التحليل الدالي

المحاضرة الثالثة

قسم الرياضيات

الصف الرابع

Definition(1.20)

Let X be normed space

1.The open ball with center $x_0 \in X$ and radius $r > 0$ denoted by $\beta_r(x_0)$ and define as $\beta_r(x_0)= \{x \in X: ||x-x_0||< r \}$ and the closed ball is $\overline{\beta_r}(x_0)=\{x \in X : ||x-x_0|| \le r\}$.

2. A subset A of X is said to be bounded if there exists $k > 0$ such that $||x|| \le k$ for all $x \in A$.

Remarks

1.Every open ball and closed ball are nonempty sets because $x_0 \in \beta_r (x_0)$

and $x_0 \in \overline{\beta_r}(x_0)$.

2.$\beta_r (x_0)= x_0 + \beta_r (0)= x_0 + r\beta_1(0)$

Indeed

$\beta_r(x_0)=\{x \in X: ||x-x_0 ||< r\}= \{x_0+y: ||y||<r\}=x_0+ \{y: ||y||<r\}=x_0+\beta_r(0)$

Also, $\beta_r(0)=\{x \in X: ||x||< r\}=\{x \in X: \frac{||x||}{r}< 1\}= \{r\, y:||y||< 1\}=r\{y: ||y||<1\}=r\beta_1(0)$.

Definition(1.21)

Let X be normed space. A subset A is said to be open set if given any point $x \in A$, there exists $r > 0$ such that $\beta_r (x) \subseteq A$. and we say that A is called a closed set if A^c is open set.

Remark

Since every normed space is metric space and every metric space is a topological space, then every normed space is topological space. $\beta_r(x_0)$ is a neighbourhood of x_0 . This topology is called a norm topology on X , and the space X is called the normed topological space.

Definition(1.22)

A metric linear space X is said to be normable if the metric function is induced by a norm.

Theorem (1.23)

Let X be normed space.

1. *Every open ball in X is open set.*
2. *Every closed ball in X is closed set.*
3. *A subset of X is open iff it is union of a family of open balls.*
4. *Any finite subset of X is closed.*

Proof: **H.W**

Definition (1.24)

Let X be normed space and let $A \subseteq X$:

1. *The union of all open sets in X contained in A is called the interior of A, denoted by int(A).*

i.e. $int(A) = \cup\{B \subseteq X : B \in T,\ B \subseteq A\}$*.Thus int(A) is the largest open set contained in A, and* $int(A) \subseteq A$.

Hence $int(A) = \{x \in A : \exists r > 0,\ \beta_r(x) \subseteq A\}$,

$$int(A) = \{x \in A : \exists r > 0,\ x + rB_1(0) \subset A\}$$

2. *The intersection of all the closed sets containing A is called the closure of A, denoted by A .*

i.e. $\bar{A} = \cap\{B \subseteq X : B^c \in T,\ A \subseteq B\}$*.Thus* $\bar{A}$ *is the smallest closed set containing A ,and* $A \subseteq \bar{A}$.

Hence $\bar{A} = \{x \in X : \forall r > 0, \exists y \in A \ni \|x - y\| < r\}$, $\bar{A} = \bigcap_{r>0} (A + r\overline{B_1}(0))$.

3. *A point $x \in X$ is called a limit point of A if each open set G in X such that $x \in G$ and $A \cap (G \mid \{x\}) \neq \phi$ .The set of all limit points of A is denoted by A' and is called the derived set of A .*

   Hence $A' = \{x \in X : \forall r > 0,\ \exists y \in A \ni y \neq x,\ \|x - y\| < r\}$

4. *The boundary of a subset A is defined as the difference between the closure and the interior of the subset A , i.e.* $\partial(A) = \bar{A} \cap (int(A))^c$.

   Hence $\partial(A) = \{x \in X : \forall r > 0, \exists y \in A,\ z \in A^c \ni \|x - y\| < r,\ \|x - z\| < r\}$

5. *The exterior of A is the complement of $\bar{A}$ and denoted by ext(A) ,*

   i.e. $ext(A) = (\bar{A})^c$.

Theorem(1.25)

Let X be normed space. If M is a subspace of X, then $\overline{M}$ is subspace of X.

Proof :

Since $0 \in M \Rightarrow M \subset \overline{M} \Rightarrow 0 \in \overline{M}$, so $\overline{M} \neq \phi$

Let $x, y \in \overline{M}$ and $\alpha, \beta \in F$. To prove $\alpha x + \beta y \in \overline{M}$

Let $r > 0$

1. If $\alpha \neq 0$ and $\beta \neq 0$, then $\frac{r}{2|\alpha|} > 0$ and $\frac{r}{2|\beta|} > 0$, there exist $a, b \in M$ such that

$$||x - a|| < \frac{r}{2|\alpha|} \quad \text{and.} \quad ||x - b|| < \frac{r}{2|\beta|}$$

Since M is subspace and $a,b \in M$, then. $\alpha a + \beta b \in M$

$(\alpha x + \beta y) - (\alpha a + \beta b) = \alpha (x - a) + \beta (y - b)$

$||(\alpha x + \beta y) - (\alpha a + \beta b)|| \leq |\alpha|\, || x - a|| + |\beta|\, || y - b|| < |\alpha| \frac{r}{2|\alpha|} + |\beta| \frac{r}{2|\beta|} = r$

Hence $\alpha x + \beta y \in \overline{M}$

Equivalent Norms

Definition(1.26) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space X. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent (or $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$), written $\|\cdot\|_1 \sim \|\cdot\|_2$, if there exists positive real numbers a and b such that $a \| x \|_1 \leq \| x \|_2 \leq b \| x \|_1$ for all $x \in X$.

Example(1.27)

Let $\| x \|_1 = \sum_{i=1}^{n} |x_i|$ and $\| x \|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$ for all $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

Show that $\|\cdot\|_1 \sim \|\cdot\|_2$.

Solution :

From Cauchy's inequality, we have

$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}$ for all $x_i, y_i \in \mathbb{R}$

Put $y_i = 1$ for all $i = 1, 2, \cdots, n$;

we have $\sum_{i=1}^{n}|x_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} 1\right)^{\frac{1}{2}}$

$\| x \|_1 \leq \| x \|_2 \cdot \sqrt{n} \Rightarrow \frac{1}{\sqrt{n}} \| x \|_1 \leq \| x \|_2$

$\Rightarrow a = \frac{1}{\sqrt{n}}$, *but* $\| x \|_2 \leq \| x \|_1 \Rightarrow b = 1.$

Hence $\|\cdot\|_1 \sim \|\cdot\|_2$.

Theorem(1.28)

On a finite dimensional vector space all norms are equivalent.

Proof:

Let X *be finite dimensional vector space with* $\dim X = n > 0$, *and* $\|\cdot\|_1, \|\cdot\|_2$ *be two norms on* X. *To prove* $\|\cdot\|_1 \sim \|\cdot\|_2$

Let $\{x_1, \cdots, x_n\}$ *be a basis for* $X \Rightarrow$ *every* $x \in X$ *has a unique representation*

$$x = \sum_{i=1}^{n} \lambda_i x_i,\ \lambda_i \in F. \qquad \cdots (1)$$

and

$$\| x \|_1 = \left\|\sum_{i=1}^{n} \lambda_i x_i\right\|_1 \leq \sum_{i=1}^{n} |\lambda_i| \|x_i\|_1 \ .....(2)$$

Put $k = max\{\| x_1\|_1, \dots, \|x_n\|_1\} \Rightarrow k$ *for all* $i = 1, \dots, n$

$\Rightarrow \sum_{i=1}^{n} |\lambda_i| \|x_i\|_1 \leq k \sum_{i=1}^{n} |\lambda_i|. \quad .... \ (3)$

From (2) and (3), we have $\| x \|_1 \leq k \sum_{i=1}^{n} |\lambda_i| \quad ....(4)$

Since the set $\{x_1, \cdots, x_n\}$ *is linear independent, by lemma of linear independent, there is* $c > 0$ *such that* $\left\|\sum_{i=1}^{n} \lambda_i x_i\right\|_2 \geq c \sum_{i=1}^{n} |\lambda_i| \quad ....(5)$

From (1) and (5), we have $\| x \|_2 \geq c \sum_{i=1}^{n} |\lambda_i| \quad .....(6)$

From (4) and (6), we have $\| x \|_1 \leq \frac{k}{c} \| x \|_2$

Put $a = \frac{c}{k}$, *we have* $a \| x \|_1 \leq \| x \|_2 \quad ....(7)$

Similarly $\| x \|_2 \leq k \sum_{i=1}^{n} |\lambda_i| \quad \cdots (8)$, *and.* $\| x \|_1 \geq c\sum_{i=1}^{n} |\lambda_i|$ ....(9)

From (8) and (9) , we have $\| x \|_2 \leq \frac{k}{c} \| x \|_1$

Put $b = \frac{k}{c}$, *we have* $\| x \|_2 \leq b \| x \|_1$ .....(10)

From (7) and (10), we have $a \| x \|_1 \leq \| x \|_2 \leq b \| x \|_1$.

Hence $\|\cdot\|_1 \sim \|\cdot\|_2$

Definition(1.29)

A semi norm on X is a function $p: X \to \mathbb{R}$ *having the following:*

1 $p(\lambda x) = |\lambda| p(x)$ *for all* $x \in X$ *and for all* $\lambda \in F$

2 $p(x + y) \leq p(x) + p(y)$ *for all* $x, y \in X$

A family F of seminorms on X is said to be separating if to each $x \neq 0$ *corresponds at least one* $p \in F$ *with* $p(x) \neq 0$.

Theorem(1.30)

Suppose p is a seminorm on a vector space X . Then:

1. $p(0)=0$

2. $p(-x)=p(x)$ *for all* $x \in X$

3. $p(y-x)=p(x-y)$ *for all* $x,y \in X$

4. $|p(x)-p(y)| \leq p(x-y)$ *for all* $x,y \in X$

5. $p(x) \geq 0$ *for all* $x \in X$

6. *The set* $N(p)=\{x \in X : p(x)=0\}$ *is a subspace of X*

7. *p is a norm if it satisfies the condition* $p(x) \neq 0$ *if* $x \neq 0$. **H.W.**

Proof :

(1),(2) and (3) direct from definition.

4. $x=(x-y)+y \Rightarrow p(x)=p((x-y)+y) \leq p(x-y)+p(y)$

$p(x) - p(y) \leq p(x-y)$ ... (1)

Also, $-p(x-y) \leq p(x) - p(y)$ .... (2)

From(1) and (2) , we have;

$-p(x-y) \leq p(x)-p(y) \leq p(x-y)$

$\Rightarrow |p(x)-p(y)| \leq p(x-y)$

5. Since $p(x)-p(y) \leq p(x-y)$ for all $x,y \in X$

Take $y=0 \Rightarrow p(x) \leq p(x)$

Since $p(x) \geq 0 \Rightarrow p(x) \geq 0$ for all $x \in X$

6. Since $p(0)=0 \Rightarrow 0 \in N(p) \Rightarrow N(p) \neq \phi$

Let $x,y \in N(p)$ and $\alpha, \beta \in F \Rightarrow p(x)=0$, $P(y)=0$

$p(\alpha x+\beta y) \leq p(\alpha x)+p(\beta y) \leq \alpha p(x)+y\, p(y)=0 \Rightarrow$

$p(\alpha x+\beta y) \leq 0$ Since $x, y \in N(p), \alpha, \beta \in F$, and X is vector space, then

$\alpha x+\beta y \in X \Rightarrow p(\alpha x+\beta y) \geq 0$

$p(\alpha x+\beta y)=0 \Rightarrow \alpha x+\beta y \in N(p) \Rightarrow N(p)$ is a subspace.

Definition(1.31)

Let X be a linear space over F. A Δ - norm on X is a function $\|\,.\,\|: X \rightarrow R$ having the following properties:

1. $\|x\| > 0$ for all $x \in X$, $x \neq 0$.

2. $\|\lambda x\| \leq \|x\|$ for all $x \in X$ and for all $0 < |\lambda| \leq 1$

3. $\lim_{\lambda \rightarrow 0} \|\lambda x\| = 0$ for all $x \in X$

4. $\|x+y\| \leq c \max\{\|x\|, \|y\|\}$ for all $x,y \in X$ where $c > 0$ is independent of x, y.

Remark

1. A Δ - norm $\|\,.\,\|$ on X is called an F-norm if it satisfies $\|x+y\| \leq \|x\|+\|y\|$ for all $x, y \in X$.

2. A Δ - norm $\|\,.\,\|$ on X is called a quasi-norm if it satisfies $\|\lambda x\| = |\lambda|\, \|x\|$ for all $x \in X$ and $\lambda \in F$.