التحليل الدالي

المحاضرة السادسة

قسم الرياضيات

الصف الرابع

Theorem 2.7.: *Let X be a Banach space, M subspace of X, M is a Banach space then M is closed in X.*

Proof:

Suppose M is Banach space $\Rightarrow$ *M is complete*

T.P. M is closed (i.e. $\overline{M}=M$*)*

Let $x \in \overline{M}$

$\Rightarrow \exists\ (x_n)$ *sequence in M s.t.* $x_n \to x$

$\Rightarrow\ (x_n)$ *is a Cauchy seq. in M*

Since M is complete

$\Rightarrow \exists\, y \in M$ *s.t.* $x_n \to y$

But the limit point is unique

$\Rightarrow x=y \Rightarrow x \in M \Rightarrow$ *M is closed*

Corollary 2.8.: *Let X be a normed space, if M is finite dimension subspace in X then M is closed.*

Proof:

M is a normed space (Every subspace of normed space is normed space)

Since M is finite dimension $\Rightarrow$ *M is complete (From theorem 2.4.)*

By using theorem 2.5. $\Rightarrow$ *M is closed.*

Theorem 2.9.: *Let X be a normed space and* ***M*** *closed subspace of X, if X is Banach space then X/M is Banach space.*

Linear Transformations

Definition 2.10.: *Let X and Y are vector spaces on F. The function* $T : X \to Y$ *is called* ***linear transformation*** *if satisfy the following conditions:*

1) $T(x+y) = T(x) + T(y)\ ,\quad \forall x,y \in X$
2) $T(\lambda x)=\lambda T(x)\ ,\quad \forall x,y \in X.$

i.e. $T(\alpha x+\beta y)= \alpha T(x)+\beta T(y)\ ,\ \forall x,y \in X\ ,\ \alpha,\beta$ *scalars.*

The linear transformation $f: X \to F$ *is called* ***linear functional*** *on X.*

Remarks 2.11:

1- *D(T) =* ***Domain*** *T*

2- $R(T)=\{T(x) : x \in X\} \subset Y$ = **Range** T, $R(T)$ is a vector space.

3- $N(T)=\{x \in D(T): T(x) = 0\}$= **Null space**, $N(T)$ is a vector space.

4- If $Y=X$, then $T: X \rightarrow X$ is called **linear operator**.

Examples 2.12:

1- ***Zero Transformation***

$O: X \rightarrow Y,\ O(x)=0,\ \forall x \in X$

Let $x,y \in X$, α, β scalars

$O(\alpha x + \beta y) = 0 = 0 + 0 = \alpha O(x) + \beta O(y)$

2- ***Identity Transformation***

$I : X \rightarrow X,\ I(x)=x,\ \forall x \in X$

Let $x,y \in X$, α, β scalars

$I(\alpha x + \beta y) = \alpha x + \beta y = \alpha I(x) + \beta I(y)$

3- ***Differential Transformation***

Let X is a space of all polynomials on $[a, b]$

$P_n(x)=a_0 + a_1 x + \ldots + a_n x^n,\ \forall x \in [a, b],\ \forall n$

$D: X \rightarrow X,\ D(P_n(x)) = P_n'(x),\ \forall P_n(x) \in X$

Let $P_n(x)$, $B_n(x) \in X$, α, β are scalars

$D(\alpha P_n(x) + \beta B_n(x)) = (\alpha P_n(x) + \beta B_n(x))'$

$= \alpha P'_n(x) + \beta B'_n(x)) = \alpha D(P_n(x)) + \beta D(B_n(x))$

4- ***Integrable Transformation (H.W.)***

Let $X=C[a, b]$, $T: C[a, b] \rightarrow C[a, b]$

$T(f(x)) = \int_0^x f(t)dt, \forall f \in C[a,b]$

T is linear transformation

5- ***Bilateral shift Transformation***

Let $X=l_2=\{ x= (x_i)_{i=1}^{\infty} : x_i \in R \text{ or } C \text{ s.t. } \sum_{i=1}^{\infty} |x_i|^2 < \infty \}$

$B= l_2 \rightarrow l_2,\ B(z_1, z_2, \ldots, z_n, \ldots) = (z_2, \ldots, z_n, \ldots),\ \forall z= (z_1, z_2, \ldots, z_n, \ldots) \in l_2$

Let $z= (z_1, z_2, \ldots, z_n, \ldots)$, $w= (w_1, w_2, \ldots, w_n, \ldots) \in l_2$, α, β scalars

$B(\alpha z + \beta w) = B(\alpha z_1+\beta w_1, \alpha z_2+\beta w_2, \dots, \alpha z_n+\beta w_n, \dots) = (\alpha z_2+\beta w_2, \dots, \alpha z_n+\beta w_n, \dots) = \alpha(z_2, \dots, z_n, \dots)+\beta(w_2, \dots, w_n, \dots) = \alpha B(z) + \beta B(w)$

6- Unilateral Shift Transformation (H.W.)

$U: l_2 \to l_2$, $U(z_1, z_2, \dots, z_n, \dots) = (0, z_1, z_2, \dots, z_n, \dots)$, $\forall z = (z_1, z_2, \dots, z_n, \dots) \in l_2$

Definition 2.13: *Let $T: X \to Y$ be a linear transformation, T is said to be* ***bounded linear transformation*** *if there exists a real number $M > 0$ s.t. $\| Tx \|_Y \le M \| x \|_X$, $\forall x \in X$.*

Definition 2.14: *Let $T: X \to Y$ be a bounded linear transformation:*

$||| T ||| = l.u.b.\{ \frac{\| T(x) \|_y}{\| x \|_x} : x \ne 0, x \in X\}$ *is the norm of the bounded linear transformation.*

Remarks 2.15:

1- $||| T ||| \ge \frac{\| T(x) \|_Y}{\| x \|_X}$, $\forall x \ne 0 \in X$, $\qquad \| T_x \|_Y \le ||| T ||| \, \| x \|_X$

2- *If* $T = 0 \Rightarrow ||| T ||| = 0$

3- *If* $\| x \|_X = 1 \Rightarrow ||| T ||| = l.u.b.\{ \| T(x) \|_Y : x \in X\}$

Examples 2.16:

1- $O: X \to Y$, $O(x) = 0$, $\forall x \in X$, *O is bounded linear transformation,* $\| 0 \| = 0$.

2- $I: X \to X$, $I(x) = x$, $\forall x \in X$, *I is bounded linear transformation,* $\| I \| = 1$.

3- *Let X be a normed space of all polynomials on* $[0, 1]$

$D: X \to X$, $D(P_n(x)) = P'_n(x)$, $\forall P_n(x) \in X$

D unbounded linear transformation

Proof:

Let $P_n(x) = x^n$, $x \in [0, 1]$, $\forall n$

$\| P_n \| = 1$

$D(P_n(x)) = D(x^n) = n x^{n-1}$

$\| D(P_n(x)) \| = \| n x^{n-1} \| = n \| x^{n-1} \| \ge n \| x^n \| \ge n \| P_n(x) \| = n$

$\Rightarrow$ *D unbounded linear transformation*

(because there is not exist $M > 0$ *s.t.* $\| D(P_n(x)) \| \le M \| P_n(x) \|$ *)*