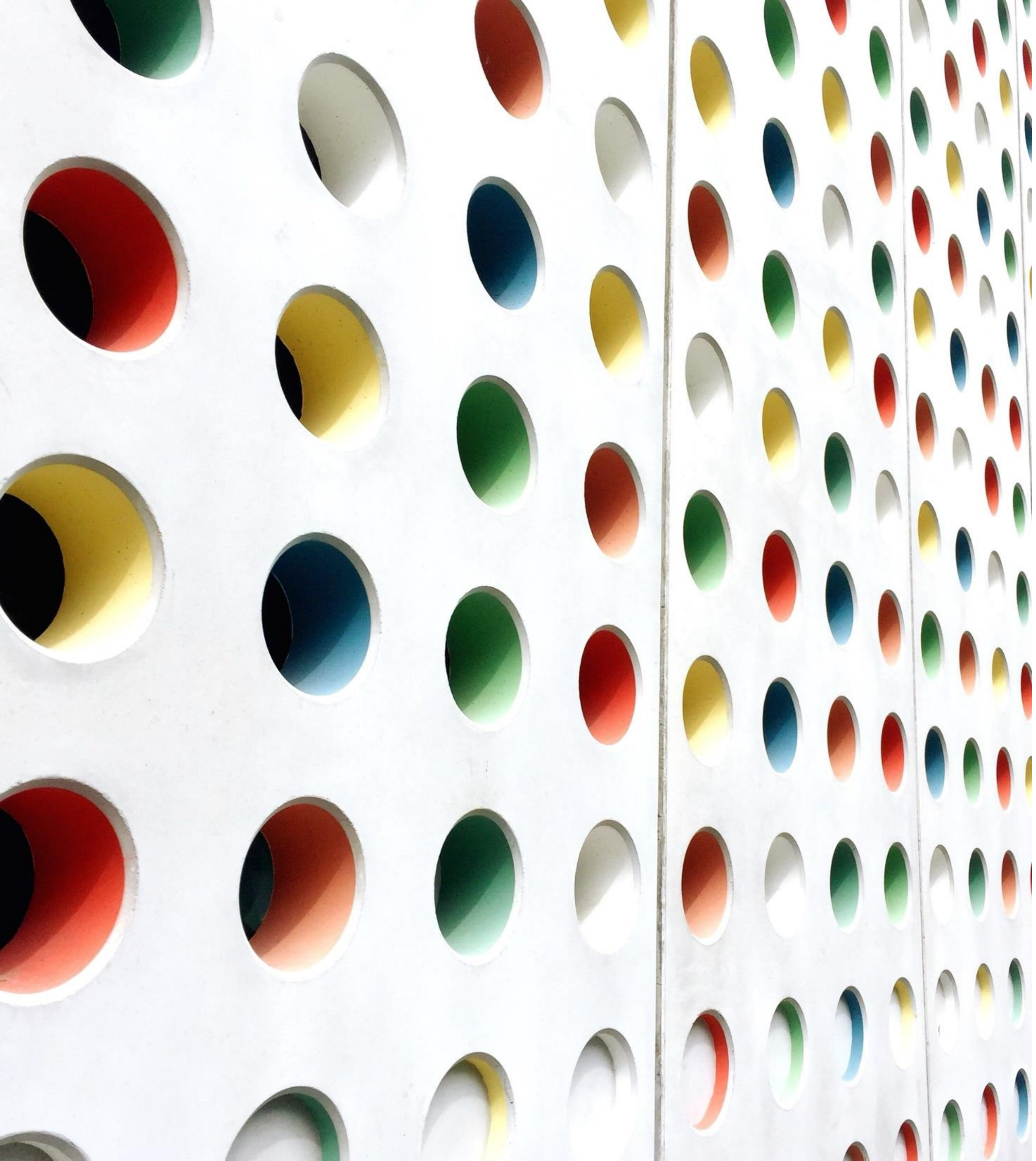

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Chapter 3

Trigonometric Polynomials

Introduction

A (real) *trigonometric polynomial*, or trig polynomial for short, is a function of the form

$$a_0 + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx \right), \tag{3.1}$$

where $a_0, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers. The *degree* of a trig polynomial is the highest frequency occurring in any representation of the form (3.1); thus, (3.1) has degree n provided that one of a_n or b_n is nonzero. We will use $\mathcal{T}_n$ to denote the collection of trig polynomials of degree at most n, and $\mathcal{T}$ to denote the collection of all trig polynomials (i.e., the union of the $\mathcal{T}_n$ over all n).

Theorem 3.1. (Weierstrass's Second Theorem, 1885) *Let* $f \in C^{2\pi}$. *Then, for every* $\varepsilon > 0$, *there exists a trig polynomial* T *such that* $\|f - T\| < \varepsilon$.

Lemma 3.2. $\cos nx$ *and* $\sin(n+1)x/\sin x$ *can be written as polynomials of degree exactly* n *in* $\cos x$ *for any integer* $n \geq 0$.

Corollary 3.3. *Any real trig polynomial (3.1) may be written as* $P(\cos x) + Q(\cos x)\sin x$, *where* P *and* Q *are algebraic polynomials of degree at most* n *and* $n-1$, *respectively. If the sum (3.1) represents an even function, then it can be written using only cosines.*

Corollary 3.4. *The collection* $\mathcal{T}$, *consisting of all trig polynomials, is both a subspace and a subring of* $C^{2\pi}$ (that is, $\mathcal{T}$ is closed under both linear combinations and products). *In other words,* $\mathcal{T}$ *is a* subalgebra *of* $C^{2\pi}$.

Corollary 3.5. *Each $f \in C^{2\pi}$ has a best approximation* (on all of $\mathbb{R}$) *out of $\mathcal{T}_n$. If f is an even function, then it has a best approximation which is also even.*

Proof. We only need to prove the second claim, so suppose that $f \in C^{2\pi}$ is even and that $T^* \in \mathcal{T}_n$ satisfies

$$\|f - T^*\| = \min_{T \in \mathcal{T}_n} \|f - T\|.$$

Then, because f is even, $\widetilde{T}(x) = T^*(-x)$ is also a best approximation to f out of $\mathcal{T}_n$; indeed,

$$\begin{aligned} \|f - \widetilde{T}\| &= \max_{x \in \mathbb{R}} |f(x) - T^*(-x)| \\ &= \max_{x \in \mathbb{R}} |f(-x) - T^*(x)| \\ &= \max_{x \in \mathbb{R}} |f(x) - T^*(x)| = \|f - T^*\|. \end{aligned}$$

But now, the *even* trig polynomial

$$\widehat{T}(x) = \frac{\widetilde{T}(x) + T^*(x)}{2} = \frac{T^*(-x) + T^*(x)}{2}$$

is also a best approximation out of $\mathcal{T}_n$ because

$$\|f - \widehat{T}\,\| = \left\| \frac{(f - \widetilde{T}\,) + (f - T^*)}{2} \right\| \le \frac{\|f - \widetilde{T}\,\| + \|f - T^*\|}{2} = \min_{T \in \mathcal{T}_n} \|f - T\|. \qquad \square$$

Weierstrass's Second Theorem

We next give (de La Vallée Poussin's version of) Lebesgue's proof of Weierstrass's second theorem; specifically, we will deduce the second theorem from the first.

Theorem 3.6. *Let $f \in C^{2\pi}$ and let $\varepsilon > 0$. Then, there is a trig polynomial T such that $\|f - T\| = \max_{x\in\mathbb{R}} |f(x) - T(x)| < \varepsilon$.*

Proof. We will prove that Weierstrass's first theorem for $C[-1, 1]$ implies his second theorem for $C^{2\pi}$.

Step 1. If $f \in C^{2\pi}$ is even, then f may be uniformly approximated by even trig polynomials.

If f is even, then it's enough to approximate f on the interval $[0, \pi]$. In this case, we may consider the function $g(y) = f(\arccos y)$, $-1 \le y \le 1$, in $C[-1, 1]$. By Weierstrass's first theorem, there is an algebraic polynomial $p(y)$ such that

$$\max_{-1\le y\le 1} |f(\arccos y) - p(y)| = \max_{0\le x\le \pi} |f(x) - p(\cos x)| < \varepsilon.$$

But $T(x) = p(\cos x)$ is an even trig polynomial! Hence,

$$\|f - T\| = \max_{x\in\mathbb{R}} |f(x) - T(x)| < \varepsilon.$$

Let's agree to abbreviate $\|f - T\| < \varepsilon$ as $f \approx T + \varepsilon$.

Step 2. Given $f \in C^{2\pi}$, there is a trig polynomial T such that $2f(x)\sin^2 x \approx T(x) + 2\varepsilon$.

Each of the functions $f(x) + f(-x)$ and $[f(x) - f(-x)]\sin x$ is *even*. Thus, we may choose even trig polynomials T_1 and T_2 such that

$$f(x) + f(-x) \approx T_1(x) \qquad \text{and} \qquad [f(x) - f(-x)]\sin x \approx T_2(x).$$

Multiplying the first expression by $\sin^2 x$, the second by $\sin x$, and adding, we get

$$2f(x)\sin^2 x \approx T_1(x)\sin^2 x + T_2(x)\sin x \equiv T_3(x),$$

where $T_3(x)$ is still a trig polynomial, and where $f \approx T_3 + 2\varepsilon$ because $|\sin x| \leq 1$.

Step 3. Given $f \in C^{2\pi}$, there is a trig polynomial T such that $2f(x)\cos^2 x \approx T(x) + 2\varepsilon$.

Repeat Step 2 for $f(x - \pi/2)$ and translate: We first choose a trig polynomial $T_4(x)$ such that

$$2f\left(x - \frac{\pi}{2}\right)\sin^2 x \approx T_4(x).$$

That is,

$$2f(x)\cos^2 x \approx T_5(x),$$

where $T_5(x)$ is a trig polynomial.

Finally, by combining the conclusions of Steps 2 and 3, we find that there is a trig polynomial $T_6(x)$ such that $f \approx T_6(x) + 2\varepsilon$. □

Theorem 3.7. *Given $f \in C[-1,1]$ and $\varepsilon > 0$, there exists an algebraic polynomial p such that $\|f - p\| < \varepsilon$.*

Proof. Given $f \in C[-1,1]$, the function $f(\cos x)$ is an even function in $C^{2\pi}$. By our proof of Weierstrass's second theorem (Step 1 of the proof), we may approximate $f(\cos x)$ by an even trig polynomial:

$$f(\cos x) \approx a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx.$$

But, as we've seen, $\cos kx$ can be written as an algebraic polynomial in $\cos x$. Hence, there is some algebraic polynomial p such that $f(\cos x) \approx p(\cos x)$. That is,

$$\max_{0 \le x \le \pi} |f(\cos x) - p(\cos x)| = \max_{-1 \le t \le 1} |f(t) - p(t)| < \varepsilon. \qquad \square$$

Chebyshev Polynomials

The algebraic polynomials $T_n(x)$ satisfying

$$T_n(\cos x) = \cos nx, \text{ for } n = 0, 1, 2, \ldots,$$

are called the *Chebyshev polynomials of the first kind*. Please note that this formula *uniquely* defines T_n as a polynomial of degree exactly n (with leading coefficient 2^{n-1}), and hence uniquely determines the values of $T_n(x)$ for $|x| > 1$, too. The algebraic polynomials $U_n(x)$ satisfying

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x}, \text{ for } n = 0, 1, 2, \ldots,$$

are called the *Chebyshev polynomials of the second kind*. Likewise, note that this formula uniquely defines U_n as a polynomial of degree exactly n (with leading coefficient 2^n).

We will discover many intriguing properties of the Chebyshev polynomials in the next chapter. For now, let's settle for just one: The recurrence formula we gave earlier

$$\cos nx = 2\cos x \, \cos(n-1)x - \cos(n-2)x$$

now becomes

$$T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x), \quad n \geq 2,$$

where $T_0(x) = 1$ and $T_1(x) = x$. This recurrence relation (along with the initial cases T_0 and T_1) may be taken as a definition for the Chebyshev polynomials of the first kind. At any rate, it's now easy to list any number of the Chebyshev polynomials T_n; for example, the next few are $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, and $T_5(x) = 16x^5 - 20x^3 + 5x$.

Properties of the Chebyshev Polynomials

As we've seen, the Chebyshev polynomial $T_n(x)$ is the (unique, real) polynomial of degree n (having leading coefficient 1 if $n = 0$, and 2^{n-1} if $n \geq 1$) such that $T_n(\cos\theta) = \cos n\theta$ for all θ. The Chebyshev polynomials have dozens of interesting properties and satisfy all sorts of curious equations. We'll catalogue just a few.

C1. $T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$.

Proof. It follows from the trig identity $\cos n\theta = 2\cos\theta\,\cos(n-1)\theta - \cos(n-2)\theta$ that $T_n(\cos\theta) = 2\cos\theta\,T_{n-1}(\cos\theta) - T_{n-2}(\cos\theta)$ for all θ; that is, the equation $T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)$ holds for all $-1 \leq x \leq 1$. But because both sides are polynomials, equality must hold for all x. □

The next two properties are proved in essentially the same way:

C2. $T_m(x) + T_n(x) = \frac{1}{2}\left[T_{m+n}(x) + T_{m-n}(x)\right]$ for $m > n$.

C3. $T_m(T_n(x)) = T_{mn}(x)$.

C4. $T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2-1}\right)^n + \left(x - \sqrt{x^2-1}\right)^n\right]$.

Proof. First notice that the expression on the right-hand side is actually a polynomial because, on combining the binomial expansions of $(x + \sqrt{x^2-1}\,)^n$ and $(x - \sqrt{x^2-1}\,)^n$, the odd powers of $\sqrt{x^2-1}$ cancel. Next, for $x = \cos\theta$,

$$\begin{aligned} T_n(x) = T_n(\cos\theta) &= \cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \\ &= \frac{1}{2}\Big[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n \Big] \\ &= \frac{1}{2}\Big[(x + i\sqrt{1-x^2}\,)^n + (x - i\sqrt{1-x^2}\,)^n \Big] \\ &= \frac{1}{2}\Big[(x + \sqrt{x^2-1}\,)^n + (x - \sqrt{x^2-1}\,)^n \Big]. \end{aligned}$$

We've shown that these two polynomials agree for $|x| \le 1$, hence they must agree for all x (real or complex, for that matter). □

For real x with $|x| \ge 1$, the expression $\frac{1}{2}\big[(x + \sqrt{x^2-1}\,)^n + (x - \sqrt{x^2-1}\,)^n\big]$ equals $\cosh(n\cosh^{-1} x)$. In other words, we have

C5. $T_n(\cosh x) = \cosh nx$ for all real x.

The next property also follows from property **C4**.

C6. $T_n(x) \le \big(|x| + \sqrt{x^2-1}\,\big)^n$ for $|x| \ge 1$.

An approach similar to the proof of property **C4** allows us to write x^n in terms of the Chebyshev polynomials $T_0, T_1, \ldots, T_n$.

C7. For n odd, $2^n x^n = \sum_{k=0}^{[n/2]} \binom{n}{k} 2\,T_{n-2k}(x)$; for n even, $2\,T_0$ should be replaced by T_0.

Proof. For $-1 \le x \le 1$,

$$\begin{aligned} 2^n x^n &= 2^n(\cos\theta)^n = (e^{i\theta}+e^{-i\theta})^n \\ &= e^{in\theta} + \binom{n}{1} e^{i(n-2)\theta} + \binom{n}{2} e^{i(n-4)\theta} + \cdots \\ &\qquad \cdots + \binom{n}{n-2} e^{-i(n-4)\theta} + \binom{n}{n-1} e^{-i(n-2)\theta} + e^{-in\theta} \\ &= 2\cos n\theta + \binom{n}{1} 2\cos(n-2)\theta + \binom{n}{2} 2\cos(n-4)\theta + \cdots \\ &= 2\,T_n(x) + \binom{n}{1} 2\,T_{n-2}(x) + \binom{n}{2} 2\,T_{n-4}(x) + \cdots, \end{aligned}$$

where, if n is even, the last term in this last sum is $\binom{n}{[n/2]} T_0$ (because the central term in the binomial expansion, namely $\binom{n}{[n/2]} = \binom{n}{[n/2]} T_0$, isn't doubled in this case). □

C8. The zeros of T_n are $x_k^{(n)} = \cos((2k-1)\pi/2n)$, $k = 1, \ldots, n$. They're real, simple, and lie in the open interval $(-1, 1)$.

Proof. Just check! But notice, please, that the zeros are listed here in *decreasing* order (because cosine decreases). □

C9. Between two consecutive zeros of T_n, there is precisely one root of T_{n-1}.

Proof. It's not hard to check that

$$\frac{2k-1}{2n} < \frac{2k-1}{2\,(n-1)} < \frac{2k+1}{2n},$$

for $k = 1, \ldots, n-1$, which means that $x_k^{(n)} > x_k^{(n-1)} > x_{k+1}^{(n)}$. □

C10. T_n and T_{n-1} have no common zeros.

Proof. Although this is immediate from property **C9**, there's another way to see it: $T_n(x_0) = 0 = T_{n-1}(x_0)$ implies that $T_{n-2}(x_0) = 0$ by property **C1**. Repeating this observation, we would have $T_k(x_0) = 0$ for every $k < n$, including $k = 0$. No good! $T_0(x) = 1$ has no zeros. □

C11. The set $\{x_k^{(n)} : 1 \le k \le n,\ n = 1, 2, \ldots\}$ is dense in $[-1, 1]$.

Proof. Because $\cos x$ is (strictly) monotone on $[0, \pi]$, it's enough to know that the set $\{(2k-1)\pi/2n\}_{k,n}$ is dense in $[0, \pi]$, and for this it's enough to know that $\{(2k-1)/2n\}_{k,n}$ is dense in $[0, 1]$. (Why?) But

$$\frac{2k-1}{2n} = \frac{k}{n} - \frac{1}{2n} \approx \frac{k}{n}$$

for n large; that is, the set $\{(2k-1)/2n\}_{k,n}$ is dense among the rationals in $[0, 1]$. □

C12. The Chebyshev polynomials are mutually *orthogonal* relative to the weight $w(x) = (1-x^2)^{-1/2}$ on $[-1, 1]$.

Proof. For $m \ne n$ the substitution $x = \cos\theta$ yields

$$\int_{-1}^{1} T_n(x)\, T_m(x)\, \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} \cos m\theta \, \cos n\theta \, d\theta = 0,$$

while for $m = n$ we get

$$\int_{-1}^{1} T_n^2(x)\, \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} \cos^2 n\theta \, d\theta = \begin{cases} \pi & \text{if } n = 0 \\ \pi/2 & \text{if } n > 0. \end{cases}$$

□

C13. $|T_n'(x)| \le n^2$ for $-1 \le x \le 1$, and $|T_n'(\pm 1)| = n^2$.

Proof. For $-1 < x < 1$ we have

$$\frac{d}{dx} T_n(x) = \frac{\frac{d}{d\theta} T_n(\cos\theta)}{\frac{d}{d\theta}\cos\theta} = \frac{n \sin n\theta}{\sin\theta}.$$

Thus, $|T_n'(x)| \le n^2$ because $|\sin n\theta| \le n|\sin\theta|$ (which can be easily checked by induction, for example). At $x = \pm 1$, we interpret this derivative formula as a limit (as $\theta \to 0$ and $\theta \to \pi$) and find that $|T_n'(\pm 1)| = n^2$. □

Example 4.14. As we've seen, the Chebyshev polynomals can be generated by a recurrence relation. By reversing the procedure, we could solve for x^n in terms of $T_0, T_1, \dots, T_n$. Here are the first few terms in each of these relations:

$$\begin{aligned} T_0(x) &= 1 & 1 &= T_0(x) \\ T_1(x) &= x & x &= T_1(x) \\ T_2(x) &= 2x^2 - 1 & x^2 &= (T_0(x) + T_2(x))/2 \\ T_3(x) &= 4x^3 - 3x & x^3 &= (3\,T_1(x) + T_3(x))/4 \\ T_4(x) &= 8x^4 - 8x^2 + 1 & x^4 &= (3\,T_0(x) + 4\,T_2(x) + T_4(x))/8 \\ T_5(x) &= 16x^5 - 20x^3 + 5x & x^5 &= (10\,T_1(x) + 5\,T_3(x) + T_5(x))/16 \end{aligned}$$

Note the separation of even and odd terms in each case. Writing ordinary garden variety polynomials in their equivalent Chebyshev form has some distinct advantages for numerical computations. Here's why:

$$1 - x + x^2 - x^3 + x^4 = \frac{15}{6}\,T_0(x) - \frac{7}{4}\,T_1(x) + T_2(x) - \frac{1}{4}\,T_3(x) + \frac{1}{8}\,T_4(x)$$

(after some simplification). Now we see at once that we can get a cubic approximation to $1 - x + x^2 - x^3 + x^4$ on $[-1, 1]$ with error at most $1/8$ by simply dropping the T_4 term on the right-hand side (because $|T_4(x)| \le 1$), whereas simply using $1 - x + x^2 - x^3$ as our cubic approximation could cause an error as big as 1. Pretty slick! This gimmick of truncating the equivalent Chebyshev form is called *economization.*