جامعة بغداد
كلية العلوم للبنات
قسم الرياضيات

نظرية التقريب

Lemma 6.3. (a) *D_n is even,*

(b) $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} D_n(t)\,dt = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} D_n(t)\,dt = 1,$

(c) $|D_n(t)| \le n + \frac{1}{2}$ *and* $D_n(0) = n + \frac{1}{2},$

(d) $\dfrac{|\sin(n+\frac{1}{2})t|}{t} \le |D_n(t)| \le \dfrac{\pi}{2t}$ *for* $0 < t < \pi,$

(e) *If* $\lambda_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |D_n(t)|\,dt,$ *then* $\dfrac{4}{\pi^2}\log n \le \lambda_n \le 3 + \log n.$

Proof. (a), (b), and (c) are relatively clear from the fact that

$$D_n(t) = \tfrac{1}{2} + \cos t + \cos 2t + \cdots + \cos nt.$$

(Notice, too, that (b) follows from the fact that $s_n(1) = 1$.) For (d) we use a more delicate estimate: Because $2\theta/\pi \le \sin\theta \le \theta$ for $0 < \theta < \pi/2$, it follows that $2t/\pi \le 2\sin(t/2) \le t$ for $0 < t < \pi$. Hence,

$$\frac{\pi}{2t} \ge \frac{|\sin(n+\frac{1}{2})t|}{2\sin\frac{1}{2}t} \ge \frac{|\sin(n+\frac{1}{2})t|}{t}$$

for $0 < t < \pi$. Next, the upper estimate in (e) is easy:

$$\begin{aligned} \frac{2}{\pi}\int_0^\pi |D_n(t)|\,dt &= \frac{2}{\pi}\int_0^\pi \frac{|\sin(n+\frac{1}{2})t|}{2\sin\frac{1}{2}t}\,dt \\ &\le \frac{2}{\pi}\int_0^{1/n}(n+\tfrac{1}{2})\,dt + \frac{2}{\pi}\int_{1/n}^\pi \frac{\pi}{2t}\,dt \\ &= \frac{2n+1}{\pi n} + \log\pi + \log n \\ &< 3 + \log n. \end{aligned}$$

The lower estimate takes some work:

$$
\begin{aligned}
\frac{2}{\pi}\int_0^{\pi} |D_n(t)|\,dt &= \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})\,t\,|}{2\sin\frac{1}{2}t}\,dt \\
&\geq \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})\,t\,|}{t}\,dt \\
&= \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi} \frac{|\sin x\,|}{x}\,dx \\
&\geq \frac{2}{\pi}\int_0^{n\pi} \frac{|\sin x\,|}{x}\,dx \\
&= \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin x\,|}{x}\,dx \\
&\geq \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin x\,|\,dx \\
&= \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \;\geq\; \frac{4}{\pi^2}\log n,
\end{aligned}
$$

because $\sum_{k=1}^{n}\frac{1}{k} \geq \log n$. □

Corollary 6.4. *If $f \in C^{2\pi}$, then*

$$|s_n(f)(x)| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t)|\,|D_n(t)|\,dt \le \lambda_n \|f\|. \tag{6.1}$$

In particular, $\|s_n(f)\| \le \lambda_n \|f\| \le (3 + \log n)\|f\|$.

Theorem 6.5. (Kharshiladze, Lozinski) *For each n, let L_n be a continuous, linear projection from $C^{2\pi}$ onto $\mathcal{T}_n$. Then, there is some $f \in C^{2\pi}$ for which $\|L_n(f) - f\|$ is unbounded.*

Theorem 6.6. *If $f \in C^{2\pi}$ and if we set $E_n^T(f) = \min_{T \in \mathcal{T}_n} \|f - T\|$, then*

$$E_n^T(f) \;\le\; \|f - s_n(f)\| \;\le\; (4 + \log n)\, E_n^T(f).$$

Proof. Let T^* be the best uniform approximation to f out of $\mathcal{T}_n$. Then, because $s_n(T^*) = T^*$, we get

$$\|f - s_n(f)\| \le \|f - T^*\| + \|s_n(T^* - f)\| \le (4 + \log n)\,\|f - T^*\|. \qquad \square$$

Because the sequence of partial sums (s_n) need not converge to f, we might try looking at their arithmetic means (or Cesàro sums):

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_{n-1}}{n}.$$

(These averages typically have better convergence properties than the partial sums themselves. Consider σ_n in the (scalar) case $s_n = (-1)^n$, for example.) Specifically, we set

$$\begin{aligned} \sigma_n(f)(x) &= \frac{1}{n}\Big[s_0(f)(x) + \cdots + s_{n-1}(f)(x)\Big] \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left[\frac{1}{n}\sum_{k=0}^{n-1} D_k(t)\right] dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, K_n(t)\, dt, \end{aligned}$$

where $K_n = (D_0 + D_1 + \cdots + D_{n-1})/n$ is called *Fejér's kernel*. The same techniques we used earlier can be applied to find a closed form for $\sigma_n(f)$ which, of course, reduces to simplifying $(D_0 + D_1 + \cdots + D_{n-1})/n$. As before, we begin with a trig identity:

$$\begin{aligned} 2\sin\theta \sum_{k=0}^{n-1} \sin\,(2k+1)\theta &= \sum_{k=0}^{n-1} \big[\cos 2k\theta - \cos\,(2k+2)\theta\big] \\ &= 1 - \cos 2n\theta = 2\sin^2 n\theta. \end{aligned}$$

Thus,

$$K_n(t) = \frac{1}{n}\sum_{k=0}^{n-1} \frac{\sin\,(2k+1)\,t/2}{2\sin\,(t/2)} = \frac{\sin^2(nt/2)}{2n\sin^2(t/2)}.$$

Please note that K_n is even, *nonnegative*, and $\frac{1}{\pi}\int_{-\pi}^{\pi} K_n(t)\,dt = 1$. Thus, $\sigma_n(f)$ is a positive, linear map from $C^{2\pi}$ onto $\mathcal{T}_n$ (but it's not a projection—why?), satisfying $\|\sigma_n(f)\|_2 \le \|f\|_2$

Now the arithmetic mean operator $\sigma_n(f)$ is still a good approximation f in L_2 norm. Indeed,

$$\|f - \sigma_n(f)\|_2 = \frac{1}{n}\left\| \sum_{k=0}^{n-1} (f - s_k(f)) \right\|_2 \le \frac{1}{n} \sum_{k=0}^{n-1} \|f - s_k(f)\|_2 \to 0$$

as $n \to \infty$ (because $\|f - s_k(f)\|_2 \to 0$). But, more to the point, $\sigma_n(f)$ is actually a good *uniform* approximation to f, a fact that we'll call *Fejér's theorem*:

Theorem 6.8. *If $f \in C^{2\pi}$, then $\sigma_n(f)$ converges uniformly to f as $n \to \infty$.*

Note that, because $\sigma_n(f) \in \mathcal{T}_n$, Fejér's theorem implies Weierstrass's second theorem. Curiously, Fejér was only 19 years old when he proved this result (about 1900) while Weierstrass was 75 at the time he proved his approximation theorems.

Theorem 6.9. *Suppose that $k_n \in C^{2\pi}$ satisfies*

(a) $k_n \ge 0$,

(b) $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} k_n(t)\,dt = 1$, *and*

(c) $\displaystyle \int_{\delta \le |t| \le \pi} k_n(t)\,dt \to 0$ *for every $\delta > 0$.*

Then, $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,k_n(t)\,dt \rightrightarrows f(x)$ *for each $f \in C^{2\pi}$.*

Proof. Let $\varepsilon > 0$. Because f is uniformly continuous, we may choose $\delta > 0$ so that $|f(x) - f(x+t)| < \varepsilon$, for any x, whenever $|t| < \delta$. Next, we use the fact that k_n is nonnegative and integrates to 1 to write

$$\begin{aligned}\left| f(x) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,k_n(t)\,dt \right| &= \frac{1}{\pi}\left|\int_{-\pi}^{\pi} [f(x) - f(x+t)]\,k_n(t)\,dt\right| \\ &\le \frac{1}{\pi}\int_{-\pi}^{\pi} \big|f(x) - f(x+t)\big|\,k_n(t)\,dt \\ &\le \frac{\varepsilon}{\pi}\int_{|t|<\delta} k_n(t)\,dt \;+\; \frac{2\|f\|}{\pi}\int_{\delta\le|t|\le\pi} k_n(t)\,dt \\ &< \varepsilon + \varepsilon = 2\varepsilon,\end{aligned}$$

for n sufficiently large. □

To see that Fejér's kernel satisfies the conditions of the Theorem is easy: In particular, (c) follows from the fact that $K_n(t) \rightrightarrows 0$ on the set $\delta \le |t| \le \pi$. Indeed, because $\sin(t/2)$ increases on $\delta \le t \le \pi$ we have

$$K_n(t) \;=\; \frac{\sin^2(nt/2)}{2n\sin^2(t/2)} \;\le\; \frac{1}{2n\sin^2(\delta/2)} \;\to 0.$$

Our second proof, or sketch, really, is based on a variant of the Bohman-Korovkin theorem for $C^{2\pi}$ due to Korovkin. In this setting, the three "test cases" are

$$f_0(x) = 1, \qquad f_1(x) = \cos x, \qquad \text{and} \qquad f_2(x) = \sin x.$$

Theorem 6.10. *Let (L_n) be a sequence of positive, linear maps on $C^{2\pi}$. If $L_n(f) \rightrightarrows f$ for each of the three functions $f_0(x) = 1$, $f_1(x) = \cos x$, and $f_2(x) = \sin x$, then $L_n(f) \rightrightarrows f$ for every $f \in C^{2\pi}$.*