

(1.3.6) Charpit's Method (General Method of Solving PDEs of Order One but of any Degree)

Let the given PDE of first order and nonlinear in p and q be

$$f(x, y, z, p, q) = 0 \quad \dots(1)$$

To solve this equation, we will use the following charpit's auxiliary equations.

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

After substituting the partial derivatives in charpit's auxiliary equations select the proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q .

Then, putting p and q in the relation $dz = p dx + q dy$ which on integration gives the complete integral of the given equation.

Example 1: Solve $px + qy = pq$ by charpit's method.

Sol. Let $f(x, y, z, p, q) = px + qy - pq = 0 \dots(2)$

charpit's auxiliary equation are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

From (2) $f_x = p$, $f_y = q$, $f_z = 0$, $f_p = x - q$, $f_q = y - p$

$$\therefore \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x - q) - q(y - p)} = \frac{dx}{-x + q} = \frac{dy}{-y + p}$$

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \ln p - \ln q = \ln a \Rightarrow \frac{p}{q} = a \Rightarrow p = aq \dots(3)$$

Substituting (3) in (2)

$$aqx + qy = aq^2 \Rightarrow ax + y = aq \Rightarrow$$

$$q = \frac{ax+y}{a} \text{ then from (3), we get } p = ax + y \dots(4)$$

Putting (4) in $dz = p dx + q dy$

$$dz = (ax + y)dx + \frac{ax + y}{a} dy$$

$$dz = ax dx + y dx + x dy + \frac{y}{a} dy$$

$$dz = ax dx + d(xy) + \frac{y}{a} dy$$

$$z = a \frac{x^2}{2} + xy + \frac{y^2}{2a} + c$$

where a and c are arbitrary constants.

Example 2: Solve $2zx - px^2 - 2qxy + pq = 0$ by charpit's method.

Sol. Let $f(x, y, z, p, q) = 2zx - px^2 - 2qxy + pq = 0 \dots(5)$

$$f_x = 2z - 2px - 2qy, f_y = -2qx, f_z = 2x, f_p = -x^2 + q,$$

$$f_q = -2xy + p$$

Substituting in charpit's auxiliary equations, we get

$$\frac{dp}{2z-2px-2qy+2px} = \frac{dq}{-2qx+2qx} = \frac{dz}{-p(-x^2+q)-q(-2xy+p)} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} \dots(6)$$

Taking the second fraction of (6)

$$dq = 0 \rightarrow q = c \dots(7)$$

Substituting (7) in (5)

$$2zx - px^2 - 2cxy + cp = 0$$

$$p = \frac{2xz-2cxy}{x^2-c} \rightarrow p = \frac{2x(z-cy)}{x^2-c} \dots (8)$$

Putting (7) and (8) in $dz = pdx + qdy$

$$dz = \frac{2x(z-cy)}{x^2-c} dx + cdy \Rightarrow dz - cdy = \frac{2x(z-cy)}{x^2-c} dx$$

$$\frac{dz-cdy}{(z-cy)} = \frac{2x dx}{x^2-c} \dots (9)$$

Integrating (9), $\ln|z - cy| = \ln|x^2 - c| + \ln b$

$$z - cy = b(x^2 - c)$$

$$\boxed{z = b(x^2 - c) + cy}$$

which is a complete integral where b and c are two arbitrary constants.

... Exercises ...

Solve the following equations:

1. $q = 3p^2$
2. $zpq = p + q$
3. $p^2 - y^2q = y^2 - x^2$
4. $(y^2 + 4)xpq - (x^2 + 2) = 0$
5. $q - px - p^2 = 0$
6. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{y}{x}$
7. $p^2 - q^2 = z$

(1.3.7) Using Some Hypotheses in the Solution

Sometimes we need some hypotheses to solve the partial differential equation, here we will give three types of hypotheses.

A) When the equation contains the term (px) or its' powers we use the hypothesis $\boxed{X = \ln x}$

as follows

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x} \quad (\text{since } X = \ln x \Rightarrow \frac{\partial X}{\partial x} = \frac{1}{x})$$

$$\Rightarrow xp = \frac{\partial z}{\partial X}$$

Then substituting this result in the given equation and solve it by previous methods.

Example 1: Solve $z = px$ by hypotheses

Sol. From $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$... (1)

Substituting (1) in the given equation, we get

$$z = \frac{\partial z}{\partial X} \Rightarrow \partial X = \frac{\partial z}{z}$$
 ... (2)

Integrating (2), $X = \ln z + \ln \phi(y)$... (3)

where ϕ is an arbitrary function for y

replacing X in (3) to get the general solution

$$\ln x = \ln(\phi(y) \cdot z)$$

$$\Rightarrow \boxed{z = \frac{x}{\phi(y)}} \dots (4)$$

Example 2: Solve $q = px + p^2x^2$ by hypotheses

Sol. Given that $q = px + (px)^2$... (5)

from $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$... (6)

Substituting (6) in (5), we get

$$q = \frac{\partial z}{\partial X} + \left(\frac{\partial z}{\partial X}\right)^2 \quad \dots (7)$$

Let $\frac{\partial z}{\partial X} = t$ then (7) will be

$$q = t + t^2 \quad \dots (8)$$

The equation (8) is of the form $f(t, q) = 0$

Then let $t = a$ and $q = b$, putting in (8) $b = a + a^2$

Substituting in $z = aX + by + c$

$$\Rightarrow z = aX + (a + a^2)y + c \dots (9)$$

where c is an arbitrary constant

replacing X in (9) to get the complete integral

$$\boxed{z = a \ln x + (a + a^2)y + c}$$