التحليل الدالي

المحاضرة الثانية

قسم الرياضيات

الصف الرابع

Definition1.10 : *(Metric Space) Suppose X is a set. A map d:* XxX→ *IR is called a metric on X if the following properties hold:*

(i) $d(x, y) \geq 0$ *for all* $x, y \in X$;

(ii) $d(x, y) = 0$ *if and only if* $x = y$;

(iii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$ *(triangle inequality).*

We call (X, d) a metric space. If it is clear what metric is being used, we simply say X is a metric space.

Examples *(H.W. 2-6)*

1) ***Real line IR***: *this is the set of all real numbers, taken with the usual metric defined by:*

$$d(x, y) = | x-y | \qquad \forall x,y \in IR$$

2) ***Euclidean plane*** IR^2: *The metric space* IR^2, *with Euclidean metric:*

   if $x=(x_1, x_2)$, $y=(y_1,y_2)$, *then:*

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

3) ***Euclidean Space*** IR^n: *If* $x=(x_1, x_2, \ldots,x_n)$, $y=(y_1,y_2, \ldots,y_n)$, *then:*

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^n}$$

4) ***Function space C[a, b]:*** *As a set X we take the set of all real-valued functions x, y, ... which are functions of an independent real variable t and are defined and continuous on a given closed interval J = [a, b]. Choosing the metric defined by*

$$d(x, y) = \max_{t \in J} | x(t) - y(t) |$$

5) ***Discrete metric space***: *We take any set X and on it the so-called discrete metric for X, defined by:*

$$d(x, x) = 0, \qquad d(x,y)=1 \qquad (x \neq y).$$

   This space (X, d) is called a discrete metric space.

6) ***Space B(A) of bounded functions***: *By definition, each element* $x \in B(A)$ *is a function defined and bounded on a given set A, and the metric is defined by:*

$$d(x, y) = \sup_{t \in A} | x(t) - y(t) |$$

Sol.(1)

[1] 1- d is real, finite & $d=| x-y | \geq 0$

2) $d(x,y)=0 \leftrightarrow | x-y | =0 \leftrightarrow x-y=0 \leftrightarrow x=y \qquad \forall x, y \in IR$

3) $d(x,y)= | x-y | = |-(y-x)| = | y-x | = d(y, x) \qquad \forall x, y \in IR$

4) $d(x,y)= | x-y |= | x-z + z - y| \leq | x-z| + | z-y| = d(x, z)+d(z, y) \qquad \forall x, y, z \in IR$

Then (IR, d) is a metric space.

2. Normed Spaces

The first to introduce the concept for the standard was the Austrian scientist E. Helly (1844 - 1943), but he did not use the name of the standard nor its symbol, it was known as any function that fulfills certain conditions (the same conditions of the standard)

* *A **norm** on a vector space is a way of measuring distance between vectors.*

Definition 1.11.: A **norm** on a linear space V over F is a function $|| \,.\, || : V \to R$ with the properties that :

(1) $|| x || \geq 0$ for all x

(2) $|| x || = 0 \leftrightarrow x = 0$ (positive definite)

(3) $|| x + y|| \leq || x || + || y ||$ for all $x, y \in V$. (triangle inequality)

(4) $||\alpha x|| = |\alpha| \, ||x||$ for all $x \in V$ and $\alpha \in F$.

In Definition 1.11(3) we are assuming that F is R or C and $| \,.\, |$ denotes the usual absolute value. If $|| \,.\, ||$ is a function with properties (2) and (3) only it is called a **semi-norm**.

Definition 1.12. A **normed linear space** is a linear space V with a norm $|| \,.\, ||$ (sometimes we write $|| \,.\, ||_V$).

Theorem 1.13. If V is a normed space, then:

1) $|| 0 ||=0$

2) $|| x || = || -x ||$ for every $x \in V$.

3) $|| x-y || = || y-x ||$ for every $x \in V$.

4) $| \, || x || - || y || \, | \leq || x-y ||$ for every $x \in V$.

Proof :

Properties (1), (2) and (3) conclude directly from the definition, to prove property (4):

$x = (x-y)+y$

$\| x \| = \| (x-y)+y \| \leq \| x-y \| + \| y \| \rightarrow \| x \| - \| y \| \leq \| x-y \| \quad ....(1)$

Similarly:

$\| y \| - \| x \| \leq \| x-y \|$

$-(\| x \| - \| y \|) \leq \| x-y \| \rightarrow (\| x \| - \| y \|) \geq - \| x-y \| \quad ....(2)$

From (1) & (2), we get:

$-\| x-y \| \leq \| x \| - \| y \| \leq \| x-y \| \rightarrow | \| x \| - \| y \| | \leq \| x-y \|$

Examples 1.14.:- **[H.W 3,5,6,7]**

[1] The vector space V is normed v.s. with the norm $\| x \| = | x |$ *for all* $x \in V$.

Proof:

1) Since $| x | \geq 0 \rightarrow \| x \| \geq 0$.
2) $\| x \| = 0 \leftrightarrow | x | = 0 \leftrightarrow x=0$
3) Let $x \in V$, $\alpha \in F$, then

   $\| \alpha x \| = | \alpha x | = |\alpha| \, | x | = | \alpha | \, \| x \|$
4) Let $x,y \in V$, then:

   $\| x+y \| = | x+y | \leq | x | + | y | = \| x \| + \| y \|$

[2] Let $V = R^n$ *with the usual Euclidean norm*

$$\| x \| = \| x \|_2 = \left(\sum_{j=1}^{n} | x_j |^2\right)^{1/2}$$

proof:

1) Since $x_j^2 \geq 0$ for all $j = 1,2,\ldots,n \rightarrow \| x \| \geq 0$
2) $\| x \| = 0 \leftrightarrow \left(\sum_{j=1}^{n} | x_j |^2\right)^{1/2} = 0 \leftrightarrow \sum_{j=1}^{n} | x_j |^2 = 0$

   $\leftrightarrow x_j^2 = 0$ for all $j = 1,2,\ldots,n \leftrightarrow x_j = 0$ for all $j = 1,2,\ldots,n \leftrightarrow x=0$
3) Let $x \in IR^n$, $\alpha \in IR$:

   $\alpha x = \alpha (x_1, \ldots , x_n) = (\alpha x_1, \ldots , \alpha x_n)$

$$|| \alpha x || = (\sum_{j=1}^{n} | \alpha x_j |^2)^{1/2} = | \alpha | (\sum_{j=1}^{n} | x_j |^2)^{1/2} = | \alpha | \, || x ||.$$

4) *Let* $x,y \in IR^n$:

$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

$$|| x+y || = (\sum_{j=1}^{n} | x_j + y_j |^2)^{1/2}$$

By using MinKowski's inquality where p=2, we have:

$$\| x + y \| = (\sum_{i=1} | x_i + y_i |^2)^{1/2} \leq (\sum_{i=1} | x_i |^2)^{1/2} + (\sum_{i=1} | y_i |^2)^{1/2} = \| x \| + \| y \|$$

[3] *There are many other norms on* R^n, *called the p-norms. For* $1 \leq p < \infty$ *defined by:*

$$|| x ||_p = (\sum_{j=1}^{n} | x_j |^p)^{1/p}$$

Then $|| \cdot ||_p$ *is a norm on V (to check the triangle inequality use MinKowski's Inequality)*

$$(\sum_{j=1}^{n} | x_j + y_j |^p)^{1/p} \leq (\sum_{j=1}^{n} | x_j |^p)^{1/p} + (\sum_{j=1}^{n} | y_j |^p)^{1/p}$$

[4] *There is another norm corresponding to* $p = \infty$, *defined by:*

$$|| x ||_\infty = \max_{1 \leq j \leq n} \{ | x_j | \}$$

where $|| \cdot ||: IR^n \to IR$ *and* $x = (x_1, \ldots, x_n)$.

proof:

1) *Since* $| x_i | \geq 0$ *for all* $i=1, \ldots, n \to || x || \geq 0$.
2) $|| x || = 0 \leftrightarrow \max \{ | x_1 |, \ldots, | x_n | \} = 0 \leftrightarrow | x_i | = 0$ *for all* $i=1, \ldots, n$

   $\leftrightarrow x_i = 0$ *for all* $i=1, \ldots, n \quad \leftrightarrow x=0$
3) *Let* $x \in IR^n$ *and* $\alpha \in IR$, *then*

   $\alpha x = \alpha (x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$

   $|| \alpha x || = \max \{ | \alpha x_1 |, \ldots, | \alpha x_n | \}$

   $= \max \{ | \alpha | | x_1 |, \ldots, | \alpha | | x_n | \}$

   $= | \alpha | \max \{ | x_1 |, \ldots, | x_n | \}$

   $= | \alpha | \, || x ||$

4) Let $x, y \in IR^n$

$x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (, \ldots,)$

$|| x + y|| = \max \{ | x_1+y_1|, \ldots , | x_n+y_n | \}$

$\leq \max \{ | x_1| + | y_1|, \ldots , | x_n |+| y_n | \}$

$\leq \max \{ | x_1|, \ldots , | x_n | \}+ \max \{ | y_1|, \ldots , | y_n | \}$

$= || x || + || y ||$

[5] *Let* $X = C[a; b]$, *and put* $|| f || = \sup_{t\in[a,b]} | f(t) |$. *This is called the uniform or supremum norm.*

[6] *Let* $X = C[a; b]$, *and choose* $1 \leq p < \infty$. *Then (using the integral form of Minkowski's inequality) we have the p-norm*

$$|| f ||_p = \left(\int_a^b | f |^p \right)^{1/p}$$

[7] *Let V be the set of Riemann-integrable functions* $f : (0; 1) \to R$ *which are square-integrable. Let* $|| f ||_2 = \left(\int_0^1 | f(x) |^2\, dx\right)^{1/2} < \infty$. *Then V is a normed linear space.*

Definition 1.15. *A set C in a linear space is convex if for any two points* $x, y \in C$,

$tx + (1 - t)y \in C$ *for all* $t \in [0; 1]$.

Definition 1.16. *A norm* $|| \,.\, ||$ *is strictly convex if* $|| x || = 1$, $|| y || = 1$, $|| x+y || = 2$ *together imply that* $x = y$.

Definition 1.17. *: Let X and Y be two sets . The Cartesian product of X and Y is the set of all ordered pairs (x, y) where* $x \in X$, $y \in Y$ *and is denoted by* $X \times Y$, *i.e.*

$$X\times Y = \{(x,y) : x \in X, y \in Y\}$$

Theorem 1.18.

If $(X, || \,.\, ||_X)$ *and* $(Y, || \,.\, ||_Y)$ *are normed spaces, then* $(X \times Y, || \,.\, ||)$ *is normed space where* $||(x, y)|| = \max\{|| x ||_X, || y ||_Y \}$.

proof:

1) $|| (x,y) || = 0 \leftrightarrow \max\{|| x ||_X, || y ||_Y\}=0 \leftrightarrow || x ||_X=0, || y ||_Y=0 \leftrightarrow x=0, y=0 \leftrightarrow (x, y)=0$

2) let $(x_1,y_1), (x_2,y_2) \in XxY$, then

$(x_1,y_1) + (x_2,y_2)=(x_1+ x_2, y_1+ y_2)$

$\|(x_1+x_2, y_1+y_2)\| = \max\{\|x_1+x_2\|_X, \|y_1+y_2\|_Y\} \le \max\{\|x_1\|_X+\|x_2\|_X, \|y_1\|_Y+\|y_2\|_Y\}$

$\le \max\{\|x_1\|_X, \|y_1\|_Y\}+\max\{\|x_2\|_X, \|y_2\|_Y\} = \|(x_1, y_1)\|+\|(x_2, y_2)\|$

3) let $(x, y) \in X\text{x}Y$ and $\alpha \in F$, then: $\|\alpha(x, y)\| = \max\{\|\alpha x\|_X, \|\alpha y\|_Y\} = \max\{|\alpha|\,\|x\|_X, |\alpha|\,\|y\|_Y\} = |\alpha| \max\{\|x\|_X, \|y\|_Y\} = |\alpha|\,\|(x, y)\|$

H.W. If $\|(x, y)\| = (\|x\|_X + \|y\|_Y)^{1/p}$, prove that $(X \times Y, \|\,.\,\|)$ is normed space.

Theorem 1.19. *Every normed linear space is metric space.*

proof:

let $(X, \|.\|)$ is a normed space. We define the function $d: X\text{x}X \to IR$ as:

$d(x,y)=\|x-y\|$ for all $x,y \in X$, since this function satisfies all the conditions of metric :

1) let $x,y \in X \to x-y \in X$ (since X is vector space) $\to \|x-y\| \ge 0 \to d(x, y) \ge 0$.

2) $d(x, y) = 0 \leftrightarrow \|x-y\| = 0 \leftrightarrow x-y = 0 \leftrightarrow x=y$

3) $d(x, y) = \|x-y\| = \|y-x\| = d(y, x)$

4) let $x,y,z \in X$:

$\|x-y\| = \|(x-z)+(z-y)\| \le \|x-z\| + \|z-y\| \to d(x, y) \le d(x, z)+ d(z, y)$

Remark : The converse may be not true, for example:

If X be a v.s., define $d: X \text{ x } X \to IR$ as:

$$d(x,y)=\begin{Bmatrix} 0 & x=y \\ 2 & x\neq y \end{Bmatrix}$$

And define $\|\,.\,\|: X \to IR$ as $\|x\| = d(x, 0)$

$(X, \|\,.\,\|)$ fails to be normed space.

Since if $x \neq 0 \to \|x\| = d(x, 0) = 2$

$\|2x\| = d(2x, 0) \to |2|\,\|x\| = 2 \to 2.2=2 \to 4=2$ C!