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A Brief Introduction to Fourier Series

The *Fourier series* of a 2π-periodic (bounded, integrable) function f is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where the coefficients are defined by

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt\, dt \qquad \text{and} \qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt\, dt.$$

Please note that if f is Riemann integrable on $[-\pi, \pi]$, then each of these integrals is well-defined and finite; indeed,

$$|a_k| \le \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)|\, dt$$

and so, for example, we would have $|a_k| \le 2\|f\|$ for $f \in C^{2\pi}$.

We write the partial sums of the series as

$$s_n(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Remarks

1. The collection of functions $1, \cos x, \cos 2x, \ldots$, and $\sin x, \sin 2x, \ldots$, are *orthogonal* on $[-\pi, \pi]$. That is,

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \, dx = \int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \int_{-\pi}^{\pi} \cos mx \, \sin nx \, dx = 0$$

for any $m \neq n$ (and the last equation even holds for $m = n$),

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi$$

for any $m \neq 0$, and, of course, $\int_{-\pi}^{\pi} 1 \, dx = 2\pi$.

2. What this means is that if $T(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx)$, then

$$\frac{1}{\pi} \int_{-\pi}^{\pi} T(x) \cos mx \, dx = \frac{\alpha_m}{\pi} \int_{-\pi}^{\pi} \cos^2 mx \, dx = \alpha_m$$

for $m \neq 0$, while

$$\frac{1}{\pi} \int_{-\pi}^{\pi} T(x) \, dx = \frac{\alpha_0}{2\pi} \int_{-\pi}^{\pi} dx = \alpha_0.$$

That is, if $T \in \mathcal{T}_n$, then T is actually equal to its own Fourier series.

3. The partial sum operator $s_n(f)$ is a *linear projection* from $C^{2\pi}$ onto $\mathcal{T}_n$.

4. If $T(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx)$ is a trig polynomial, then

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) T(x) \, dx &= \frac{\alpha_0}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx + \sum_{k=1}^{n} \frac{\alpha_k}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx \\ &\quad + \sum_{k=1}^{n} \frac{\beta_k}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx \\ &= \frac{\alpha_0 a_0}{2} + \sum_{k=1}^{n} (\alpha_k a_k + \beta_k b_k), \end{aligned}$$

where (a_k) and (b_k) are the Fourier coefficients for f. [This should remind you of the *dot product* of the coefficients.]

5. Motivated by Remarks 1, 2, and 4, we define the *inner product* of two elements f, $g \in C^{2\pi}$ by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx.$$

(Be forewarned: Some authors prefer the normalizing factor $1/2\pi$ in place of $1/\pi$ here.) Note that from Remark 4 we have $\langle f, s_n(f) \rangle = \langle s_n(f), s_n(f) \rangle$ for any n. (Why?)

6. If some $f \in C^{2\pi}$ has $a_k = b_k = 0$ for all k, then $f \equiv 0$.

Proof. Indeed, by Remark 4 (or linearity of the integral), this means that

$$\int_{-\pi}^{\pi} f(x)\, T(x)\, dx = 0$$

for any trig polynomial T. But from Weierstrass's second theorem we know that f is the uniform limit of some sequence of trig polynomials (T_n). Thus,

$$\int_{-\pi}^{\pi} f(x)^2\, dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x)\, T_n(x)\, dx = 0.$$

Because f is continuous, this easily implies that $f \equiv 0$. □

7. If $f, g \in C^{2\pi}$ have the same Fourier series, then $f \equiv g$. Hence, the Fourier series for an $f \in C^{2\pi}$ provides a *unique representation* for f (even if the series fails to converge to f).

8. The coefficients $a_0, a_1, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ minimize the expression

$$\varphi(a_0, a_1, \ldots, b_n) = \int_{-\pi}^{\pi} \left[f(x) - s_n(f)(x) \right]^2 dx.$$

9. The partial sum $s_n(f)$ is the best approximation to f out of $\mathcal{T}_n$ *relative to the L_2 norm*

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx \right)^{1/2}.$$

That is,

$$\|f - s_n(f)\|_2 = \min_{T \in \mathcal{T}_n} \|f - T\|_2.$$

Moreover, using Remarks 4 and 5, we have

$$\begin{aligned} \|f - s_n(f)\|_2^2 &= \langle f - s_n(f), f - s_n(f) \rangle \\ &= \langle f, f \rangle - 2 \langle f, s_n(f) \rangle + \langle s_n(f), s_n(f) \rangle \\ &= \|f\|_2^2 - \|s_n(f)\|_2^2 \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx - \frac{a_0^2}{2} - \sum_{k=1}^{n} (a_k^2 + b_k^2). \end{aligned}$$

[This should remind you of the *Pythagorean theorem*.]

10. It follows from Remark 9 that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} s_n(f)(x)^2\,dx \;=\; \frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2) \;\le\; \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx.$$

In other symbols, $\|s_n(f)\|_2 \le \|f\|_2$. In particular, the Fourier coefficients of any $f \in C^{2\pi}$ are square summable. (Why?)

11. If $f \in C^{2\pi}$, then its Fourier coefficients (a_n) and (b_n) tend to zero as $n \to \infty$.

12. It follows from Remark 10 and Weierstrass's second theorem that $s_n(f) \to f$ *in the L_2 norm* whenever $f \in C^{2\pi}$. Indeed, given $\varepsilon > 0$, choose a trig polynomial T such that $\|f - T\| < \varepsilon$. Then, because $s_n(T) = T$ for large enough n, we have

$$\begin{aligned}\|f - s_n(f)\|_2 &\le \|f - T\|_2 + \|s_n(T - f)\|_2\\ &\le 2\|f - T\|_2\\ &\le 2\sqrt{2}\,\|f - T\| < 2\sqrt{2}\,\varepsilon,\end{aligned}$$

where the penultimate inequality follows from the easily verifiable fact that $\|f\|_2 \le \sqrt{2}\,\|f\|$ for any $f \in C^{2\pi}$. (Compare this calculation with Lebesgue's Theorem 5.9.)

By way of comparison, let's give a simple class of functions whose Fourier partial sums provide good *uniform* approximations.

(because f' is 2π-periodic). Thus, $|a_k| \le 2\|f''\|/k^2 \to 0$ as $k \to \infty$. More importantly, this inequality (along with the Weierstrass M-test) implies that the Fourier series for f is both uniformly and absolutely convergent:

$$\left| \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right| \le \left| \frac{a_0}{2} \right| + \sum_{k=1}^{\infty} (|a_k| + |b_k|) \le C \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

But why should the series actually converge to f? Well, if we call the sum

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

then $g \in C^{2\pi}$ (why?) and g has the same Fourier coefficients as f (why?). Hence (by Remarks 6.1 (7), above, $g = f$. □

Our next chore is to find a closed expression for $s_n(f)$. For this we'll need a couple of trig identities; the first two need no explanation.

$$\cos kt \cos kx + \sin kt \sin kx = \cos k(t - x)$$
$$2 \cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$$
$$\tfrac{1}{2} + \cos \theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin\,(n + \frac{1}{2})\,\theta}{2 \sin \frac{1}{2}\theta}$$

Here's a short proof for the third:

$$\begin{aligned}\sin \tfrac{1}{2}\theta + \textstyle\sum_{k=1}^{n} 2\cos k\theta \, \sin \tfrac{1}{2}\theta &= \sin \tfrac{1}{2}\theta + \textstyle\sum_{k=1}^{n} \left[\sin (k+\tfrac{1}{2})\theta - \sin (k-\tfrac{1}{2})\theta\right] \\ &= \sin (n+\tfrac{1}{2})\theta.\end{aligned}$$

The function

$$D_n(t) = \frac{\sin (n+\frac{1}{2})\, t}{2\sin \frac{1}{2}\, t}$$

is called *Dirichlet's kernel*. It plays an important role in our next calculation.

We're now ready to re-write our formula for $s_n(f)$.

$$\begin{aligned}s_n(f)(x) &= \tfrac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n} \cos kt \cos kx + \sin kt \sin kx\right] dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n} \cos k(t-x)\right] dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot \frac{\sin (n+\frac{1}{2})\,(t-x)}{2\sin \frac{1}{2}\,(t-x)}\, dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, D_n(t-x)\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, D_n(t)\, dt.\end{aligned}$$