Section(1.1): Origin of Partial Differential Equations

(1.1.1) Introduction:

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

(1.1.2) Definition Partial Differential Equations(PDE)

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a (PDE).

For examples of partial differential equations we list the following:

1. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy$
2. $(\frac{\partial z}{\partial x})^2+\frac{\partial^3 z}{\partial y^3}=2x(\frac{\partial z}{\partial y})$
3. $z(\frac{\partial z}{\partial x})+\frac{\partial z}{\partial y}=x$
4. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz$
5. $\frac{\partial^2 z}{\partial x^2}=(1+\frac{\partial z}{\partial y})^{\frac{1}{2}}$
6. $y\{(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2\}=z(\frac{\partial z}{\partial y})$

(1.1.3) <u>Definition: Order of a Partial Differential Equation (O.P.D.E.)</u>

The order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

The equations in examples (1),(3),(4) and (6) are of the first order ,(5) is of the second order and (2) is of the third order.

(1.1.4)<u>Definition: Degree of a Partial Differential Equation (DPDE)</u>

The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized, i.e. made free from radicals and fractions so for as derivatives are concerned. in (1.1.2), equations (1),(2),(3) and (4) are of first degree while equations (5) and (6) are of second degree.

(1.1.5) <u>Definition: Linear and Nonlinear Partial Differential Equations</u>

A partial differential equation is said to be (Linear) if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied . A partial differential equation which is not linear is called a (nonlinear) partial differential equation.

In (1.1.2), equations (1) and (4) are linear while equation (2),(3),(5) and (6) are non-linear.

(1.1.6) Notations:

When we consider the case of two independent variables we usually assume them to be x and y and assume (z) to be the dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y} \text{ and } t = \frac{\partial^2 z}{\partial y^2}$$

In case there are n independent variables, we take them to be $x_1, x_2, \dots \dots \dots, x_n$ and z is than regarded as the dependent variable. In this case we use the following notations:

$$p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, \dots \dots \dots, p_n = \frac{\partial z}{\partial x_n}$$

Sometimes the partial differentiations are also denoted by making use of suffixes. Thus we write :

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

and so on.

(1.1.7) Classification of First Order PDEs into:

linear, semi-linear, quasi-linear and nonlinear equations

*__linear equation:__ A first order equation $f(x, y, z, p, q) = 0$

Is known as linear if it is linear in p , q and z , that is ,if given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

for example:

1. $yx^2p + xy^2q = xyz + x^2y^3$
2. $p + q = z + xy$

are both first order LPDEs.

***Semi-linear equation:** A first order PDE $f(x, y, z, p, q) = 0$

Is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only. i.e if the given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

for example:

1. $xyp + x^2yq = x^2y^2z^2$
2. $yp + xq = \frac{x^2y^2}{z^2}$

are both semi-linear equations.

***Quasi-linear equation:** A first order PDE $f(x, y, z, p, q) = 0$

Is known as quasi-linear equation, if it is linear in p and q. i.e if the given equation is of the form:

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

for example:

1. $x^2zp + y^2zq = xy$
2. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

are both quasi-linear equation.

***Nonlinear equation:** A first order PDE $f(x, y, z, p, q) = 0$, if the degree of the dependent variable or its partial derivatives is not equal to one or if they are multiply by each other, the equation will be nonlinear.

for example:

1. $p^2 + q^2 = 1$
2. $pq = z$
3. $x^2p^2 + y^2q^2 = z^2$

are all nonlinear PDEs.

Note: The two classifications (semi-linear) and (quasi-linear) are classifications of the nonlinear equation.
