Example:

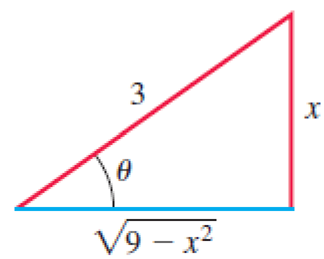
Evaluate $\int \frac{x^2dx}{\sqrt{9-x^2}}$.

Solution:

We set

$$x = 3\sin\theta\,,\ dx = 3\cos\theta\,d\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}\,,$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$



Then

$$\int \frac{x^2dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta\cdot 3\cos\theta\,d\theta}{|3\cos\theta|} = 9\int \sin^2\theta d\theta \qquad \boldsymbol{\cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$$

$$= 9\int \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \boldsymbol{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$= \frac{9}{2}\left(sin^{-1}\frac{x}{3} - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C. \blacksquare$$

Example:

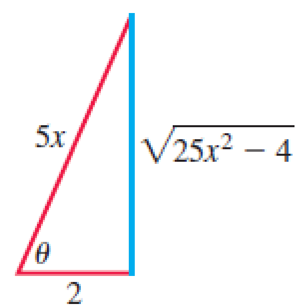
Evaluate $\int \frac{dx}{\sqrt{25x^2-4}}$, $x > \frac{2}{5}$.

Solution:

We first rewrite the radical as

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2-\frac{4}{25}\right)} = 5\sqrt{x^2-\left(\frac{2}{5}\right)^2} \qquad \sqrt{x^2-a^2} \text{ with } a = \frac{2}{5}$$

to put the radicand in the form $x^2 - a^2$. We then substitute



$$x = \frac{2}{5}\sec\theta,\ dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, 0 < \theta < \frac{\pi}{2}.$$

We then get

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25} = \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

and $\sqrt{x^2-\left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$ $\quad$ **$\tan\theta > 0$ for $0 < \theta < \frac{\pi}{2}$**

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{\left(x^2-\frac{4}{25}\right)}} = \int \frac{\left(\frac{2}{5}\right)\sec\theta\tan\theta\, d\theta}{5.\left(\frac{2}{5}\right)\tan\theta} = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C = \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C. ■$$

Exercises:

1. Evaluate the integrals in following Exercises

a) $\int \frac{dx}{\sqrt{9+x^2}}$ **b)** $\int \frac{3dx}{\sqrt{1+9x^2}}$ **c)** $\int_{-2}^{2} \frac{dx}{4+x^2}$

d) $\int_0^2 \frac{dx}{8+2x^2}$ **e)** $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$ **f)** $\int_0^{1/2\sqrt{2}} \frac{2dx}{\sqrt{1-4x^2}}$

g) $\int \sqrt{25-t^2}dt$ **h)** $\int \sqrt{1-9t^2}dt$ **i)** $\int \frac{dx}{\sqrt{4x^2-49}}$, $x > \frac{7}{2}$

j) $\int \frac{5dx}{\sqrt{25x^2-9}}$, $x > \frac{3}{5}$ **k)** $\int \frac{\sqrt{y^2-49}}{y}dy,\ y > 7$ **l)** $\int \frac{\sqrt{y^2-25}}{y^3}dy,\ y > 5$

m) $\int \frac{dx}{x^2\sqrt{x^2-1}}$, $x > 1$ **n)** $\int \frac{2dx}{x^3\sqrt{x^2-1}}$, $x > 1$

2. Use any method to evaluate the integrals in following Exercises. Most will require trigonometric substitutions, but some can be evaluated by other methods.

a) $\int \frac{x}{\sqrt{9-x^2}}dx$ **b)** $\int \frac{x^2}{\sqrt{4+x^2}}dx$ **c)** $\int \frac{x^3}{\sqrt{x^2+4}}dx$

d) $\int \frac{dx}{x^2\sqrt{x^2+1}}$ **e)** $\int \frac{8dw}{w^2\sqrt{4-w^2}}$ **f)** $\int \frac{\sqrt{9-w^2}}{w^2}dw$

g) $\int \sqrt{\frac{x+1}{1-x}}dx$ **h)** $\int x\sqrt{x^2-4}\,dx$ **i)** $\int_0^{\sqrt{3}/2} \frac{4x^2dx}{(1-x^2)^{3/2}}$

j) $\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$ **k)** $\int \frac{dx}{(x^2-1)^{3/2}}, x>1$ **l)** $\int \frac{x^2dx}{(x^2-1)^{5/2}}, x>1$

m) $\int \frac{(1-x^2)^{3/2}}{x^6}dx$ **n)** $\int \frac{(1-x^2)^{1/2}}{x^4}dx$ **o)** $\int \frac{8dx}{(4x^2+1)^2}$

p) $\int \frac{6dt}{(9t^2+1)^2}$ **q)** $\int \frac{x^3dx}{x^2-1}$ **r)** $\int \frac{xdx}{25+4x^2}$

s) $\int \frac{v^2dv}{(1-v^2)^{5/2}}$ **t)** $\int \frac{(1-r^2)^{5/2}}{r^8}dr$

3. In following Exercises, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

a) $\int_0^{\ln 4} \frac{e^tdt}{\sqrt{e^{2t}+9}}$ **b)** $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^tdt}{(1+e^{2t})^{3/2}}$ **c)** $\int_{1/12}^{1/4} \frac{2dt}{\sqrt{t}+4t\sqrt{t}}$

d) $\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}}$ **e)** $\int \frac{dx}{x\sqrt{x^2-1}}$ **f)** $\int \frac{dx}{1+x^2}$

g) $\int \frac{xdx}{\sqrt{x^2-1}}$ **h)** $\int \frac{dx}{\sqrt{1-x^2}}$ **i)** $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx$

j) $\int \sqrt{\frac{4-x}{x}}dx$ **k)** $\int \sqrt{\frac{x}{1-x}}dx$ **l)** $\int \sqrt{x}\sqrt{1-x}\,dx$

m) $\int \frac{\sqrt{x-2}}{\sqrt{x-1}}dx$

4. For the following Exercises, complete the square before using an appropriate trigonometric substitution.

a) $\int \sqrt{8-2x-x^2}dx$ **b)** $\int \frac{1}{\sqrt{x^2-2x+5}}dx$

c) $\int \frac{\sqrt{x^2+4x+3}}{x+2}dx$ **d)** $\int \frac{\sqrt{x^2+2x+2}}{x^2+2x+1}dx$

Calculus II

Second Semester

Lecturer 5

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5.5 Integration of Rational Functions by Partial Fractions:

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called ***partial fractions***, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

We can verify this equation algebraically by placing the fractions on the right side over a common denominator $(x+1)(x-3)$. The skill acquired in writing rational functions such as a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x - 3)/(x^2 - 2x - 3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx.$$

The method for rewriting rational functions as a sum of simpler fractions is called ***the method of partial fractions***. In the case of the preceding example, it consists of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3} \qquad (1)$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x+1)$ and $B/(x-3)$ partial fractions because their denominators are only part of the original denominator $(x^2 - 2x - 3)$. We call A and B undetermined coefficients until suitable values for them have been found.

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$5x - 3 = A(x - 3) + B(x + 1) = (A + B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

$$A + B = 5, -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

5.5.1 General Description of the Method:

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.
- The values of the undetermined coefficients form a system of n linear equations in n unknowns. For large n, solving such systems may require linear algebra methods (such as Gaussian Elimination).

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known. A quadratic polynomial (or factor) is ***irreducible*** if it cannot be written as the product of two linear factors with real coefficients. That is, the polynomial has no real roots.

Method of Partial Fractions When $f(x)/g(x)$ Is Proper

1. Let $x-r$ be a linear factor of $g(x)$. Suppose that $(x-r)^m$ is the highest power of $x-r$ that divide $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}.$$

Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be an irreducible quadratic factor of $g(x)$ so that x^2+px+q has no real roots. Suppose that $(x^2+px+q)^n$ is the highest power of this factor that divides g(x). Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{(x^2+px+q)}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}.$$

Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example:

Use partial fractions to evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$.

Solution:

Note that each of the factors $(x-1)$, $(x+1)$, and $(x+3)$ is raised only to the first power. Therefore, the partial fraction decomposition has the form

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$x^2 + 4x + 1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$
$$= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1)$$
$$= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C).$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of *x*, obtaining

Coefficient of x^2: $A + B + C = 1$

Coefficient of x^1: $4A + 2B = 4$

Coefficient of x^0: $3A - 3B - C = 1$

There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. The solution is $A = 3/4, B = 1/2$, and $C = -1/4$. Hence, we have

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\, dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$
$$= \frac{3}{4}\ln|X-1| + \frac{1}{2}\ln|X+1| - \frac{1}{4}\ln|X+3| + K,$$

where *K* is the arbitrary constant of integration (we call it *K* here to avoid confusion with the undetermined coefficient we labeled as C). ■

Example:

Use partial fractions to evaluate $\int \frac{6x+7}{(x+2)^2}\, dx$.

Solution:

First, we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$ **Two terms because $(x+2)$ is squared**

$$6x + 7 = A(x+2) + B$$ **Multiply both sides by $(x+2)^2$.**

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of *x* gives $A = 6$ and $2A + B = 12 + B = 7$, or $A = 6$ and $B = -5$. Therefore,

$$\int \frac{6x+7}{(x+2)^2} = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx = 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\, dx$$

$$= 6\ln|x+2| + 5\int (x+2)^{-1} + C. \blacksquare$$

Remark:

The next example shows how to handle the case when $f(x)/g(x)$ is an improper fraction. It is a case where the degree of f is larger than the degree of g.

Example:

Use partial fractions to evaluate $\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\, dx$.

Solution:

First, we divide the denominator into the numerator to get a polynomial plus a proper fraction.

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\begin{array}{r|l} & 2x \\ \hline x^2-2x-3 & 2x^3-4x^2-x-3 \\ & 2x^3-4x^2-6x-3 \\ \hline & 5x-3 \end{array}$$

$$\frac{2x^3-4x^2-x-3}{x^2-2x-3} = 2x + \frac{5x-3}{x^2-2x-3}.$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\int \frac{2x^3-4x^2-x-3}{x^2-2x-3}\, dx = \int 2x\, dx + \int \frac{5x-3}{x^2-2x-3}\, dx$$

$$= \int 2x\, dx + \int \frac{2}{x+1}\, dx + \int \frac{3}{x-3}\, dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C. \blacksquare$$

Example:

Use partial fractions to evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx$.

Solution:

The denominator has an irreducible quadratic factor x^2+1 as well as a repeated linear factor $(x-1)^2$, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2} \qquad (2)$$

Clearing the equation of fractions gives

$$-2x + 4 = (Ax + B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$

$$= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D).$$

Equating coefficients of like terms gives

Coefficients of x^3: $0 = A + C$

Coefficients of x^2: $0 = -2A + B - C + D$

Coefficients of x^1: $-2 = A - 2B + C$

Coefficients of x^0: $4 = B - C + D$

We solve these equations simultaneously to find the values of *A*, *B*, *C*, and *D*:

$-4 = -2A, A = 2$ **Subtract fourth equation from second.**

$C = -A = -2$ **From the first equation**

$B = (A + C + 2)/2 = 1$ **From the third equation and $C = -A$**

$D = 4 - B + C = 1.$ **From the fourth equation**

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

Finally, using the expansion above we can integrate:

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\, dx = \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx$$

$$= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx$$

$$= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \blacksquare$$

Example:

Use partial fractions to evaluate $\int \frac{dx}{x(x^2+1)^2}$.

Solution:

The form of the partial fraction decomposition is

$$\frac{dx}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A. \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, C=0, 2A+B+D=0, C+E=0, A=1.$$

Solving this system gives $A=1, B=-1, C=0, D=-1$, and $E=0$. Thus,

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx = \int \frac{dx}{x} - \int \frac{xdx}{x^2+1} - \int \frac{xdx}{(x^2+1)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad \boldsymbol{u = x^2+1, du = 2x\,dx}$$

$$= \ln|x| - \frac{1}{2}ln|u| + \frac{1}{2u} + K$$

$$= \ln|x| - \frac{1}{2}ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \blacksquare$$

Remark (The Heaviside "Cover-up" Method for Linear Factors):

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and $g(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$ is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

Example:

Find A, B, and C in the partial fraction expansion

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}. \qquad (3)$$

Solution:

If we multiply both sides of Equation (3) by $(x-1)$ to get

$$\frac{x^2+1}{(x-2)(x-3)} = A + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x-3}$$

and set $x=1$, the resulting equation gives the value of A:

$$\frac{(1)^2+1}{(1-2)(1-3)} = A + 0 + 0 \Rightarrow A = 1.$$

In the same way, we can multiply both sides by $(x-2)$ and then substitute in $x=2$. This gives $\frac{(2)^2+1}{(2-1)(2-3)} = B$. So, $B=-5$. Finally, we multiply both sides by $(x-3)$ and then substitute in $x=3$, which yields $\frac{(3)^2+1}{(3-1)(3-2)}=C$, and $C=5$. ■

Heaviside Method

1. Write the quotient with $g(x)$ factored:

$$\frac{f(x)}{g(x)}=\frac{f(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)}$$

2. Cover the factors $(x-r_i)$ of $g(x)$ one at a time, each time replacing all the uncovered x's by the number r_i . This gives a number A_i for each root r_i:

$$A_1=\frac{f(r_1)}{(r_1-r_2)\cdots(r_1-r_n)}$$

$$A_2=\frac{f(r_2)}{(r_2-r_1)(r_2-r_3)\cdots(r_2-r_n)}$$

$$\vdots$$

$$A_n=\frac{f(r_n)}{(r_n-r_1)(r_n-r_2)\cdots(r_n-r_{n-1})}.$$

3. Write the partial fraction expansion of $f(x)/g(x)$ as

$$\frac{f(x)}{g(x)}=\frac{A_1}{(x-r_1)}+\frac{A_2}{(x-r_2)^2}+\cdots+\frac{A_n}{(x-r_n)^n}.$$

Example:

Use the Heaviside Method to evaluate

$$\int\frac{x+4}{x^3+3x^2-10x}\,dx.$$

Solution:

The degree of $f(x)=x+4$ is less than the degree of the cubic polynomial $g(x)=x^3+3x^2-10x$, and, with $g(x)$ factored,

$$\frac{x+4}{x^3+3x^2-10x}=\frac{x+4}{x(x-2)(x+5)}$$

The roots of $g(x)$ are $r_1=0, r_2=2$, and $r_3=-5$. We find

$$A_1=\frac{f(0)}{(0-2)(0+5)}=\frac{0+4}{(-2)(5)}=\frac{4}{-10}=-\frac{2}{5}$$

$$A_2=\frac{f(2)}{2(2+5)}=\frac{2+4}{(2)(7)}=\frac{6}{14}=\frac{3}{7}$$

$$A_3=\frac{f(-5)}{(-5)(-5-2)}=\frac{-5+4}{(-5)(-7)}=\frac{-1}{35}.$$

Therefore, $\frac{x+4}{x^3+3x^2-10x}=-\frac{2}{5\,x}+\frac{3}{7(x-2)}-\frac{1}{35(x+5)}$, and

$$\int\frac{x+4}{x^3+3x^2-10x}\,dx=-\frac{2}{5}\ln|x|+\frac{3}{7}\ln|x-2|-\frac{1}{35}\ln|x+5|+C.$$ ■

5.5.2 Other Ways to Determine the Coefficients:

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

Example:

Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution:

We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining $1 = 2A(x+1) + B$.

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence, $\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}$. ■

Remark:

In some problems, assigning small values to x, such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

Example:

Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

by assigning numerical values to x.

Solution:

Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$x = 1$: $(1)^2 + 1 = A(-1)(-2) + B(0) + C(0) \Rightarrow 2 = 2A \Rightarrow$ **A = 1**

$x = 2$: $(2)^2 + 1 = A(0) + B(1)(-1) + C(0) \Rightarrow 5 = -B \Rightarrow$ **B = -5**

$x = 3$: $(3)^2 + 1 = A(0) + B(0) + C(2)(1) \Rightarrow 10 = 2C \Rightarrow$ **C = 5**.

Conclusion: $\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}$. ■

Exercises:

1. Expand the quotients in the following Exercises by partial fractions.

a) $\frac{5x-13}{(x-3)(x-2)}$ b) $\frac{5x-7}{x^2-3x+2}$ c) $\frac{x+4}{(x+1)^2}$ d) $\frac{2x+2}{x^2-2x+1}$

e) $\frac{z+1}{z^2(z-1)}$ f) $\frac{z}{z^3-z^2-6z}$ g) $\frac{t^2+8}{t^2-5t+6}$ h) $\frac{t^4+9}{t^4+9t^2}$

2. In following Exercises, express the integrand as a sum of partial fractions and evaluate the integrals.

a) $\int \frac{dx}{1-x^2}$ b) $\int \frac{dx}{x^2+2x}$ c) $\int \frac{x+4}{x^2+5x-6}\,dx$ d) $\frac{2x+1}{x^2-7x+12}dx$

e) $\int_4^8 \frac{y\,dy}{y^2-2y-3}$ f) $\int_{1/2}^1 \frac{y+4}{y^2+y}$ g) $\int \frac{dt}{t^3+t^2-2t}$ h) $\int \frac{x+3}{2x^3-8x}dx$

3. In following Exercises, express the integrand as a sum of partial fractions and evaluate the integrals.

a) $\int_0^1 \frac{x^3\,dx}{x^2+2x+1}$ b) $\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$ c) $\int \frac{dx}{(x^2-1)^2}$ d) $\int \frac{x^2 dx}{(x-1)(x^2+2x+1)}$

4. In following Exercises, express the integrand as a sum of partial fractions and evaluate the integrals.

a) $\int_0^1 \frac{dx}{(x+1)(x^2+1)}$ b) $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+t}dt$ c) $\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy$

d) $\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$ e) $\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$ f) $\int \frac{s^4+81}{s(s^2+9)^2}\,ds$

g) $\int \frac{x^2-x+2}{x^3-1}\,dx$ h) $\int \frac{1}{x^4+x}\,dx$ i) $\int \frac{x^2}{x^4-1}\,dx$

j) $\int \frac{x^2+x}{x^4-3x^3-4}\,dx$ k) $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta$ l) $\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta$

5. In following Exercises, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

a) $\int \frac{2x^3-2x^2+1}{x^2-x}\,dx$ b) $\int \frac{x^4}{x^2-1}\,dx$ c) $\int \frac{9x^3-3x+1}{x^3-x^2}\,dx$

d) $\int \frac{16x^3}{4x^2-4x+1}\,dx$ e) $\int \frac{y^4+y^2-1}{y^3+y}\,dy$ f) $\int \frac{2y^4}{y^3-y^2+y-1}\,dy$

6. Evaluate the integrals in following Exercises.

a) $\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$ b) $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt$ c) $\int \frac{\cos y\,dy}{\sin^2 y+\sin y-6}$

d) $\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$ e) $\int \frac{1}{x^{3/2}-\sqrt{x}}\,dx$ f) $\int \frac{1}{(x^{1/3}-1)\sqrt{x}}\,dx$

g) $\int \frac{\sqrt{x+1}}{x}\,dx$ h) $\int \frac{1}{x\sqrt{x+9}}\,dx$ i) $\int \frac{1}{x(x^4+1)}\,dx$

j) $\int \frac{1}{x^6(x^5+1)}\,dx$ k) $\int \frac{1}{\cos 2\theta\sin\theta}\,d\theta$ l) $\int \frac{1}{\cos\theta+\sin 2\theta}\,d\theta$

m) $\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$ **n)** $\int \frac{\sqrt{x}}{\sqrt{2-\sqrt{x}}+\sqrt{x}}\,dx$

o) $\int \frac{(x-2)^2 tan^{-1}(2x)-12x^3-3x}{(4x^2+1)(x-2)^2}\,dx$

p) $\int \frac{(x+1)^2 tan^{-1}(3x)+9x^3+x}{(9x^2+1)(x+1)^2}\,dx$

7. Use any method to evaluate the integrals in following Exercises

a) $\int \frac{x^3-2x^2-3x}{x+2}\,dx$ **b)** $\int \frac{x+2}{x^3-2x^2-3x}\,dx$ **c)** $\int \frac{2^x-2^{-x}}{2^x+2^{-x}}\,dx$

d) $\int \frac{2^x}{2^{2x}+2^x-2}\,dx$ **e)** $\int \frac{1}{x^4-1}\,dx$ **f)** $\int \frac{x^4-1}{x^5-5x+1}\,dx$

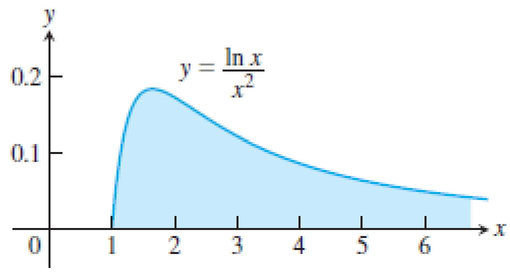
g) $\int \frac{2}{x(\ln x-2)^3}\,dx$ **h)** $\int \frac{1}{\sqrt{x^2-1}}\,dx$ **i)** $\int \frac{x}{x+\sqrt{x^2+2}}\,dx$

j) $\int x^5\sqrt{x^3+1}\,dx$ **k)** $\int x^2\sqrt{1-x^2}\,dx$ **l)** $\int \frac{\ln x+2}{x(\ln x+1)(\ln x+3)}\,dx$

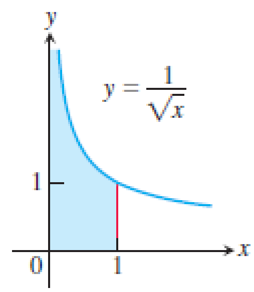
5.6 Improper Integrals:

Up to now, we have required definite integrals to satisfy two properties. First, the domain of integration $[a, b]$ must be finite. Second, the range of the integrand must be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions.

The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite.



The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite.



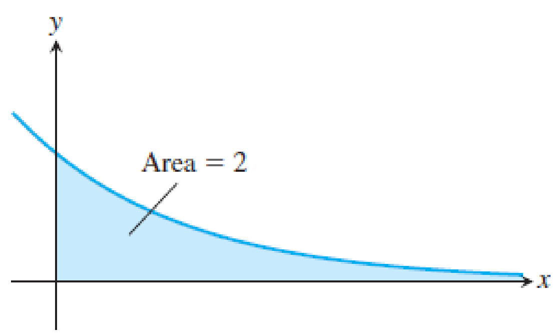
In either case, the integrals are said to be improper and are calculated as limits.

5.6.1 Infinite Limits of Integration:

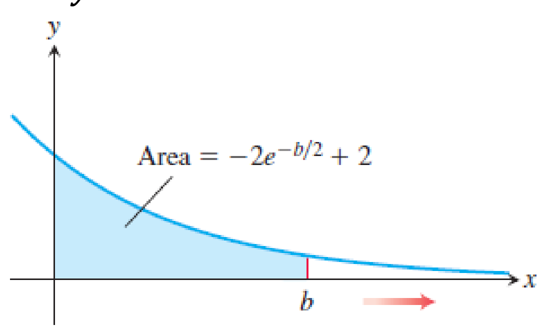
Consider the infinite region (unbounded on the right) that lies under the curve $y = e^{-x/2}$ in the first quadrant. You might think this region has infinite area, but we will see that the value is finite. We assign value to the area in the following way. First find the area A(b) of the portion of the region that is bounded on the right by $x = b$.

$$A(b) = \int_0^b e^{-\frac{x}{2}}\,dx = -2e^{-\frac{x}{2}}\Big]_0^b$$

$$= -2e^{-\frac{b}{2}} + 2.$$



The area in the first quadrant under the curve $y = e^{-x/2}$.



The area is an improper integral of the first type.

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-\frac{b}{2}} + 2) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^\infty e^{-\frac{x}{2}}\,dx = \lim_{b\to\infty} \int_0^b e^{-\frac{x}{2}}\,dx = 2.$$

Definition:

Integrals with infinite limits of integration are ***improper integrals of Type I***.

1. If $f(x)$ is continuous on $[a,\infty)$, then $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$.
2. If $f(x)$ is continuous on $(-\infty, b]$, then $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$.
3. If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

The choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^\infty f(x)\,dx$ with any convenient choice.

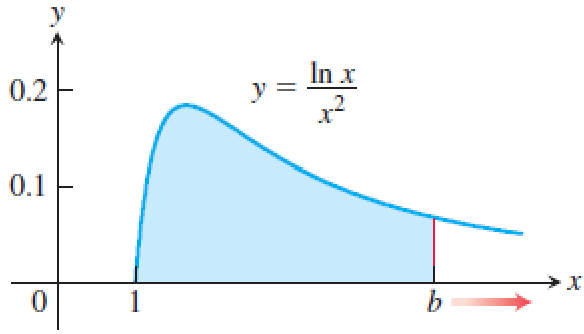
Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in $y = e^{-x/2}$ as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is ***infinite***.

Example:

Is the area under the curve $y = (ln\ x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution:

We find the area under the curve from $x = 1$ to $x = \infty$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve. The area from 1 to b is



$$\int_1^b \frac{\ln x}{x^2}dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

Integration by parts with $\boldsymbol{u = \ln x, dv = dx/x^2,}$ $\boldsymbol{du = dx/x, v = -1/x}$

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b = -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]$$

$$= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1$$

$$= -\left[\lim_{b\to\infty}\frac{\frac{1}{b}}{1}\right] + 1 = 0 + 1 = 1.$$ **l'Hôpital's Rule**

Thus, the improper integral converges, and the area has finite value 1. ■

Example:

Evaluate $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$.

Solution:

According to the definition (Part 3), we can choose $c = 0$ and write

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{\infty}\frac{dx}{1+x^2}\,.$$

Next we evaluate each improper integral on the right side of the equation above.

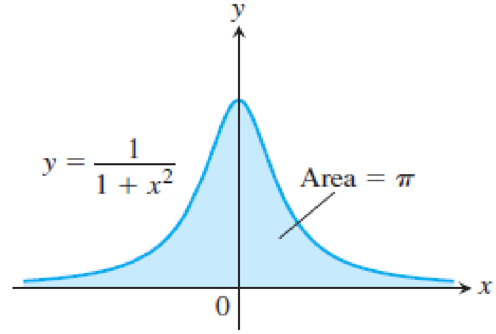
$$\int_{-\infty}^{0}\frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2}dx = \lim_{a\to-\infty}\tan^{-1}x\Big]_a^0$$

$$= \lim_{a\to-\infty}(\tan^{-1}0 - \tan^{-1}a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

$$\int_0^{\infty}\frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\tan^{-1}x\Big]_0^b$$

$$= \lim_{b\to\infty}(\tan^{-1}b - \tan^{-1}0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$



$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1 + x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis.

5.6.2 The Integral $\int_1^{\infty} \frac{dx}{x^p}$:

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

Example:

For what values of p do the integral $\int_1^{\infty} \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

Solution:

If $p \neq 1$,

$$\int_1^{b} \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}(\frac{1}{b^{p-1}} - 1).$$

Thus,

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^p} = \lim_{b\to\infty} [\frac{1}{1-p}(b^{-p+1} - 1)] = \begin{cases} \frac{1}{p-1} & p > 1 \\ \infty & p < 1 \end{cases}$$

Because $\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0 & p > 1 \\ \infty & p < 1 \end{cases}$. Therefore, the integral converges to the value $1/(p - 1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\int_1^{\infty} \frac{dx}{x^p} = \int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to\infty} \ln x]_1^b = \lim_{b\to\infty} (\ln b - \ln 1) = \infty. \blacksquare$$

5.6.3 Integrands with Vertical Asymptotes:

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$. First, we find the area of the portion from a to 1:

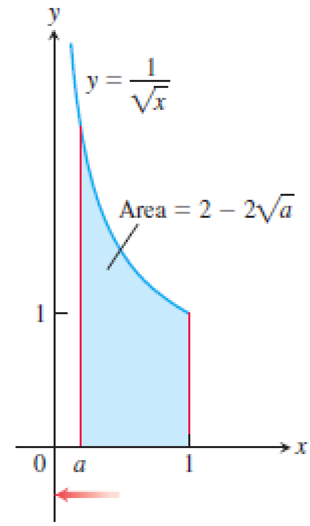
$$\int_{\mathrm{a}}^{1} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}\,.$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0}(2 - 2\sqrt{a}) = 2.$$

Therefore, the area under the curve from 0 to 1 is finite and is defined to be

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = 2.$$



The area under this curve is an example of an improper integral of the second kind.

Definition:

Integrals of functions that become infinite at a point within the interval of integration are ***improper integrals of Type II***.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit exists and is finite, we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist, the integral ***diverges***.

In Part 3 of the definition, the integral on the left side of the equation converges if both integrals on the right side converge; otherwise, it diverges.

Example:

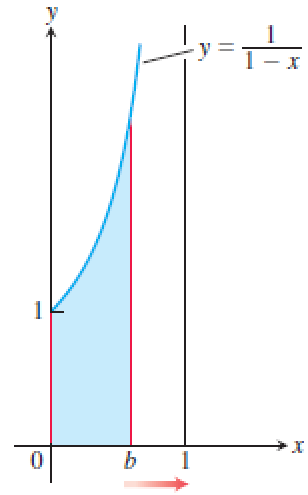
Investigate the convergence of $\int_0^1 \frac{1}{1-x}\,dx$

Solution:

The integrand $f(x) = 1/(1-x)$ is continuous on $[0,1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$. We evaluate the integral as

$$\lim_{b\to 1^-}\int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-}[-ln|1-x|]_0^b$$
$$= \lim_{b\to 1^-}[-ln|1-x|]_0^b$$
$$= \lim_{b\to 1^-}[-\ln(1-b)+0] = \infty.$$

The limit is infinite, so the integral diverges. ■



The area beneath the curve and above the x-axis for [0, 1) is not a real number.

Example:

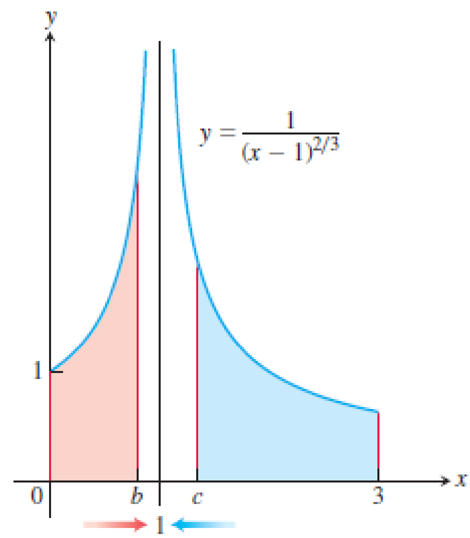
Evaluate $\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

Solution:

The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0,1)$ and $(1,3]$. Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}\,.$$

Next, we evaluate each improper integral on the right-hand side of this equation.



The area under the curve exists (so it is a real number).

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-}\int_0^b \frac{dx}{(x-1)^{2/3}} = \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b = \lim_{b\to 1^-}[3(x-1)^{1/3}+3] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{c\to 1^+}\int_c^3 \frac{dx}{(x-1)^{2/3}} = \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3$$
$$= \lim_{c\to 1^+}[3(3-1)^{1/3} - 3(c-1)^{1/3}] = 3\sqrt[3]{2}$$

We conclude that $\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}$. ■

5.6.4 Tests for Convergence and Divergence:

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

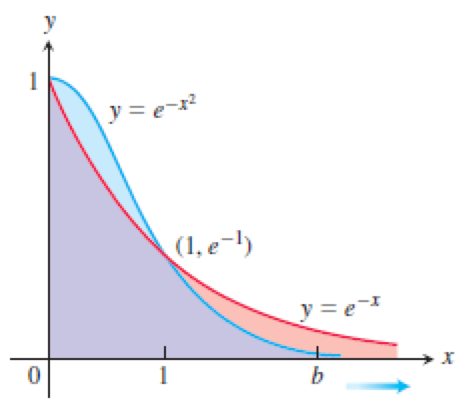
Example:

Does the integral $\int_1^\infty e^{-x^2}dx$ converge?

Solution:

By definition, $\int_1^\infty e^{-x^2}dx = \lim_{b\to\infty}\int_1^b e^{-x^2}dx$.

We cannot evaluate this integral directly because it is nonelementary. But we can show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2}dx$ is an increasing function of b because the area under the curve increases as b increases. Therefore, either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. For our function it does not become infinite: For every value of $x \geq 1$, we have $e^{-x^2} \leq e^{-x}$ so that



The graph of e^{-x^2} lies below the graph of $e^{-x}$for $x > 1$

$$\int_1^b e^{-x^2}dx \leq \int_1^b e^{-x}dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence, $\int_1^\infty e^{-x^2}dx = \lim_{b\to\infty}\int_1^b e^{-x^2}dx$ converges to some finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. ■

Remark:

The comparison of e^{-x^2} and e^{-x} in previous Example is a special case of the following test

Theorem (Direct Comparison Test):

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. **If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ also converges.**
2. **If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ also diverges.**

Remark:

Although the theorem is stated for Type I improper integrals, a similar result is true for integrals of Type II as well.

Example:

These examples illustrate how we use Theorem 2.

a) $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ converges because

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x^2} dx \text{ converges.}$$

b) $\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}} dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \text{ on } [1, \infty) \text{ and } \int_1^\infty \frac{1}{x} dx \text{ diverges.}$$

c) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$ converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \text{ on } [0, \tfrac{\pi}{2}]$$

$0 \leq \cos x \leq 1$ on $[0, \frac{\pi}{2}]$

$$\text{and } \int_1^{\frac{\pi}{2}} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^{\frac{\pi}{2}} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \sqrt{4x}\Big]_a^{\frac{\pi}{2}}$$

$2\sqrt{x} = \sqrt{4x}$

$$= \lim_{a \to 0^+} (\sqrt{2\pi} - \sqrt{4a}) = \sqrt{2\pi} \text{ converges.}$$ ■

Theorem (Limit Comparison Test):

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Remark:

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

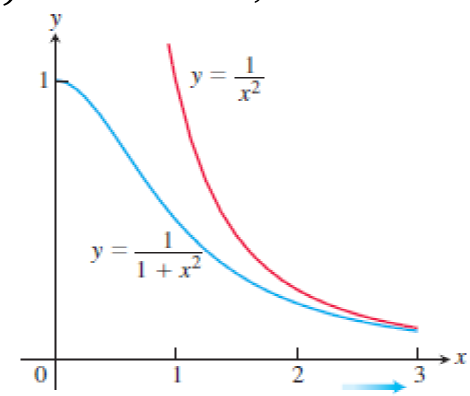
Example:

Show that $\int_1^{\infty} \frac{dx}{1+x^2}$ converges by comparison with $\int_1^{\infty} \frac{dx}{x^2}$. Find and compare the two integral values.

Solution:

The functions $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{1+x^2}$ are positive and continuous on $[1,\infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{1+x^2}} = \lim_{x\to\infty} \frac{1+x^2}{x^2} = \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$



which is a positive finite limit. Therefore, $\int_1^{\infty} \frac{dx}{1+x^2}$ converges because $\int_1^{\infty} \frac{dx}{x^2}$ converges.

The integrals converge to different values, however:

$$\int_1^{\infty} \frac{dx}{x^2} = \frac{1}{2-1} = 1,$$

and $\int_1^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{1+x^2} = \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. ■

Example:

Investigate the convergence of $\int_1^{\infty} \frac{1-e^{-x}}{x} dx$.

Solution:

The integrand suggests a comparison of $f(x) = (1 - e^{-x})/x$ with $g(x) = 1/x$. However, we cannot use the Direct Comparison Test because $f(x) \leq g(x)$ and the integral of $g(x)$ diverges. On the other hand, using the Limit Comparison Test we find that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1-e^{-x}}{x}\right)\left(\frac{x}{1}\right) = \lim_{x\to\infty} (1 - e^{-x}) = 1,$$

which is a positive finite limit. Therefore, $\int_1^{\infty} \frac{1-e^{-x}}{x}\,dx$ diverges because $\int_1^{\infty} \frac{dx}{x}$ diverges. ■

Exercises:

1. The integrals in the following Exercises converge. Evaluate the integrals.

1) $\int_0^{\infty} \frac{dx}{x^2+1}$ 2) $\int_1^{\infty} \frac{dx}{x^{1.001}}$ 3) $\int_0^{1} \frac{dx}{\sqrt{x}}$

4) $\int_0^{4} \frac{dx}{\sqrt{4-x}}$ 5) $\int_{-1}^{1} \frac{dx}{x^{2/3}}$ 6) $\int_{-8}^{1} \frac{dx}{x^{1/3}}$

7) $\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$ 8) $\int_0^{1} \frac{dr}{r^{0.999}}$ 9) $\int_{-\infty}^{-2} \frac{2dx}{x^2-1}$

10) $\int_{-\infty}^{2} \frac{2dx}{x^2+4}$ 11) $\int_2^{\infty} \frac{2dv}{v^2-v}$ 12) $\int_2^{\infty} \frac{2dt}{t^2-1}$

13) $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$ 14) $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$ 15) $\int_0^{1} \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16) $\int_0^{2} \frac{s+1}{\sqrt{4-s^2}}\,ds$ 17) $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$ 18) $\int_1^{\infty} \frac{dx}{x\sqrt{x^2-1}}$

19) $\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$ 20) $\int_0^{\infty} \frac{16\tan^{-1} x}{x^2+1}\,dx$ 21) $\int_{-\infty}^{0} \theta e^{\theta}\,d\theta$

22) $\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta$ 23) $\int_{-\infty}^{0} e^{-|x|}dx$ 24) $\int_{-\infty}^{\infty} 2xe^{-x^2}dx$

25) $\int_0^{1} x\ln x\,dx$ 26) $\int_0^{1}(-\ln x)\,dx$ 27) $\int_0^{2} \frac{ds}{\sqrt{4-s^2}}$

28) $\int_0^{1} \frac{4r\,dr}{\sqrt{1-r^4}}$ 29) $\int_1^{2} \frac{ds}{s\sqrt{s^2-1}}$ 30) $\int_2^{4} \frac{dt}{t\sqrt{t^2-4}}$

31) $\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$ 32) $\int_0^{2} \frac{dx}{\sqrt{|x-1|}}$ 33) $\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6}$

34) $\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

2. In the following Exercises, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

1) $\int_{1/2}^{2} \frac{dx}{x\ln x}$ 2) $\int_{-1}^{1} \frac{d\theta}{\theta^2-2\theta}$ 3) $\int_{1/2}^{\infty} \frac{dx}{x(\ln x)^3}$

4) $\int_0^{\infty} \frac{d\theta}{\theta^2-1}$ 5) $\int_0^{\pi/2} \tan\theta\ d\theta$ 6) $\int_0^{\pi/2} \cot\theta\ d\theta$

7) $\int_0^{1} \frac{\ln x}{x^2}\,dx$ 8) $\int_1^{2} \frac{dx}{x\ln x}$ 9) $\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

10) $\int_0^{1} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$ 11) $\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$ 12) $\int_0^{1} \frac{dt}{t-\sin t}$

13) $\int_0^2 \frac{dx}{1-x^2}$

14) $\int_0^2 \frac{dx}{1-x}$

15) $\int_{-1}^1 \ln|x|\ dx$

16) $\int_{-1}^1 -x\ln|x|\ dx$

17) $\int_1^\infty \frac{dx}{x^3+1}$

18) $\int_4^\infty \frac{dx}{\sqrt{x}-1}$

19) $\int_2^\infty \frac{dv}{\sqrt{v-1}}$

20) $\int_0^\infty \frac{d\theta}{1+e^\theta}$

21) $\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$

22) $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

23) $\int_1^\infty \frac{\sqrt{x+1}}{x^2} dx$

24) $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$

25) $\int_\pi^\infty \frac{2+\cos x}{x} dx$

26) $\int_\pi^\infty \frac{1+\sin x}{x^2} dx$

27) $\int_4^\infty \frac{2dt}{t^{3/2}-1}$

28) $\int_2^\infty \frac{1}{\ln x} dx$

29) $\int_1^\infty \frac{e^x}{x} dx$

30) $\int_{e^e}^\infty \ln(\ln x)\, dx$

31) $\int_1^\infty \frac{1}{\sqrt{e^x-x}} dx$

32) $\int_1^\infty \frac{1}{e^x-2^x} dx$

33) $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

34) $\int_{-\infty}^\infty \frac{1}{e^x-e^{-x}} dx$