Functional_Analysis

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Theorem 3.2.:

If X is a pre-Hilbert space, then :

1) $< x, 0> = < 0, x > = 0$

2) $< x, \alpha y + \beta z > = \bar{\alpha} < x, y > + \bar{\beta} < x, z >, \quad \forall x, y, z \in X \ \& \ \forall \alpha, \beta \in F$

Proof:

1- $< 0, x > = < 0.0, x > = 0 < 0, x > = 0$

2- $< x, \alpha y + \beta z > = \overline{< \alpha y + \beta z, x >}$

$= \overline{< \alpha y, x > + < \beta z, x >}$

$= \overline{\alpha < y, x >} + \overline{\beta < z, x >}$

$= \bar{\alpha} < x, y > + \bar{\beta} < x, z >$

Corollary 3.3.:

If X is a pre- Hilbert space , then:

1) $< \sum_{i=1}^{n} \alpha_i x_i, y > = \sum_{i=1}^{n} \alpha_i < x_i, y >$

2) $< x, \sum_{j=1}^{m} \beta_j y_j > = \sum_{j=1}^{m} \overline{\beta_j} < x, y_j >$

3) $< \sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j > = \sum_{i,j} \alpha_i \overline{\beta_j} < x_i, y_j >$

Theorem 3.4.:(Chauchy- Schwarz Inequality)

Let X be a pre-Hilbert Space and the function $\| . \|: X \to R$ defined by:

$\| x \| = \sqrt{< x, x >}, \forall x \in X$ then $| < x, y > | \leq \| x \| \, \| y \|$, $\forall \ x, y \in X$

Proof:

If $x = 0$ or $y = 0 \Rightarrow < x, y > = 0$.

If $y \neq 0$, we put $z = \frac{y}{\| y \|}$

$\Rightarrow \| z \|^2 = < z, z > = < \frac{y}{\| y \|}, \frac{y}{\| y \|} > = \frac{1}{\| y \|^2} < y, y > = \frac{1}{\| y \|^2} \| y \|^2 = 1$

We must prove $| < x, z > | \leq \| x \|$

Let $\lambda \in F$, then:

$< x-\lambda z, x-\lambda z > \geq 0$

$\| x \|^2 - \bar{\lambda} < x, z > - \lambda < z, x > + | \lambda |^2 \| z \|^2 \geq 0$

$\| x \|^2 - \bar{\lambda} < x, z > - \lambda < z, x > + | \lambda |^2 \geq 0$

$\| x \|^2 - < x, z > \overline{< x,z >} + < x, z > \overline{< x,z >} - \bar{\lambda} < x,z > - \lambda < z,x > + \lambda\bar{\lambda} \geq 0$

$\| x \|^2 - |< x,z >|^2 + < x,z > (\overline{< x,z >} - \bar{\lambda}) - \lambda(< z,x > - \bar{\lambda}) \geq 0$

$\| x \|^2 - |< x,z >|^2 + (< x,z > - \lambda)\overline{(< x,z > - \lambda)} \geq 0$

$\| x \|^2 - | < x, z >|^2 + | < x, z > - \lambda^2 | \geq 0 \ , \ \forall \lambda \in F$

Since $< x, z > \in F$, put $\lambda =< x, z >$, then

$\| x-< x, z> z\|^2 = \| x \|^2 - | < x, z >|^2 + | < x, z> - < x, z>|^2$

$= \| x \|^2 - |< x, z >|^2 \geq 0$

$\Rightarrow |< x, z >| \leq \| x \|$

$\Rightarrow |< x, \frac{y}{\| y \|} >| \leq \| x \|$

$\Rightarrow |< x, y >| \leq \| x \| \, \| y \|$

Theorem 3.5.: Every Pre-Hilbert space is a normed space (metric space).

Proof:

Let X be a Pre-Hilbert space and let the function $\| . \|: X \to R$ such that:

$\| x \| = \sqrt{< x,x >}$, $\forall x \in X$

T.P. the space X satisfies the conditions of the norm:

1- Since $< x, x> \geq 0$, $\forall x \in X \Rightarrow \| x \| \geq 0, \forall x \in X$.

2-$\| x \| =0 \Leftrightarrow \sqrt{< x,x >} =0 \Leftrightarrow < x, x >=0 \Leftrightarrow x =0$

3-let $x \in X$, $\lambda \in F$:

$\| \lambda x \| = \sqrt{< \lambda x, \lambda x >} = \sqrt{\lambda\bar{\lambda} < x,x >} = \sqrt{| \lambda |^2 < x,x >} = | \lambda | \sqrt{< x,x >} = | \lambda | \, \| x \|$

4- let $x,y \in X$:

$\| x+y \|^2 = < x+y, x+y > = < x, x > + < x, y > + < y, x > + < y, y >$

$= \| x \|^2 + < x, y > + \overline{< x, y >} + \| y \|^2$

Since $< x, y > + \overline{< x, y >} = 2 \text{ Re}(< x, y >)$

$\Rightarrow \| x + y \|^2 = \| x \|^2 + 2 \text{ Re} (< x, y >) + \| y \|^2$

Since $\text{Re} (< x, y >) \leq | < x, y > |$

$\Rightarrow \| x + y \|^2 \leq \| x \|^2 + 2 | < x, y > | + \| y \|^2$

By Cauchy – Schwars inequality $| < x, y > | \leq \| x \| \| y \|$, we get:

$\Rightarrow \| x + y \|^2 \leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 = (\| x \| + \| y \|)^2$

Then $\| x + y \| \leq \| x \| + \| y \|$

Theorem 3.6.: If x, y are vectors on Pre-Hilbert space X, then:

1- $\| x+y \|^2 = \| x \|^2 + 2 \text{ Re } (< x, y >) + \| y \|^2$ (Polar inequality)

2- $\| x + y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$ (Parallel Law)

3- $< x, y > = \frac{1}{4} [\| x + y \|^2 - \| x- y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2]$ (Identical Polarization)

Proof:

1- We get from theorem 3.5.

2- $\| x + y \|^2 = < x+y, x+y > = \| x \|^2 + < x, y > + < y, x > + \| y \|^2$

$\| x - y \|^2 = < x - y, x - y > = \| x \|^2 - < x, y > - < y, x > + \| y \|^2$

$\Rightarrow \| x + y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$

3- From (2), we get:

$\| x + y \|^2 - \| x - y \|^2 = 2 < x,y > +2 < x, y >$

$\| x + iy \|^2 = \| x \|^2 - i < x, y > + i < y, x > + \| y \|^2$

$\| x - iy \|^2 = \| x \|^2 + i < x, y > - i < y, x > + \| y \|^2$

$i \| x + iy \|^2 - i \| x - iy \|^2 = 2 < x, y > - 2 < y, x >$

$\| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2 = 4 < x, y >$

$\Rightarrow < x, y > = \frac{1}{4} [\| x + y \|^2 - \| x- y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2]$

Theorem 3.7.: Let $(X, \| . \|)$ is a normed space such that:

$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2 \ , \forall x, y \in X$

And let $< , >$ is defined by:

$< x, y > = \frac{1}{4} [\| x + y \|^2 - \| x- y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2]$

Then $< , >$ is an inner product on X (i.e. X is a Pre-Hilbert space).

Remark:

If X is a normed space then it is not necessary X is a Pre-Hilbert space. For example:

Let X = C[a, b] and $\| f \| = \max\{ | f(x) | : a \leq x \leq b\} \ , \forall f \in X$

Since X is normed space

T.P. it is not Pre-Hilbert space, we need prove that:

$\| f + g \|^2 + \| f - g \|^2 \neq 2 \| f \|^2 + 2 \| g \|^2$, $f, g \in X$

Let $f(x) = 1$, $g(x) = \frac{x-a}{b-a}$, $\forall x \in [a, b]$

$\| f \| = 1$, $\| g \| = 1$

$f(x) + g(x) = 1 + \frac{x-a}{b-a} \Rightarrow \| f + g \| = 2$

$f(x) - g(x) = 1 - \frac{x-a}{b-a} \Rightarrow \| f - g \| = 1$

$\Rightarrow \| f + g \|^2 + \| f - g \|^2 = 4+1=5 \neq 2 \| f \|^2 + 2 \| g \|^2 = 2+2=4$

Theorem 3.8.: On Pre-Hilbert space X:

1- If $x_n \to x$, $y_n \to y$ then $< x_n , y_n > \to < x, y >$

2- If $\{ x_n \}$ and $\{ y_n \}$ are Cauchy sequence on X then $\{ < x_n , y_n > \}$ is Cauchy sequence on F.

Proof:

1- $< x_n , y_n > = < x+(x_n - x), y+(y_n - y) >$

$= < x, y > + < x, y_n - y > + < x_n - x, y > + < x_n - x, y_n - y >$

$< x_n , y_n > - < x, y > = < x, y_n - y > + < x_n - x, y > + < x_n - x, y_n - y >$

$| < x_n , y_n > - < x, y > | \leq | < x, y_n - y > | + | < x_n - x, y > | + | < x_n - x, y_n - y > |$

$\leq \| x \| \, \| y_n - y \| + \| x_n - x \| \, \| y \| + \| x_n - x \| \, \| y_n - y \|$

Since $\| x_n - x \| \to 0$, $\| y_n - y \| \to 0$ where $n \to \infty$

$\Rightarrow | < x_n - y_n > - < x - y > | \to 0$ where $n \to \infty$

$\Rightarrow < x_n - y_n > \to < x - y >$

2- Similarly to (1):

$| < x_n , y_n > - < x_m, y_m > | \leq \| x_m \| \, \| y_n - y_m \| + \| x_n - x_m \| \, \| y_m \| + \| x_n - x_m \| \, \| y_n - y_m \|$

Since $\| x_n \|$, $\| y_n \|$ is bounded

Definition 3.9.: The complete Pre-Hilbert space is called ***Hilbert space***. In other words if X a vector space on F with an inner product $<,>$, then X is Hilbert space if the metric space which is generated by the norm $\| x \|^2 = < x, x >$ complete Hilbert space.

Examples:

1- The space F^n with an inner product which defined by:

$$< x, y > = \sum_{i=1}^{n} x_i y_i, \quad \forall x, y \in F^n$$ is a Hilbert space.

2- The space $l^2 = \{x = (x_1, x_2, \dots, x_n, \dots) : x_i \in F, \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with an inner product defined by: $< x, y > = \sum_{i=1}^{\infty} x_i \overline{y_i}$ is Hilbert space.

3- The space X=C[-1, 1] with an inner product defined by : $< f, g > = \int_{-1}^{1} f(x)\overline{g(x)}$ is not Hilbert space.

Proof:

The space X with an inner product is not complete space because if we take the sequence $\{ f_n \}$ such that:

$$f_n(x) = \begin{cases} 0 & -1 \le x \le 0 \\ nx & 0 < x < \frac{1}{n} \\ 1 & \frac{1}{n} \le x \le 1 \end{cases}$$

$$\| f_n - f_m \|^2 = < f_n - f_m, f_n - f_m > = \frac{(n-m)^2}{3n^2 m}$$

$\Rightarrow \| f_n - f_m \| \to 0$ where $n,m \to \infty$

i.e. $\{ f_n \}$ is Cauchy sequence but it is not convergent in X

if we suppose that $f_n \to f$

$\Rightarrow f \notin$ X because it is not continuous.

4- Every Hilbert space is Banach space but the inverse is not true.

Sol.:

If X is Hilbert space then its Pre-Hilbert space and complete.

Since every Pre-Hilbert space is normed space then X is complete normed space i.e. Banach space.

And the space l_p ($p \neq 2$) is Banach space.

T.P. l_p ($p \neq 2$) not Hilbert space , we prove it is not satisfying Parallel Law.

Let $x = (1, 1, 0, 0, \ldots)$, $y = (1, -1, 0, 0,\ldots)$

$\Rightarrow x, y \in l_p$, $\| x \| = \| y \| = 2^{1/p}$ & $\| x + y \| = \| x - y \| = 2$

$\Rightarrow \| x + y \|^2 + \| x - y \|^2 \neq 2 \| x \|^2 + 2\| y \|^2$

Definition 3.10.:

Let X be Pre-Hilbert space and let $x, y \in$ X, we say x ***orthogonal*** on y if $< x, y >=0$ (write $x \perp y$).

Remarks:

1- The orthogonal is symmetric, i.e. if $x \perp y$ then $y \perp x$.

Since $x \perp y \Rightarrow < x, y >=0 \Rightarrow \overline{< x, y >} = \bar{0} = 0 \Rightarrow < y, x >=0 \Rightarrow y \perp x.$

2- Zero vector orthogonal on all vectors, i.e. $0 \perp x$, $\forall x \in$X, because $<0, x>=0$, $\forall x \in$X.

3- If $x \perp x \Rightarrow x = 0$, because if $x \perp x \Rightarrow < x, x > =0 \Rightarrow x = 0$.

4- If $x \perp y \Rightarrow \lambda x \perp y$, $\forall \lambda \in$F. because $< \lambda x, y > = \lambda < x, y > = \lambda (0)=0$

Examples:

1- Let X = R^2 with an inner product and $x=(1, 2)$, $y=(2, -1)$, $z= (-6, 3)$

Since $< x, y >=(1)(2)+(2)(-1)=0 \Rightarrow x \perp y.$

$< x, z > = (1)(-6)+(2)(3)=0 \Rightarrow x \perp z.$

$<y, z > = (2)(-6)+(-1)(3)=-15 \neq 0 \Rightarrow y$ not orthogonal on z.

2- If the vector x is orthogonal on all the vectors $x_1,x_2,\ldots,x_n$ in Pre-Hilbert space X, then x is orthogonal on every linear combination of x_i.

Let $z=\sum_{i=1}^{n} \lambda_i x_i \quad , \lambda_i \in$F

$< x, z >=< x, \sum_{i=1}^{n} \lambda_i x_i > = \sum_{i=1}^{n} \overline{\lambda_i} < x, x_i > =0$ (because $x \perp x_i$, $\forall i=1, 2, .., n$).

3- Find the values of a which make the vectors $x = (1, 2, a)$, $y = (-1, 3, 5)$ orthogonal in R^3.

$< x, y > = (1)(-1)+(2)(3)+5a = -1+6+5a = 5+5a = 0 \Rightarrow 5a = -5 \Rightarrow a = -1.$

4- Let x, y vectors in Pre-Hilbert space X s.t. $\| x \| = \| y \| =1$, then $x+y$ orthogonal on $x-y$.

$< x+y, x-y > = < x, x > - < x, y > + < y, x > - < y, y >$

$= \| x \|^2 - < x, y > + < x, y > - \| y \|^2 = 1-1= 0.$

Theorem 3.11.:

If x, y are orthogonal vectors in Pre-Hilbert space X, then:

$$\| x + y \|^2 = \| x - y \|^2 = \| x \|^2 + \| y \|^2$$

Proof:

Since $x \perp y \Rightarrow \langle x, y \rangle = \langle y, x \rangle = 0$

$\Rightarrow \| x + y \|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

$= \| x \|^2 + 0 + 0 + \| y \|^2$

$= \| x \|^2 + \| y \|^2$

Similarity prove that $\| x - y \|^2 = \| x \|^2 + \| y \|^2$.

Corollary 3.12.: If $x_1, x_2, \ldots, x_n$ are orthogonal vectors (i.e. $x_i \perp x_j$, $\forall i \neq j$) in Pre-Hilbert space X, then : $\| \sum_{i=1}^{n} x_i \|^2 = \sum_{i=1}^{n} \| x_i \|^2$.

Definition 3.13.: Let A nonempty subset in Pre-Hilbert space X. The vector $x \in$X is called orthogonal vector on A (write $x \perp$ A) if $x \perp y$, $\forall y \in$A.

Definition 3.14.: Let A & B are nonempty subsets of Pre-Hilbert space X, We say A orthogonal on B (write A $\perp$ B) if $x \perp y$, $\forall\, x \in$A & $\forall y \in$B.

Remark: If M_1 & M_2 are subspaces of Pre- Hilbert space X such that $M_1 \perp M_2$, then $M_1 \cap M_2 = \{0\}$.

Definition 3.15.: Let A nonempty subset of Pre-Hilbert space X. The Orthogonal complement of A denoted by $A^\perp$ and defined by:

$A^\perp = \{ x \in X: x \perp y , \forall y \in A\} = \{ x \in X: x \perp A\}$

And define $(A^\perp)^\perp = A^{\perp\perp} = \{x \in X: x \perp y , \forall y \in A^\perp)$.

Theorem 3.16.: Let X be a Pre-Hilbert space, then:

1- $\{0\}^\perp = X$, 2- $X^\perp = \{0\}$

Proof:

1- $\{0\}^\perp = \{ x \in X: x \perp 0\} = X$, 2- $X^\perp = \{ x \in X: x \perp x\} = \{0\}$

Theorem 3.17.: Let A and B be two nonempty subsets of Pre-Hilbert space X. Then:

1- $A \cap A^\perp \subset \{0\}$

2- $A \subseteq A^{\perp\perp}$

3- If $A \subset B$ then $B^\perp \subset A^\perp$

4- $A \subseteq B^{\perp} \Leftrightarrow B \subset A^{\perp}$

Proof:

1- Let $x \in A \cap A^{\perp} \Rightarrow x \in A$ & $x \in A^{\perp} \Rightarrow x \perp x \Rightarrow x = 0 \Rightarrow A \cap A^{\perp} \subset \{0\}$

2- Let $x \in A \Rightarrow x \perp y,\ \forall\, y \in A^{\perp} \Rightarrow x \perp A^{\perp} \Rightarrow x \in A^{\perp\perp} \Rightarrow A \subseteq A^{\perp\perp}$

3- Let $x \in B^{\perp} \Rightarrow x \perp y,\ \forall y \in B$

Since $A \subset B \Rightarrow x \perp y,\ \forall y \in A \Rightarrow x \in A^{\perp} \Rightarrow B^{\perp} \subset A^{\perp}$

4- Let $A \subseteq B^{\perp}$ T.P. $B \subset A^{\perp}$

Since $A \subseteq B^{\perp} \Rightarrow B^{\perp\perp} \subset A^{\perp}$ (by part 3)

But $B \subset B^{\perp\perp}$ (by part 2)

$\Rightarrow B \subset A^{\perp}$

Similarity prove if $B \subset A^{\perp} \Rightarrow A \subseteq B^{\perp}$

Theorem 3.18.: If A is nonempty subset of Pre-Hilbert space X, then $A^{\perp}$ is closed subspace of X.

Proof:

Since $0 \perp x,\ \forall x \in A \Rightarrow 0 \in A^{\perp} \Rightarrow A^{\perp} \neq \phi$

Let $x, y \in A^{\perp}$, $\alpha, \beta \in F$,

$\forall z \in A$, we have

$< \alpha x + \beta y, z > = \alpha < x, z > + \beta < y, z > = \alpha\,(0) + \beta\,(0) = 0$

$\Rightarrow \alpha x + \beta y \in A^{\perp} \Rightarrow A^{\perp}$ is subspace of X

T.P. $A^{\perp}$ is closed subspace (i.e. $\overline{A^{\perp}} = A^{\perp}$)

Let $x \in \overline{A^{\perp}} \Rightarrow$ there exist a sequence $\{x_n\}$ in $A^{\perp}$ such that $x_n \to x$

$\forall\, y \in A \Rightarrow < x_n , y > = 0$, $\forall\, n \in Z^{+}$

Since $< x_n , y > \to < x, y >$

$\Rightarrow < x, y > = 0$, $\forall\, y \in A \Rightarrow x \in A^{\perp} \Rightarrow \overline{A^{\perp}} = A^{\perp} \Rightarrow A^{\perp}$ is closed subspace of X.

Definition 3.19.: Let A be subset of Pre-Hilbert space. The set A is called ***orthogonal*** if $x \perp y$, $\forall x, y \in A, x \neq y$, and called A ***orthonormal*** if A is orthogonal and $\| x \| = 1$, $\forall x \in A$. In the other word, we say A is orthonormal if :

$$< x, y > = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases},\ \forall\, x, y \in A.$$

The sequence $\{ x_n \}$ is called orthogonal if $x_n \perp x_m$, $\forall n \neq m$, and called orthonormal if:

$$< x_n, y_m > = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

Remark: The orthonormal set not contained the zero vector because $\| 0 \| = 0 \neq 1$.

Examples:

1- Let X = R^3, and A = {(1, 2, 2), (2, 1, -2), (2, -2, 1)} then A is orthogonal set in R^3.

Sol.: x = (1, 2, 2), y = (2, 1, -2), z = (2, -2, 1)

$< x, y > = \sum_{i=1}^{3} x_i y_i$ = (1)(2)+(2)(1)+(2)(-2)= 2+2-4=0

Similarity prove $<x, z>$ =0 & $< y, z >$=0

2- Let X= C[-π, π] and $f_n(x)$ = sin (nx) then { f_n } is orthogonal sequence.

Sol.:

$< f_n, f_m > = \int_{-\pi}^{\pi} f_n(x) f_m(x) dx = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0$

If $g_n(x)$= cos (nx) then the sequence { g_n } is orthogonal.

Theorem 3.20.: Let $x_1, \dots , x_n$ are orthonormal vectors in Pre-Hilbert space X, $\forall x \in$ X:

1- $\| x - \sum_{i=1}^{n} < x, x_i > x_i \|^2 = \| x \|^2 - \sum_{i=1}^{n} |< x, x >|^2$, $\forall x \in$ X

2- $\sum_{i=1}^{n} |< x, x_i >|^2 \leq \| x \|^2$, $\forall x \in$ X

3- (x- $\sum_{i=1}^{n} < x, x_i > x_i) \perp x_j$, $\forall x \in$ X and for all j.

Proof:

Let $\lambda_i = < x, x_i >$

$\| x - \sum_{i=1}^{n} < x, x_i > x_i \|^2 = \| x - \sum_{i=1}^{n} \lambda_i x_i \|^2 = < x - \sum_{i=1}^{n} \lambda_i x_i , x - \sum_{i=1}^{n} \lambda_i x_i >$

$= < x, x > - < x, \sum_{i=1}^{n} \lambda_i x_i > - < \sum_{i=1}^{n} \lambda_i x_i , x > + < \sum_{i=1}^{n} \lambda_i x_i , \sum_{i=1}^{n} \lambda_i x_i >$

$= \| x \|^2 - \sum_{i=1}^{n} \overline{\lambda_i} < x, x_i > - \sum_{i=1}^{n} \lambda_i < x_i, x > + \| \sum_{i=1}^{n} \lambda_i x_i \|^2$

$= \| x \|^2 - \sum_{i=1}^{n} \overline{\lambda_i} \lambda_i - \sum_{i=1}^{n} \lambda_i \overline{\lambda_i} + \sum_{i=1}^{n} | \lambda_i |^2 \| x_i \|^2$

$= \| x \|^2 - \sum_{i=1}^{n} | \lambda_i |^2 - \sum_{i=1}^{n} | \lambda_i |^2 + \sum_{i=1}^{n} | \lambda_i |^2$

$$= \| x \|^2 - \sum_{i=1}^{n} | \lambda_i |^2 = \| x \|^2 - \sum_{i=1}^{n} |< x, x_i >|^2$$

2-since $\| x - \sum_{i=1}^{n} < x, x_i > x_i \|^2 \geq 0$

$$\Rightarrow \| x \|^2 - \sum_{i=1}^{n} |< x, x_i >|^2 \geq 0$$

$$\Rightarrow \sum_{i=1}^{n} |< x, x_i >|^2 \leq \| x \|^2$$

3- $< x - \sum_{i=1}^{n} < x, x_i > x_i, x_j > = < x, x_j > - < \sum_{i=1}^{n} < x, x_i > x_i, x_j >$

$$= < x, x_j > - \sum_{i=1}^{n} < x, x_i >< x_i, x_j >$$

Since $< x_i, x_j > = \begin{Bmatrix} 0 & i \neq j \\ 1 & i = j \end{Bmatrix}$

$\Rightarrow < x - \sum_{i=1}^{n} < x, x_i > x_i, x_j > = < x, x_j > - < x, x_j > = 0$

Then $x - \sum_{i=1}^{n} < x, x_i > x_i) \perp x_j$, $\forall x \in X$ and for all j.

Corollary 3.21.: Let $\{ x_n \}$ be an orthonormal sequence in Pre-Hilbert space X,

then: $\sum_{i=1}^{n} |< x, x_i >|^2 \leq \| x \|^2$, $\forall x \in X$.

Theorem 3.22.: (Gram- Schmidt Theorem)

If $\{ y_n \}$ is a sequence of independent linear vectors in Pre-Hilbert space X, then there exist an orthogonal sequence $\{ x_n \}$ in X such that:

$[x_1, x_2, \ldots, x_n] = [y_1, y_2, \ldots, y_n]$ for all n.