Partial Differential Equations

المعادلات التفاضلية الجزئية

المرحلة الثالثة

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Section(1.1): Origin of Partial Differential Equations

(1.1.1) Introduction:

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

(1.1.2) Definition Partial Differential Equations(PDE)

An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a (PDE).

For examples of partial differential equations we list the following:

1. $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=z+xy$
2. $(\frac{\partial z}{\partial x})^2+\frac{\partial^3 z}{\partial y^3}=2x(\frac{\partial z}{\partial y})$
3. $z(\frac{\partial z}{\partial x})+\frac{\partial z}{\partial y}=x$
4. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz$
5. $\frac{\partial^2 z}{\partial x^2}=(1+\frac{\partial z}{\partial y})^{\frac{1}{2}}$
6. $y\{(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2\}=z(\frac{\partial z}{\partial y})$

(1.1.3) <u>Definition: Order of a Partial Differential Equation (O.P.D.E.)</u>

The order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

The equations in examples (1),(3),(4) and (6) are of the first order ,(5) is of the second order and (2) is of the third order.

(1.1.4)<u>Definition: Degree of a Partial Differential Equation (DPDE)</u>

The degree of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized, i.e. made free from radicals and fractions so for as derivatives are concerned. in (1.1.2), equations (1),(2),(3) and (4) are of first degree while equations (5) and (6) are of second degree.

(1.1.5) <u>Definition: Linear and Nonlinear Partial Differential Equations</u>

A partial differential equation is said to be (Linear) if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied . A partial differential equation which is not linear is called a (nonlinear) partial differential equation.

In (1.1.2), equations (1) and (4) are linear while equation (2),(3),(5) and (6) are non-linear.

(1.1.6) Notations:

When we consider the case of two independent variables we usually assume them to be x and y and assume (z) to be the dependent variable. We adopt the following notations throughout the study of partial differential equations.

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \, \partial y} \text{ and } t = \frac{\partial^2 z}{\partial y^2}$$

In case there are n independent variables, we take them to be $x_1, x_2, \dots \dots \dots, x_n$ and z is than regarded as the dependent variable. In this case we use the following notations:

$$p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, \dots \dots \dots, p_n = \frac{\partial z}{\partial x_n}$$

Sometimes the partial differentiations are also denoted by making use of suffixes. Thus we write :

$$u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

and so on.

(1.1.7) Classification of First Order PDEs into:

linear, semi-linear, quasi-linear and nonlinear equations

*__linear equation:__ A first order equation $f(x, y, z, p, q) = 0$

Is known as linear if it is linear in p , q and z , that is ,if given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y)z + S(x, y)$$

for example:

1. $yx^2p + xy^2q = xyz + x^2y^3$
2. $p + q = z + xy$

are both first order LPDEs.

***Semi-linear equation:** A first order PDE $f(x, y, z, p, q) = 0$

Is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only. i.e if the given equation is of the form:

$$P(x, y)p + Q(x, y)q = R(x, y, z)$$

for example:

1. $xyp + x^2yq = x^2y^2z^2$
2. $yp + xq = \frac{x^2y^2}{z^2}$

are both semi-linear equations.

***Quasi-linear equation:** A first order PDE $f(x, y, z, p, q) = 0$

Is known as quasi-linear equation, if it is linear in p and q. i.e if the given equation is of the form:

$$P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$$

for example:

1. $x^2zp + y^2zq = xy$
2. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

are both quasi-linear equation.

***Nonlinear equation:** A first order PDE $f(x, y, z, p, q) = 0$, if the degree of the dependent variable or its partial derivatives is not equal to one or if they are multiply by each other, the equation will be nonlinear.

for example:

1. $p^2 + q^2 = 1$
2. $pq = z$
3. $x^2p^2 + y^2q^2 = z^2$

are all nonlinear PDEs.

Note: The two classifications (semi-linear) and (quasi-linear) are classifications of the nonlinear equation.

Section (1.2): Derivation of Partial Differential Equation by the Elimination of Arbitrary Constants

For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a and b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations $F(x, y, z, a, b) = 0$, $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$

The equation free from a and b will be the required partial differential equation.

Three situations may arise:

Situation (1):

When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.

Example: Consider $z = ax + y$ …………(1)

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. x, we get

$\frac{\partial z}{\partial x} = a$ …………(2)

Differentiating (1) partially w.r.t. y, we get

$\frac{\partial z}{\partial y} = 1$ …………(3)

Eliminating abetween (1) and (2) yields

$z = x\left(\frac{\partial z}{\partial x}\right) + y$ ....(4)

Since (3) does not contain arbitrary constant, so (3) is also PDE under consideration thus, we get two PDEs (3) and (4).

Situation (2):

When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique PDE of order one.

Example: Eliminate a and b from

$az + b = a^2x + y$ ....(1)

Differencing (1) partially w.r.t. x and y, we have

$a\left(\frac{\partial z}{\partial x}\right) = a^2$ ....(2)

$a\left(\frac{\partial z}{\partial y}\right) = 1$ ....(3)

Eliminating a from (2) and (3), we have

$$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

which is the unique PDE of order one.

Situation (3):

When the number of arbitrary constants is greater than the number of independent variables. Then the elimination of arbitrary constants leads to a partial differential equations of order usually greater than one.

Example: Eliminate a, b and c from

$z = ax + by + cxy$ ....(1)

Differentiating (1) partially w.r.t. x and y, we have

$\frac{\partial z}{\partial x} = a + cy$ ....(2) $\frac{\partial z}{\partial y} = b + cx$ ....(3)

from (2) and (3) $\frac{\partial^2 z}{\partial x^2} = 0,\ \frac{\partial^2 z}{\partial y^2} = 0$ ....(4)

$\frac{\partial^2 z}{\partial x\,\partial y} = c$ ....(5)

Now, multiply (2) by x and (3) by y, we get

$x\frac{\partial z}{\partial x} = ax + cxy$ ....(6)

and

$y\frac{\partial z}{\partial y} = by + cxy$ ....(7), by adding (6) to (7), we obtain

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underbrace{ax + by + cxy} + cxy$$

from (1) and (5)

$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\frac{\partial^2 y}{\partial x\,\partial y}$ ....(8)

Thus, we get three PDEs given by (4) and (8) which are all of order two.

Examples:

Example1: Find a PDE by eliminating a and b from

$$z = ax + by + a^2 + b^2$$

Sol. Given $z = ax + by + a^2 + b^2$ …………(1)

Differentiating (1) partially with respect to x and y,

we get $\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$

Substituting these values of a and b in (1), we see that the arbitrary constants a and b are eliminated and we obtain

$$z = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2$$

which is required PDE.

Example2: Eliminate arbitrary constants a and b from

$z = (x - a)^2 + (y - b)^2$ to form the PDE.

Sol. Given $z = (x - a)^2 + (y - b)^2$ …………(1)

Differentiating (1) partially with respect to x and y, to get

$\frac{\partial z}{\partial x} = 2(x - a)$, $\frac{\partial z}{\partial y} = 2(y - b)$

Squaring and adding these equations, we have

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4(x - a)^2 + 4(y - b)^2$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4[(x - a)^2 + (y - b)^2]$$

$\left(\frac{\partial z}{\partial x}\right)^2 + (\frac{\partial z}{\partial y})^2 = 4z$ using (1)

Example 3: Find the PDEs by eliminating arbitrary constants a and b from the following relations:

(a) $z = a(x + y) + b$ (b) $z = ax + by + ab$

(c) $z = ax + a^2y^2 + b$ (d) $z = (x + a)(y + b)$

Sol. (a) Given $z = a(x + y) + b$ …………(1)

Differentiating (1) w.r.t. x and y, we get

$\frac{\partial z}{\partial x} = a \quad , \frac{\partial z}{\partial y} = a$

Eliminating a between these, we get

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ which is the required PDE.

(b)Try by yourself (c)Try by yourself (d)Try by yourself

… Exercises …

Ex.(1): Eliminate a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ to form the PDE.

Ex.(2): Eliminate h and k from the equation $(x - h)^2 + (y - k)^2 + z^2 = \alpha^2$to form the PDE.

Ex.(3): Eliminate a and b from the equation $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ to form the PDE.

Ex.(4): Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding PDEs

(1) $z = ax^3 + by^3$,(a and b)

(2) $4z = \left[ax + \left(\frac{y}{a}\right) + b\right]^2$,(a and b)

(3) $z = ax^2 + bxy + cy^2$, (a,b,c)

Section (1.3): Methods for solving linear and nonlinear partial differential equations of order one

(1.3.1) Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are function of x, y, z.

A quasi-linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are function of x, y, z. Such a partial differential equation is known as (Lagrange equation), for example: * $xyp + yzq = zx$

* $(x - y)p + (y - z)q = z - x$

(1.3.2) Working Rule for solving $Pp + Qq = R$ by Lagrange's method

Step 1. Put the given quasi-linear PDE of the first order in the standard form $Pp + Qq = R$ …………(1)

Step 2. Write down Lagrange's auxiliary equations for (1) namely $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ …………(2)

Step 3. Solve (2) by using the method for solving ordinary differential equation of order one. The equation (2) gives three ordinary differential equations. every two of them are independent and give a solution.

Let $u(x, y, z) = a$ and $v(x, y, z) = b$, then the (general solution) is $\emptyset(u, v) = 0$, wher $\emptyset$ is an arbitrary function and the complete solution is $u = \alpha v + \beta$ where α, β are arbitrary constant.

<u>Ex.1:</u> Solve $2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x$

Sol. Given $2\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = 2x$ ....(1)

The Lagrange's auxiliary for (1) are

$\frac{dx}{2} = \frac{dy}{-3} = \frac{dz}{2x}$ ....(2)

Taking the first two fractions of (2), we have

$\frac{dx}{2} = \frac{dy}{-3} \rightarrow -3dx - 2dy = 0$ ....(3)

Integrating (3), $-3x - 2y = a$ ....(4)

a being an arbitrary constant

Next, taking the first and the last fractions of (2), we get

$\frac{dx}{2} = \frac{dz}{2x} \rightarrow xdx = dz \rightarrow xdx - dz = 0$ ......(5)

Integrating (5), $\frac{x^2}{2} - z = b$ ....(6)

b being an arbitrary constant

From (4) and (6) the required general solution is

$$\emptyset(a, b) = 0 \rightarrow \emptyset\left(-3x - 2y, \frac{x^2}{2} - z\right) = 0$$

Where $\emptyset$ is an arbitrary function.

Ex.2: Solve $\left(\frac{y^2z}{x}\right)p + xzq = y^2$

Sol. Given $\left(\frac{y^2z}{x}\right)p + xzq = y^2$ …………(1)

The Lagrange's auxiliary equation for (1) are

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{…………(2)}$$

Taking the first two fractions of (2), we have

$$x^2zdx = y^2zdy \quad \rightarrow \; x^2dx - y^2dy = 0 \quad \text{…………(3)}$$

Integrating (3), $\frac{x^3}{3} - \frac{y^3}{3} = a \;\rightarrow\; x^3 - y^3 = a_1$ ……(4)

a_1 being an arbitrary constant.

Next, taking the first and the last fractions of (2), we get

$$xy^2dx = y^2zdz \;\rightarrow\; xdx - zdz = 0 \quad \text{…………(5)}$$

Integrating (5), $\frac{x^2}{2} - \frac{z^2}{2} = b \;\rightarrow x^2 - z^2 = b_1$ …(6)

b_1 being an arbitrary constant

From (4) and (6) the general solution is

$$\emptyset(a_1, b_1) = 0 \;\rightarrow\; \emptyset(x^3 - y^3, x^2 - z^2) = 0$$

Ex.3: Solve $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$

Sol. Given $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$ …………(1)

The Lagrange's auxiliary equation for (1) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{xyt} \quad \text{…………(2)}$$

Taking the first two fractions of (2), we have

$$\frac{dx}{x} = \frac{dy}{y} \;\rightarrow\; \frac{dx}{x} - \frac{dy}{y} = 0 \quad \text{…………(3)}$$

Integrating (3), $\ln x - \ln y = \ln a \;\; \rightarrow \frac{x}{y} = a \;\; \ldots\ldots(4)$

Taking the second and the third fractions of (2), we get

$\frac{dy}{y} = \frac{dt}{t} \;\; \rightarrow \;\; \frac{dy}{y} - \frac{dt}{t} = 0 \;\; \ldots\ldots\ldots\ldots(5)$

Integrating (5), $\ln y - \ln t = \ln b \;\; \rightarrow \frac{y}{t} = b \;\; \ldots\ldots(6)$

Next, taking the second and the last fractions of (2), we get

$\frac{dy}{y} = \frac{dz}{xyt} \;\; \rightarrow \;\; xtdy - dz = 0 \;\;\; \ldots\ldots\ldots\ldots(7)$

Substituting (4) and (6) in (7), we get

$\frac{a}{b} y^2 dy - dz = 0 \qquad \ldots\ldots\ldots\ldots(8)$

Integrating (8), $\frac{a}{3b} y^3 - z = c$

Using (4) and (6), $\frac{1}{3} xyt - z = c \qquad \ldots\ldots\ldots\ldots(9)$

Where a, b and c are an arbitrary constant

The general solution is

$$\emptyset(a, b, c) = 0 \;\; \rightarrow \;\; \emptyset\left(\frac{x}{y}, \frac{y}{t}, \frac{1}{3} xyt - z\right) = 0$$

$\emptyset$ being an arbitrary function.

<u>Rule</u>: for any equal fractions, if the sum of the denominators equal to zero, then the sum of the numerators equal to zero also.

Now, Return to the last example depending on the Rule above we will find the constant c.

Multiplying each fraction in Lagrange's auxiliary (2) by $yt, xt, xy, -3$ respectively, we get the sum of the denominators is

$xyt + xyt + xyt - 3xyt = 0$ …………(10)

Then the sum of the numerators equal to zero also:

$ytdx + xtdy + xydt - 3dz = 0 \rightarrow \quad d(xyt) - 3dz = 0$……(11)

Integrating (11), $xyt - 3z = c$ …………(12)

Note that (12) and (9) are the same.

<u>Ex.4:</u> Solve $(\mathbf{y - z})\mathbf{p} + (\mathbf{z - x})\mathbf{q} = \mathbf{x - y}$

Sol. Given $(y - z)p + (z - x)q = x - y$ …………(1)

The Lagrange's auxiliary equations for (1) are

$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$ …………(2)

The sum of the denominators is

$$y - z + z - x + x - y = 0$$

Then, the sum of the numerators is equal to zero also, (by Rule)

$dx + dy + dz = 0$ …………(3)

Integrating (3), $x + y + z = a$ …………(4)

To find b, multiplying (2) by x, y, z resp. the sum of the denominators is

$x(y - z) + y(z - x) + z(x - y) = xy - xz + yz - xy + zx - yz = 0$

Then, the sum of the numerators is equal to zero

$xdx + ydy + zdz = 0$ …………(5)

Integrating (5), $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = b$ …………(6)

Where, a and b are arbitrary constants.

The general solution is

$$\emptyset(\mathrm{a,b}) = 0 \;\; \rightarrow \;\; \emptyset\left(\mathrm{x+y+z}, \frac{\mathrm{x}^2}{2} + \frac{\mathrm{y}^2}{2} + \frac{\mathrm{z}^2}{2}\right) = 0$$

… Exercises …

Solve the following partial differential equation:

1. $p\ tanx + q\ tany = tanz$.

2. $y^2p - xyq = x(z - 2y)$.

3. $(x^2 + 2y^2)p - xyq = xz$.

4. $xp + yq = z$.

5. $(-a + x)p + (-b + y)q = (-c + z)$.

6. $x^2p + y^2q + z^2 = 0$.

7. $yzp + zxq = xy$.

8. $y^2p + x^2q = x^2y^2z^2$.

9. $p - q = \frac{z}{(x+y)}$

(1.3.2) The equation of the form $f(p,q)=0$

Here we consider equations in which p and q occur other than in the first degree, that is nonlinear equations. To solve the equation

$f(p,q)=0$ ………(1)

Taking $p = constant = a$ …………(2)

$q = constant = b$ …………(3)

Substituting (2),(3) in (1), we get

$F(a,b)=0 \rightarrow b = F_1(a) \quad or \quad a = F_2(b)$………..(4)

From $dz = pdx + qdy$ …………(5)

Using (2),(3) $\rightarrow dz = adx + bdy$ ………..(6)

Integrating (6), $z = ax + by + c$ ………..(7)

Where c is an arbitrary constant

Substituting (4) in (7) to obtain the complete integral (complete solution)

$z = ax + F_1(a)y + c \quad or \quad z = F_2(b)x + by + c$ ……….(8)

Ex.1: **Solve $p^2 + p = q^2$**

Sol. $p^2 + p - q^2 = 0$ ………..(1)

The equation (1) of the form $f(p,q) = 0$

Let $p = a \ , q = b$

Substituting in (1)

$$a^2 + a - b^2 = 0 \rightarrow b^2 = a^2 + a \rightarrow b = \pm\sqrt{a^2 + a}$$

The complete integral is

$$z = ax + by + c$$

$$= ax \pm \sqrt{a^2 + a}y + c$$

Where c is an arbitrary constant.

<u>Ex.2</u>: Solve pq = k , where k is a constant.

Sol. Given that $pq = k$ …………(1)

Since (1) is of the form $f(p, q) = 0$, it's solution is

$z = ax + by + c$ …………(2)

Let $p = a$, $q = b$, substituting in (1) , then $ab = k \to b = \frac{k}{a}$ ...(3)

Putting (3) in (2), to get the complete solution

$z = ax + \frac{k}{a}y + c$; c is an arbitrary constant .

<u>Ex.3</u>: Solve $\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial y} = (\frac{\partial z}{\partial y})^3$

Sol. Given that $p - 3q = q^3$ ……….. (1)

Since (1) is of the form $f(p, q) = 0$, then

Let $p = a$, $q = b$

Substituting in (1) , $a - 3b = b^3 \rightarrow a = b^3 + 3b$ ………(2)

Putting (2) in the equation $z = ax + by + c$, we get

$$z = (b^3 + 3b)x + by + c$$

Where c is an arbitrary constant

The equation (3) is the complete integral.

(1.3.3) <u>The Equation of the form $z = px + qy + f(p, q)$</u>

A first order partial differential equation is said to be of **Clariaut** form if it can be written in the form

$$z = px + qy + f(p, q) \qquad \ldots(1)$$

to solve this equation taking $p = a$, $q = b$ and substituting in (1), so the complete integral is

$$z = ax + by + f(a, b) \qquad \ldots(2)$$

Example 1: Solve $z = px + qy + pq$

Sol. The given equation is of the form $z = px + qy + f(p, q)$

let $p = a$ and $q = b$ substituting in the given equation, so the complete integral is

$$\boxed{z = ax + by + ab}$$

where a, b being arbitrary constant.

Example 2: Solve $x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = z - 5\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$

Sol. Rearrange the given equation, we have

$$x\,p + y\,q = z - 5p + pq$$

$$z = x\,p + y\,q + 5p - pq \qquad \ldots(3)$$

Equation (3) is of Clariaut form

let $p = a$ and $q = b$ substituting in (3), then the complete integral is $\boxed{z = ax + by + 5a - ab}$

where a, b being arbitrary constant.

Example 3: Solve $px + qy = z - p^3 - q^3$

Sol. Rearrange the given equation, we have

$$z = px + qy + p^3 + q^3 \qquad \ldots(4)$$

let $p = a$ and $q = b$ substituting in (4)

$\boxed{z = ax + by + a^3 + b^3}$ that is the complete integral and a, b being arbitrary constants.

(1.3.4) The Equation of the form $f(z, p, q) = 0$

To solve the equation of the form

$f(z, p, q) = 0 \quad \ldots(1)$

1. Let $\quad u = x + ay \qquad \ldots(2)$

   where a is an arbitrary constant

2. Replace p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (1) as follows,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} = a\frac{dz}{du} \qquad \ldots(3)$

from (2) $\quad \frac{\partial u}{\partial x} = 1 \quad$ and $\quad \frac{\partial u}{\partial y} = a$

3. Substituting (3) in (1) and solve the resulting ordinary differential equation of first order by usual methods.

4. Next, replace u by $x + ay$ in the solution obtained in step 3 to get the complete solution.

Example 1: Solve $\boldsymbol{z = p + q}$

Sol. Given equation is $\quad z = p + q \qquad \ldots(4)$

which is of the form $f(z, p, q) = 0$. Let $u = x + ay$ where a is an arbitrary constant.

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (4), we get

$$z = \frac{dz}{du} + a\ \frac{dz}{du}$$

$$\Rightarrow z = (1+a)\ \frac{dz}{du}$$

$$\Rightarrow du = (1+a)\frac{dz}{z} \qquad \ldots(5)$$

Integrating (5), $u + c = (1+a)\ln z$

where c is an arbitrary constant

Replacing u,

$$x + ay + c = \ln z^{(1+a)}$$

$$\Rightarrow\ e^{x+ay+c} = z^{(1+a)}$$

$$\Rightarrow\ z = e^{\frac{x+ay+c}{1+a}} \qquad \ldots(6)$$

and that is the complete integral.

Example 2: Solve $\left(\frac{\partial z}{\partial x}\right)^2 z - \left(\frac{\partial z}{\partial y}\right)^2 = 1$

Sol. Rearrange the given equation, we have

$p^2 z - q^2 = 1 \ldots(7)$

This equation is of the form $\mathrm{f(z,p,q)} = 0$

Let $u = x + ay$, where a is an arbitrary constant

Now, replacing p and q by $\frac{dz}{du}$ and $a\frac{dz}{du}$ respectively in (7), we get

$$\left(\frac{dz}{du}\right)^2 z - \left(a\ \frac{dz}{du}\right)^2 = 1$$

$$\Rightarrow (z-a^2)\left(\frac{dz}{du}\right)^2 = 1$$

$\Rightarrow \pm\sqrt{z-a^2}\,\frac{dz}{du} = 1$ by taking the square root

$\Rightarrow \pm\sqrt{z-a^2}\,dz = du$…(8)

Integrating (8),

$\pm\frac{2}{3}(z-a^2)^{3/2} = u + c$…(9)

Replacing u in (9) to get the complete integral

$$\boxed{\pm\frac{2}{3}(z-a^2)^{\frac{3}{2}} = x + ay + c}$$

(1.3.5) The Equation of the form $f_1(x,p) = f_2(y,q) = 0$

In this form z does not appear and the terms containing x and p are on one side and those containing y and q on the other side.

To solve this equation putting

$f_1(x,p) = f_2(y,q) = a$…(1)

where a is an arbitrary constant

$\therefore\ f_1(x,p) = a \quad \Rightarrow \quad p = g_1(x,a)$…(2)

$f_2(y,q) = a \quad \Rightarrow \quad q = g_2(y,a)$…(3)

Substituting (2) and (3) in $dz = pdx + qdy$, we get

$dz = g_1(x,a)dx + g_2(y,a)dy$…(4)

Integrating (4),

$$\boxed{z = \int g_1(x,a)dx + \int g_2(y,a)dy + b}$$

which is a complete integral containing two arbitrary constants a and b.

Example 1: Solve $p = 2xq^2$

Sol. Separating p and x from q and y, the given equation reduces to$\frac{p}{x} = 2q^2$…(5)

Equating each side to an arbitrary constant a, we have

$$\frac{p}{x} = a \qquad \Longrightarrow p = ax$$

$$2q^2 = a \quad \Longrightarrow q = \pm\sqrt{\frac{a}{2}}$$

Putting these values of p and q in

$dz = pdx + qdy$, we get

$$dz = axdx \pm \sqrt{\frac{a}{2}}dy \qquad \text{…(6)}$$

Integrating (6), $z = \frac{a}{2}x^2 \pm \sqrt{\frac{a}{2}}\, y + b$

where a and b are two arbitrary constants.

Example 2: Solve $xq - y^2p - x^2y^2 = 0$

Sol. Separating p and x from q and y, the given equation reduces to $\frac{p+x^2}{x} = \frac{q}{y^2}$…(7)

Equating each side to an arbitrary constant a, we have

$$\frac{p+x^2}{x} = a \qquad \Longrightarrow \quad p = ax - x^2 \qquad \text{…(8)}$$

$$\frac{q}{y^2} = a \qquad \Longrightarrow \quad q = a\,y^2 \qquad \text{…(9)}$$

Putting (8) and (9) in $dz = pdx + qdy$, we get

$$dz = (ax - x^2)dx + ay^2dy \qquad \ldots(10)$$

Integrating (10) , $\boxed{z = \frac{ax^2}{2} - \frac{x^3}{3} + a\frac{y^3}{3} + b}$

which is a complete integral containing two arbitrary constants a and b.

Example 3: Solve $\boldsymbol{p - 3x^2 = q^2 - y}$

Sol. Equating each side to an arbitrary constant a, we get

$$p - 3x^2 = a \qquad \Longrightarrow \qquad p = a + 3x^2 \qquad \ldots(11)$$

$$q^2 - y = a \qquad \Longrightarrow \qquad q = \pm\sqrt{a + y} \qquad \ldots(12)$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (a + 3x^2)dx \pm \sqrt{a + y}dy \qquad \ldots(13)$$

Integrating (13) , $\boxed{z = ax + x^3 \pm \frac{2}{3}(a + y)^{3/2} + b}$

which is a complete integral containing two arbitrary constant a and b.

(1.3.6) Charpit's Method (General Method of Solving PDEs of Order One but of any Degree)

Let the given PDE of first order and nonlinear in p and q be

$f(x,y,z,p,q)=0$ …(1)

To solve this equation, we will use the following charpit's auxiliary equations.

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

After substituting the partial derivatives in charpit's auxiliary equations select the proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Then, putting p and q in the relation $dz=pdx+qdy$ which on integration gives the complete integral of the given equation.

Example 1: Solve $px+qy=pq$ by charpit's method.

Sol. Let $f(x,y,z,p,q)=px+qy-pq=0$…(2)

charpit's auxiliary equation are

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

From (2) $f_x=p$, $f_y=q$, $f_z=0$, $f_p=x-q$, $f_q=y-p$

$$\therefore \quad \frac{dp}{p}=\frac{dq}{q}=\frac{dz}{-p(x-q)-q(y-p)}=\frac{dx}{-x+q}=\frac{dy}{-y+p}$$

$\frac{dp}{p} = \frac{dq}{q} \Longrightarrow \ln p - \ln q = \ln a \Rightarrow \frac{p}{q} = a \quad \Rightarrow \quad p = aq \quad$ ….(3)

Substituting (3) in (2)

$aqx + qy = aq^2 \quad \Rightarrow \quad ax + y = aq \quad \Rightarrow$

$q = \frac{ax+y}{a}$ then from (3), we get p= $ax + y$ ….(4)

Putting (4) in $dz = pdx + qdy$

$$dz = (ax + y)dx + \frac{ax + y}{a}dy$$

$$dz = axdx + ydx + xdy + \frac{y}{a}dy$$

$$dz = axdx + d(xy) + \frac{y}{a}dy$$

$$z = a\frac{x^2}{2} + xy + \frac{y^2}{2a} + c$$

where a and c are arbitrary constants.

Example 2: Solve $2zx - px^2 - 2qxy + pq = 0$ by charpit's method.

Sol. Let $f(x, y, z, p, q) = 2zx - px^2 - 2qxy + pq = 0$…(5)

$f_x = 2z - 2px - 2qy$, $f_y = -2qx$, $f_z = 2x$, $f_p = -x^2 + q$,

$f_q = -2xy + p$

Substituting in charpit's auxiliary equations , we get

$\frac{dp}{2z-2px-2qy+2px} = \frac{dq}{-2qx+2qx} = \frac{dz}{-p(-x^2+q)-q(-2xy+p)} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p}$…(6)

Taking the second fraction of (6)

$dq = 0 \quad \rightarrow \quad q = c$…(7)

Substituting (7) in (5)

$$2zx - px^2 - 2cxy + cp = 0$$

$p = \frac{2xz-2cxy}{x^2-c} \quad \rightarrow \quad p = \frac{2x(z-cy)}{x^2-c}$…(8)

Putting (7) and (8) in $dz = pdx + qdy$

$$dz = \frac{2x(z-cy)}{x^2-c}dx + cdy \Longrightarrow dz - cdy = \frac{2x(z-cy)}{x^2-c}dx$$

$\frac{dz-cdy}{(z-cy)} = \frac{2x\,dx}{x^2-c}$…(9)

Integrating (9), $ln|z-cy| = \ln|x^2-c| + \ln b$

$$z - cy = b(x^2-c)$$

$$\boxed{z = b(x^2-c) + cy}$$

which is a complete integral where b and c are two arbitrary constants.

… Exercises …

Solve the following equations:

1. $q = 3p^2$
2. $zpq = p + q$
3. $p^2 - y^2q = y^2 - x^2$
4. $(y^2+4)xpq - (x^2+2) = 0$
5. $q - px - p^2 = 0$
6. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{y}{x}$
7. $p^2 - q^2 = z$

(1.3.7) Using Some Hypotheses in the Solution

Sometimes we need some hypotheses to solve the partial differential equation, here we will give three types of hypotheses.

A) When the equation contains the term (px) or its' powers we use the hypothesis $\boxed{X = \ln x}$

as follows

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial X}{\partial X} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}.\frac{1}{x} \text{ (since } X = \ln x \implies \frac{\partial X}{\partial x} = \frac{1}{x})$$

$$\implies xp = \frac{\partial z}{\partial X}$$

Then substituting this result in the given equation and solve it by previous methods.

Example 1: Solve $z = px$ by hypotheses

Sol. From $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$ …(1)

Substituting (1) in the given equation, we get

$z = \frac{\partial z}{\partial X} \implies \partial X = \frac{\partial z}{z}$ …(2)

Integrating (2), $X = \ln z + \ln \emptyset(y)$ …(3)

where $\emptyset$ is an arbitrary function for y

replacing X in (3) to get the general solution

$\ln x = \ln(\emptyset(y).z)$

$\implies \boxed{z = \frac{x}{\emptyset(y)}}$ …(4)

Example 2: Solve $q = px + p^2x^2$ by hypotheses

Sol. Given that $q = px + (px)^2$ …(5)

from $X = \ln x$ we have $xp = \frac{\partial z}{\partial X}$ …(6)

Substituting (6) in (5), we get

$q = \frac{\partial z}{\partial X} + \left(\frac{\partial z}{\partial X}\right)^2$ …(7)

Let $\frac{\partial z}{\partial X} = t$ then (7) will be

$q = t + t^2$ …(8)

The equation (8) is of the form $f(t, q) = 0$

Then let $t = a$ and $q = b$, putting in (8) $b = a + a^2$

Substituting in $z = aX + by + c$

$\Longrightarrow z = aX + (a + a^2)y + c$…(9)

where c is an arbitrary constant

replacing X in (9) to get the complete integral

$\boxed{z = a\, ln\, x + (a + a^2)y + c}$

B) When the equation contains the term (qy) or its' powers we use the hypothesis $\boxed{Y = \ln y}$

as follows:

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} . \frac{\partial Y}{\partial Y} = \frac{\partial z}{\partial Y} . \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial Y} . \frac{1}{y}$ (since $Y = \ln y \implies \frac{\partial Y}{\partial y} = \frac{1}{y}$)

$$\implies qy = \frac{\partial z}{\partial Y}$$

Then solving by the same way in (A).

Example 3: Solve $2p + qy = 4$ by hypotheses

Sol. Given that $2p + qy = 4$ …(10)

from $Y = \ln y$ we have $qy = \frac{\partial z}{\partial Y}$ …(11)

Substituting (11) in (10), we get

$$2p + \frac{\partial z}{\partial Y} = 4$$

Let $\frac{\partial z}{\partial Y} = t$ then,

$2p + t = 4$ …(12)

The equation (12) is of the form $f(p, t) = 0$

Then let $p = a$ and $t = b$, putting in (12) $2a + b = 4$

$\implies b = 4 - 2a$ …(13)

Substituting (13) in $z = ax + bY + c$

$\implies z = ax + (4 - 2a)Y + c$…(14)

where c is an arbitrary constant

replacing Y in (14) to get the complete integral

$\boxed{z = a\,x + (4 - 2a) \ln y + c}$

Example 4: Solve $p^2x^2 = z^2 + q^2y^2$ by hypotheses

Sol. Given that $p^2x^2 = z^2 + q^2y^2$ …(15)

from $X = \ln x$ and $Y = \ln y$ we have

$xp = \frac{\partial z}{\partial X}$ and $qy = \frac{\partial z}{\partial Y}$ …(16)

Substituting (16) in (15), we get

$\left(\frac{\partial z}{\partial X}\right)^2 = z^2 + \left(\frac{\partial z}{\partial Y}\right)^2$ …(17)

Let $t = \frac{\partial z}{\partial X}$ and $r = \frac{\partial z}{\partial Y}$ putting in (17)

$t^2 - r^2 = z^2$ …(18)

Note that (18) is of the form $f(t, r, z) = 0$

Taking $u = X + aY$ (a is constant)

Then $t = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial X}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial X} = \frac{dz}{du}$

$r = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial Y}.\frac{\partial u}{\partial u} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial Y} = a\frac{dz}{du}$ …(19)

because ($\frac{\partial u}{\partial X} = 1$ and $\frac{\partial u}{\partial Y} = a$)

putting (19) in (18)

$$\left(\frac{dz}{du}\right)^2 - a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$(1 - a^2)\left(\frac{dz}{du}\right)^2 = z^2$$

$\pm\sqrt{1 - a^2}\,\frac{dz}{du} = z$ (taking the square root)

$\pm\sqrt{1 - a^2}\,\frac{dz}{z} = du$ …(20)

Integrating (20),

$\pm\sqrt{1 - a^2}\ln z = u + \ln c$ (c is constant) …(21)

Now, replacing u in (21) to get the complete integral

$\pm\sqrt{1-a^2}\ln z = X + aY + \ln c$ …(22)

Next, replacing X and Y in (22) to get the complete integral

$\pm\sqrt{1-a^2}\ln z = \ln x + aln\, y + \ln c$

$\ln z^b = \ln cx\, y^a$ where $b = \pm\sqrt{1-a^2}$

$\Longrightarrow z^b = cx\, y^a$ …(23)

So, (23) is the complete integral.

C) When the equation contains the terms $\frac{p}{z}$ or $\frac{q}{z}$ or its' powers we use the hypothesis $\boxed{Z = \ln z}$

as follows:

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}.\frac{\partial Z}{\partial Z} = \frac{\partial z}{\partial Z}.\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x}$ (since $\frac{\partial z}{\partial Z} = z$)

hence $\frac{p}{z} = \frac{\partial Z}{\partial x}$

by the same way we have $\frac{q}{z} = \frac{\partial Z}{\partial y}$

then substituting this terms in the given equation and solve it by the same way in (A) and (B).

Example 5: Solve $\boldsymbol{p^2 + q^2 = z^2}$ by $\boxed{\boldsymbol{Z = \ln z}}$

Sol. Given that $\boldsymbol{p^2 + q^2 = z^2}$ …(24)

Dividing on z^2, $\frac{p^2}{z^2} + \frac{q^2}{z^2} = 1$ …(25)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$,substituting in (25)

$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = 1$ …(26)

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$ thus, (26) would be

$t^2 + r^2 = 1$ …(27)

Clear that (27) is of the form $f(t, r) = 0$

Let $t = a$, $r = b$ (a, b are constant)

Then $a^2 + b^2 = 1$ …(28)

$a = \pm\sqrt{1 - b^2}$ …(29)

Substituting a, b in $Z = ax + by + c$

$\Longrightarrow Z = \pm\sqrt{1 - b^2}x + by + c$ …(30)

Replacing Z from the hypothesis to get the complete integral

$\therefore \ln z = \pm\sqrt{1 - b^2}x + by + c$

$\Longrightarrow \boxed{z = e^{\pm\sqrt{1-b^2}x+by+c}}$ …(31)

Then (31) is the complete integral.

Example 6: Solve $p^2 + q^2 = z^2(x + y)$ by hypotheses

Sol. Dividing on z^2,

$\frac{p^2}{z^2} + \frac{q^2}{z^2} = x + y$ …(32)

using $Z = \ln z$ we have $\frac{p}{z} = \frac{\partial Z}{\partial x}$ and $\frac{q}{z} = \frac{\partial Z}{\partial y}$, substituting in (32)

$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x + y$ …(33)

Let $t = \frac{\partial Z}{\partial x}$ and $r = \frac{\partial Z}{\partial y}$ putting in (33)

$t^2 + r^2 = x + y$ …(34)

Then $t^2 - x = a \rightarrow t = \pm\sqrt{a + x}$

$y - r^2 = a \rightarrow r = \pm\sqrt{y - a}$

Substituting in $dZ = tdx + rdy$

$\Longrightarrow dZ = \pm\sqrt{a+x}dx + \pm\sqrt{y-a}dy$ …(35)

Integrating (35), we get

$Z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c$ (where c is constant)

Replacing Z from the hypothesis to get the complete integral

$$\Longrightarrow \boxed{\ln z = \pm\frac{2}{3}(a+x)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + c}$$

… Exercises …

1. $p^2x^2 = z(z - qy)$
2. $pq = z^2 y \sec x$
3. $p + q = z\, e^{x+y}$
4. $p^2 + zq = z^2(x - y)$
5. $p^2 + zp = z^2(x - y)$
6. $xp + 4q = \cos y$
7. $p^2 + q^2 = z^2 y$

Section(1.4): Homogeneous linear partial differential equations with constant coefficients and higher order

A linear partial differential equation with constant coefficients is called homogeneous if all it's derivatives are of the same order. The general form of such an equation is

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} = f(x, y) \quad \ldots\ldots\ldots\ldots(1)$$

Where $A_0, A_1, \ldots, A_n$ are constant coefficients.

For example:

1. $3\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = o$ homo. of order 2.
2. $2\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 8\frac{\partial^3 z}{\partial y^3} = x + y$ homo. of order 3.

For convenience $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ will be denoted by $D \; or \; D_x$ and D' or D_y respectively. Then (1) can be rewritten as:

$$(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n) z = f(x, y) \quad \ldots\ldots\ldots\ldots(2)$$

On the other hand, when all the derivatives in the given equation are not of the same order, then it is called a non-homogenous linear partial differential equation with constant coefficients.

In this section we propose to study the various methods of solving homogeneous linear partial differential equation with constant coefficients, namely (2).

Equation (2) may rewritten as:

$$F(D_x, D_y) z = f(x, y) \quad \ldots\ldots\ldots\ldots\ldots(3)$$

Where $F(D_x, D_y) = A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n$

Equation (3) has a general solution when $f(x,y) = 0$

i.e$F(D_x, D_y)z = 0$

$\rightarrow$ $(A_0 D_x^n + A_1 D_x^{n-1} D_{y+\cdots+} A_n D_y^n)z = 0$……………*(4)*

And a particular solution (particular integral) when $f(x,y) \neq 0$

❖ Now, we will find the general solution of (4)

Let $z = \emptyset(y + mx)$ be a solution of (4) where $\emptyset$ is an arbitrary function and m is a constant , then

$$D_x z = \emptyset'(y + mx).m$$

$$D_x^2 z = \emptyset''(y + mx).m^2$$

$$\vdots$$

$$D_x^n z = \emptyset^{(n)}(y + mx).m^n$$

$$D_y z = \emptyset'(y + mx)$$

$$D_y^2 z = \emptyset''(y + mx)$$

$$\vdots$$

$$D_y^n z = \emptyset^{(n)}(y + mx)$$

$$D_x D_y z = m\,\emptyset''(y + mx)$$

$$D_x^2 D_y z = m^2 \emptyset^{(3)}(y + mx)$$

$$\vdots$$

$$D_x^r D_y^s z = m^r \emptyset^{(r+s)}(y + mx)$$

$= m^r \emptyset^{(n)}(y + mx)$, where $r + s = n$

Substituting these values in (4) and simplifying, we get :

$(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n)\emptyset^{(n)}(y + mx) = 0$ …(5)

Which is true if m is a root of the equation

$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0$ ……..(6)

The equation (6) is known as the (characteristic equation) or the (auxiliary equation(A.E.)) and is obtained by putting $D_x = m$ and $D_y = 1$ in $F(D_x, D_y)z = 0$, and it has $n$roots.

Let $m_1, m_2, \dots, m_n$ be n roots of A.E. (6). **Three cases arise**:

Case 1 when the roots are distinct.

If $m_1, m_2, \dots, m_n$ are n distinct roots of A.E. (6) then $\emptyset_1(y + m_1 x), \emptyset_2(y + m_2 x), \dots \dots \dots, \emptyset_n(y + m_n x)$ are the linear solution corresponding to them and since the sum of any linear solutions is a solution too than the general solution in this case is:

$z = \emptyset_1(y + m_1 x) + \emptyset_2(y + m_2 x) + \cdots + \emptyset_n(y + m_n x)$ …..(7)

<u>**Ex.1**</u>: **Find the general solution of**

$$(D_x^3 + 2D_x^2 D_y - 5D_x D_y^2 - 6D_y^3)z = 0$$

Sol. The A.E. is$m^3 + 2m^2 - 5m - 6 = 0$

$$\rightarrow \ (m + 1)(m^2 + m - 6 = 0$$

$$\rightarrow \ (m + 1)(m + 3)(m - 2) = 0$$

$$m_1 = -1, \ m_2 = -3, \ m_3 = 2$$

Note that m_1, m_2 and m_3 are different roots, then the general solution is

$$z = \emptyset_1(y + m_1 x) + \emptyset_2(y + m_2 x) + \emptyset_3(y + m_3 x)$$

$$\rightarrow \quad z = \emptyset_1(y-x) + \emptyset_2(y-3x) + \emptyset_3(y+2x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

Ex.2: Find the general solution of $m^2 - a^2 = 0$ where a is a real number.

Sol. Given that $\quad m^2 - a^2 = 0 \;\rightarrow\; m^2 = a^2$

$$\rightarrow \quad m = \pm a \quad \text{different root}$$

$$m_1 = a \quad , m_2 = -a$$

The general solution is

$$z = \emptyset_1(y+ax) + \emptyset_2(y-ax)$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions.

Case 2 when the roots are repeated.

If the root m is repeated k times . i.e. $m_1 = m_2 = \cdots = m_k$, then the corresponding solution is :

$$z = \emptyset_1(y+m_1x) + x\emptyset_2(y+m_1x) + \cdots + x^{k-1}\emptyset_k(y+m_1x) \quad \ldots(8)$$

Where $\emptyset_1, \ldots, \emptyset_k$ are arbitrary functions.

Note: If some of the roots $m_1, m_2, \ldots, m_n$ are repeated and the other are not . i.e. $m_1 = m_2 = \cdots = m_k \neq m_{k+1} \neq \cdots \neq m_n$ then the general solution is :

$$z = \emptyset_1(y+m_1x) + x\emptyset_2(y+m_1x) + \cdots + x^{k-1}\emptyset_k(y+m_1x) + \emptyset_{k+1}(y+m_{k+1}x) + \cdots + \emptyset_n(y+m_nx) \quad \ldots\ldots\ldots\ldots(9)$$

Ex.3: **Solve** $(D_x^3 - D_x^2 D_y - 8D_x D_y^2 + 12D_y^3)z = 0$

Sol. The A.E. is $\quad m^3 - m^2 - 8m + 12 = 0$

$$\rightarrow \quad (m-2)(m-2)(m+3) = 0$$

$m_1 = m_2 = 2 \quad , \quad m_3 = -3$

Then, the general solution is

$$z = \emptyset_1(y+2x) + x\emptyset_2(y+2x) + \emptyset_3(y-3x)$$

Where $\emptyset_1, \emptyset_2, \emptyset_3$ are arbitrary functions.

Ex.4: **Find the general solution of the equation that it's A.E. is :**

$$(m-1)^2(m+2)^3(m-3)(m+4) = 0$$

Sol. Given that $(m-1)^2(m+2)^3(m-3)(m+4) = 0$

$$m_1 = m_2 = 1 \quad , m_3 = m_4 = m_5 = -2 \quad , m_6 = 3 \quad , m_7 = -4$$

The general solution is

$$z = \emptyset_1(y+x) + x\emptyset_2(y+x) + \emptyset_3(y-2x) + x\emptyset_4(y-2x)$$
$$+ x^2\emptyset_5(y-2x) + \emptyset_6(y+3x) + \emptyset_7(y-4x)$$

Where $\emptyset_1, \dots, \emptyset_7$ are arbitrary functions.

Case 3 when the roots are complex.

If one of the roots of the given equation is complex let be m_1 then the conjugate of m_1 is also a root, let be m_2, so the general solution is:

$$z = \emptyset_1(y + m_1 x) + \emptyset_2(y + m_2 x) + \cdots + \emptyset_n(y + m_n x)$$

Where $\emptyset_1, \dots, \emptyset_n$ are arbitrary functions.

Ex.5: Solve $(D_x^2 + D_y^2)z = 0$

Sol. The A. E. is $\quad m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$

$$\therefore \quad m_1 = i \quad , \quad m_2 = -i$$

The general solution is

$$z = \emptyset_1(y + ix) + \emptyset_2(y - ix)$$

Where $\emptyset_1 , \emptyset_2$ are arbitrary functions.

Ex.6: Solve $(D_x^2 - 2D_xD_y + 5D_y^2)z = 0$

Sol. The A. E. is $\quad m^2 - 2m + 5 = 0$

$$\rightarrow \quad m = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

$$\therefore \quad m_1 = 1 + 2i \quad , \quad m_2 = 1 - 2i$$

$$z = \emptyset_1(y + (1 + 2i)x) + \emptyset_2(y + (1 - 2i)x)$$

That is the general solution where $\emptyset_1 , \emptyset_2$ are arbitrary functions.

Ex.7: Solve $(D_x^4 - D_x^3D_y + 2D_x^2D_y^2 - 5D_xD_y^3 + 3D_y^4)z = 0$

Sol. The A.E. is $\quad m^4 - m^3 + 2m^2 - 5m + 3 = 0$

$$\rightarrow \quad (m - 1)^2(m^2 + m + 3) = 0$$

$m_1 = m_2 = 1 \quad , \quad m = \frac{-1 \pm \sqrt{1 - 12}}{2} = \frac{-1 \pm \sqrt{11} i}{2}$

$$\therefore \; m_3 = \frac{-1 + \sqrt{11} i}{2} \qquad , m_4 = \frac{-1 - \sqrt{11} i}{2}$$

Then, the general solution is

$$z = \emptyset_1(y+x) + x\emptyset_2(y+x) + \emptyset_3\left(y + (\frac{-1+\sqrt{11}i}{2})x\right)$$

$$+ \ \ \emptyset_4\left(y + (\frac{-1-\sqrt{11}i}{2})x\right)$$

Where $\emptyset_1 , \dots , \emptyset_4$ are arbitrary functions.

❖ Particular integral (P.I.) of homogeneous linear partial differential equation

When $f(x,y) \neq 0$in the equation (3) which it's$F(D_x,D_y)z = f(x,y)$ multiplying (3) by the inverse operator$\frac{1}{F(D_x,D_y)}$ of the operator $F(D_x,D_y)$ to have

$$\frac{1}{F(D_x,D_y)} \cdot F(D_x,D_y)\, z = \frac{1}{F(D_x,D_y)}\, f(x,y)$$

$$\rightarrow \boxed{z = \frac{1}{F(D_x,D_y)}\, f(x,y)} \quad \ldots\ldots\ldots (11)$$

Which it's the particular integral (P.I.)

The operator $F(D_x,D_y)$ can be written as

$$F(D_x,D_y) = (D_x - m_1 D_y)(D_x - m_2 D_y) \ldots (D_x - m_n D_y) \quad \ldots.(12)$$

Substituting (12) in (11) :

$$z = \frac{1}{(D_x - m_1 D_y)(D_x - m_2 D_y)\ldots(D_x - m_n D_y)}\, f(x,y) \quad \ldots\ldots\ldots\ldots(13)$$

Taking $u_1 = \frac{1}{(D_x - m_n D_y)}\, f(x,y)$

$$\therefore\ (D_x - m_n D_y)u_1 = f(x,y)$$

This equation can be solved by Lagrange's method .

The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m_n} = \frac{du_1}{f(x,y)} \quad \ldots\ldots\ldots..(14)$$

Taking the first two fractions of (14)

$$m_n dx + dy = 0 \quad \rightarrow \boxed{m_n x + y = a} \quad \ldots\ldots\ldots.(15)$$

Taking the first and third fractions of (14)

$$dx = \frac{du_1}{f(x,y)} \quad \rightarrow \boxed{f(x,y)dx = du_1} \quad \ldots\ldots\ldots.(16)$$

Substituting (15) in (16) we have

$$f(x, a - m_n x)dx = du_1$$

Integrating the last one we have

$$u_1 = \int f(x, a - m_n x)dx + b$$

Let $b = 0$, then we have u_1

By the same way , we take

$$u_2 = \frac{1}{D_x - m_{n-1}D_y} u_1$$

And solve it by Lagrange's method to get u_2 , then continue in this way until we get to

$$z = u_n = \frac{1}{D_x - m_1 D_y} u_{n-1}$$

And by solving this equation we get the particular integral (P.I.)

Ex.1: solve $\left(D_x^2 - D_y^2\right)z = \sec^2(x + y)$

Sol. Firstly, we will find the general solution of

$\left(D_x^2 - D_y^2\right)z = 0$ ……….(1)

The A. E. is $m^2 - 1 = 0 \rightarrow m^2 = 1 \rightarrow m = \pm 1$

$$\therefore \quad m_1 = 1 , \; m_2 = -1$$

$\therefore \; z = \emptyset_1(y + x) + \emptyset_2(y - x)$ ………….(2)

Where $\emptyset_1 , \emptyset_2$ are arbitrary functions.

Second, we will find the particular integral as follows

$$z_2 = \frac{1}{D_x^2 - D_y^2} \sec^2(x + y)$$

$$= \frac{1}{(D_x - D_y)(D_x + D_y)} \sec^2(x + y)$$

Let $u_1 = \frac{1}{(D_x+D_y)} \sec^2(x + y)$

$$(D_x + D_y)u_1 = \sec^2(x + y)$$

The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du_1}{\sec^2(x + y)}$$

Taking the first two fractions

$dx = dy \quad \rightarrow \quad x - y = a$ ……….(3)

Taking the first and third fractions

$dx = \frac{du_1}{\sec^2(x+y)} \quad \rightarrow \quad \sec^2(x + y)\, dx = du_1$ ……….(4)

Substituting (3) in (4), we have

$\sec^2(2x - a)\, dx = du_1$ …………(5)

Integrating (5), we have

$$u_1 = \frac{1}{2}\tan(2x - a) + b$$

Let $b = 0$ and replacing a , we get

$u_1 = \frac{1}{2}\tan(x + y)$ ……..(6)

Putting (6) in z_2

$$z_2 = \frac{1}{(D_x - D_y)} \cdot \frac{1}{2}\tan(x + y)$$

$$\rightarrow (D_x - D_y)z_2 = \frac{1}{2}\tan(x + y)$$

The Lagrange's auxiliary equation are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

Taking the first two fractions

$dx = -dy \;\; \rightarrow \;\; x + y = a$ ……….(7)

Taking the first and third fractions

$$dx = \frac{dz_2}{\frac{1}{2}\tan(x+y)}$$

$\frac{1}{2}\tan(x+y)\, dx = dz_2$ ……..(8)

Substituting (7) in (8)

$\frac{1}{2}\tan a \;\, dx = dz_2$ ………(9)

Integrating (9) , we get

$$\frac{1}{2} x \tan a \;= z_2 + b$$

Let $b = 0$, and replacing a from (7) we get the particular integral

$z_2 = \frac{1}{2} x \tan(x+y)$ ……………(10)

Hence the required general solution is

$$z = z_1 + z_2$$

$$= \emptyset_1(y+x) + \emptyset_2(y-x) + \frac{x}{2}\tan(x+y) \quad ………(11)$$

Short methods of finding the P.I. in certain cases :

Case 1 **When $f(x,y) = e^{ax+by}$ where a and b are arbitrary constants**

To find the P.I. when $F(a,b) \neq 0$, we derive $f(x,y)$ for x any y n times:

$$D_x e^{ax+by} = a e^{ax+by}$$
$$D_x^2 e^{ax+by} = a^2 e^{ax+by}$$
$$\vdots$$
$$D_x^n e^{ax+by} = a^n e^{ax+by}$$

$$D_y e^{ax+by} = b e^{ax+by}$$
$$D_y^2 e^{ax+by} = b^2 e^{ax+by}$$
$$\vdots$$
$$D_y^n e^{ax+by} = b^n e^{ax+by}$$

$D_x^r D_y^s e^{ax+by} = a^r b^s e^{ax+by}$ where $r + s = n$

So

$$F(D_x, D_y) e^{ax+by} = F(a, b) e^{ax+by}$$

Multiplying both sides by $\frac{1}{F(D_x,D_y)}$, we get

$$e^{ax+by} = \frac{1}{F(D_x, D_y)} F(a, b) e^{ax+by}$$

Since $F(a, b) \neq 0$, then we can divide on it :

$\frac{1}{F(a,b)} e^{ax+by} = \frac{1}{F(D_x,D_y)} e^{ax+by}$ ....*

Which it is equal to z , then the P. I. is

$$\boxed{z = \frac{1}{F(D_x,D_y)} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}}$$, where $F(a, b) \neq 0$

when $F(a, b) = 0$, then analyze $F(D_x, D_y)$ as follows

$$F(D_x, D_y) = (D_x - \frac{a}{b} D_y)^r G(D_x, D_y)$$

Where $G(a, b) \neq 0$, we get

$$z = \frac{1}{F(D_x, D_y)} e^{ax+by} = \frac{1}{(D_x - \frac{a}{b} D_y)^r G(D_x, D_y)} e^{ax+by}$$

$$= \frac{1}{(D_x - \frac{a}{b} D_y)^r} \cdot \frac{1}{G(a,b)} e^{ax+by} \quad \text{from } *$$

Since $G(a, b) \neq 0$

$$= \frac{1}{G(a, b)} \cdot \frac{1}{(D_x - \frac{a}{b} D_y)^r} e^{ax+by}$$

Then by Lagrange's method r times , we get

$$z = \frac{1}{F(D_x, D_y)} e^{ax+by} = \frac{1}{G(a, b)} \cdot \frac{x^r}{r!} e^{ax+by}$$

Which it's the P.I. where $F(a, b) = 0$, $G(a, b) \neq 0$

<u>Ex.2</u>: Solve $\boldsymbol{(D_x^2 - D_x D_y - 6D_y^2)z = e^{2x-3y}}$

Sol.

1) To find the general solution

The A.E. of the given equation is

$$m^2 - m - 6 = 0 \quad \rightarrow \quad (m-3)(m+2) = 0$$

$$\therefore \quad m_1 = 3 \ , \qquad m_2 = -2$$

$\therefore \quad z_1 = \emptyset_1(y + 3x) + \emptyset_2(y - 2x)$

Where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions

2) To find the particular Integral (P.I.)

$$a = 2 \ , b = -3$$

$$F(a, b) = a^2 - ab - 6b^2$$

$F(2, -3) = 4 + 6 - 54 = -44 \neq 0$

$$z_2 = \frac{1}{F(a,b)} e^{ax+by} = \frac{1}{-44} e^{2x-3y}$$

$$\therefore \ z = z_1 + z_2$$

$$= \emptyset_1(y + 3x) + \emptyset_2(y - 2x) - \frac{1}{44} e^{2x-3y}$$

<u>Ex.3</u>: Solve $\left(D_x^2 - D_x D_y - 6D_y^2\right) z = e^{3x+y}$

Sol.

1) The general solution is similar to that in Ex.2
2) To find P.I.

$$a = 3 \ , b = 1$$

$$F(a, b) = a^2 - ab - 6b^2$$

$F(3,1) = 9 - 3 - 6 = 0$,

analyze $F(D_x, D_y)$,$F(D_x, D_y) = D_x^2 - D_x D_y - 6D_y^2$

$$= (D_x - 3D_y)(D_x + 2D_y)$$

$(D_x - \frac{a}{b} D_y)^r \rightarrow \therefore r = 1 , \ \ 3 + 2 = 5 \neq 0 = G$

$$z_2 = \frac{1}{G(a,b)} \cdot \frac{x^r}{r!} e^{ax+by} = \frac{1}{5} \cdot \frac{x}{1} e^{3x+y} = \frac{x}{5} e^{3x+y}$$

$$\therefore \ z = z_1 + z_2$$

$$= \emptyset_1(y + 3x) + \emptyset_2(y - 2x) + \frac{x}{5} e^{3x+y}$$

Where $\emptyset_1$and$\emptyset_2$are arbitrary functions

Case 2 **when $f(x,y) = \sin(ax+by)$ *or* $\cos(ax+by)$ where a and b are arbitrary constants**

Here, we will find the P.I. of (H.L.P.D.E.) of order 2 only, by the same way that in case 1 we will derive $f(x,y)$ for x and y .

Let $f(x,y) = \sin(ax+by)$

$$D_x \sin(ax+by) = a\cos(ax+by)$$

$$D_x^2 \sin(ax+by) = -a^2 \sin(ax+by)$$

$$D_y \sin(ax+by) = b\cos(ax+by)$$

$$D_y^2 \sin(ax+by) = -b^2 \sin(ax+by)$$

$$D_x D_y \sin(ax+by) = D_x[b\cos(ax+by)]$$

$$= -ab\sin(ax+by)$$

$$F\left(D_x^2, D_x D_y, D_y^2\right)\sin(ax+by) = F(-a^2,-ab,-b^2)\sin(ax+by)$$

Multiplying both sides by $\frac{1}{F\left(D_x^2, D_x D_y, D_y^2\right)}$

$$\sin(ax+by) = \frac{1}{F\left(D_x^2, D_x D_y, D_y^2\right)} F(-a^2,-ab,-b^2)\sin(ax+by)$$

If $F(-a^2,-ab,-b^2) \neq 0$ then we can divide on it

$$\rightarrow z = \frac{1}{F\left(D_x^2, D_x D_y, D_y^2\right)}\sin(ax+by)$$

$$= \frac{1}{F(-a^2,-ab,-b^2)}\sin(ax+by)$$

Which is the particular integral.

and if $F(-a^2,-ab,-b^2) = 0$, then we write

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad , \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and follow the solution of the exponential function in case1.

Ex.4: Solve $\boldsymbol{\left(D_x^2 - D_xD_y - 6D_y^2\right)z = \sin(2x - 3y)}$

Sol.

1) The general solution $z_1$is the same in Ex.2

2) The P.I.z_2

$$a = 2 \ , b = -3$$

$$F(-a^2, -ab, -b^2) = -a^2 + ab + 6b^2$$

$$F(-4, 6, -9) = -4 - 6 + 54 = 44 \neq 0$$

$$z_2 = \frac{1}{44}\sin(2x - 3y)$$

The required general solution

$$\therefore \ z = z_1 + z_2$$

$$= \emptyset_1(y + 3x) + \emptyset_2(y - 2x) + \frac{1}{44}\sin(2x - 3y)$$

Where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

Ex. 5: Solve $\boldsymbol{\left(D_x^2 - 3D_xD_y + 2D_y^2\right)z = e^{2x+3y} + e^{x+y} + \sin(x - 2y)}$

Sol.

1) Finding the general solution z_1

The A.E. is

$$m^2 - 3m + 2 = 0 \ \Longrightarrow \ (m - 2)(m - 1) = 0$$

$$\therefore m_1 = 2 \, , m_2 = 1$$

$\therefore z_1 = \emptyset_1(y + 2x) + \emptyset_2(y + x)$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $z_2 = \frac{1}{F(D_x,D_y)} e^{2x+3y} + \frac{1}{F(D_x,D_y)} e^{x+y} + \frac{1}{F(D_x,D_y)} \sin(x-2y)$

Let $u_1 = \frac{1}{F(D_x,D_y)} e^{2x+3y}$ $\quad , a = 2, b = 3$

$F(D_x, D_y) = a^2 - 3ab + 2b^2$

$F(2,3) = 4 - 18 + 18 = 4 \neq 0$

$$\boxed{u_1 = \frac{1}{4} e^{2x+3y}}$$

$u_2 = \frac{1}{F(D_x,D_y)} e^{x+y}$ $\quad ,a = 1, b = 1$

$F(D_x, D_y) = a^2 - 3ab + 2b^2$

$F(1,1) = 1 - 3 + 2 = 0$

Analyze $F(D_x, D_y)$,

$F(D_x, D_y) = (D_x - 2D_y)(D_x - D_y)$

$$u_2 = \frac{1}{G(a,b)} \frac{x^r}{r!} e^{ax+by}$$

$$= \frac{1}{-1} \frac{x}{1} e^{x+y}$$

$$\boxed{u_2 = -x\, e^{x+y}}$$

$$u_3 = \frac{1}{F(D_x, D_y)} \sin(x-2y)$$

$F(-a^2, -ab, -b^2) = -a^2 + 3ab - 2b^2$

$F(-1,2,-4) = -1 - 6 - 8 = -15 \neq 0$

$$\boxed{u_3 = \frac{1}{-15} \sin(x-2y)}$$

Then, the required general solution is

$$z = z_1 + z_2 = \emptyset_1(y+2x) + \emptyset_2(y+x) + \frac{1}{4}e^{2x+3y} - xe^{x+y}$$
$$- \frac{1}{15}\sin(x-2y)$$

where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions.

Ex. 6: Find the P.I. of the equation

$$\left(D_x^2 - 4D_xD_y + 3D_y^2\right)z = \cos(x+y)$$

Sol. $a = 1, b = 1$

$F(-a^2, -ab, -b^2) = -a^2 + 4ab - 3b^2$

$F(-1, -1, -1) = -1 + 4 - 3 = 0$

Taking $\cos(x+y) = \frac{e^{ix+iy} + e^{-ix-iy}}{2}$

$$z = \frac{1}{2}\left[\frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy} + \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{-ix-iy}\right]$$

Let $u_1 = \frac{1}{D_x^2 - 4D_xD_y + 3D_y^2}e^{ix+iy}$

To find $u_1, a = i, b = i$

$F(a, b) = a^2 - 4ab + 3b^2$

$F(i, i) = i^2 - 4i^2 + 3i^2 = 0$

Analyze $F(D_x, D_y)$,

$F(D_x, D_y) = (D_x - D_y)(D_x - 3D_y)$

$$u_1 = \frac{1}{-2i}xe^{ix+iy}$$

By the same way $u_2 = \frac{1}{2i}xe^{-ix-iy}$

$$\therefore z = \frac{1}{2}\left[\frac{1}{-2i}xe^{ix+iy} + \frac{1}{2i}xe^{-ix-iy}\right]$$

$= \frac{-x}{2}\left[\frac{e^{ix+iy} - e^{-ix-iy}}{2i}\right] = \frac{-x}{2}sin(x+y)$ which is the P.I.

Case 3 **When $f(x,y) = x^a y^b$ where a and b are Non- Negative Integer Numbers**

The particular integral (P.I.) is evaluated by expanding the function $\frac{1}{F(D_x,D_y)}$ in an infinite series of ascending powers of D_x or D_y (i.e.) by transfer the function $\frac{1}{F(D_x,D_y)}$ according to the following

$$\frac{1}{1-\theta} = 1 + \theta + \theta^2 + \cdots$$

<u>Ex.7</u>: Find P.I. of the equation $(D_x^2 - 2D_xD_y)z = x^3y$

Sol. P.I. $= \frac{1}{D_x^2 - 2D_xD_y} x^3y$

$= \frac{1}{D_x^2(1-2\frac{D_y}{D_x})} x^3y, \qquad D_y^n y^m = 0 \; if \; n > m$

$= \frac{1}{D_x^2}\left[1 + 2\frac{D_y}{D_x} + \frac{4D_y^2}{D_x^2} + \cdots\right] x^3y \qquad , \frac{4D_y^2}{D_x^2} = 0$

$$= \frac{1}{D_x^2}\left[x^3y + \frac{1}{2}x^4\right]$$

$$= \frac{1}{D_x}\left[\frac{x^4y}{4} + \frac{x^5}{10}\right] = \frac{x^5y}{20} + \frac{x^6}{60}$$

<u>Ex.8</u>: Find P.I. of the equation $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = x^2y$

Sol. P.I. $= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3} x^2y$

$= \frac{1}{D_x^3\left[1-\left(\frac{7D_y^2}{D_x^2}+\frac{6D_y^3}{D_x^3}\right)\right]} x^2y$

$$= \frac{1}{D_x^3}\left[1 + \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right) + \left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)^2 + \cdots\right]x^2y$$

$= \frac{1}{D_x^3}[x^2y]$ since $\left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right) = 0$, $\left(\frac{7D_y^2}{D_x^2} + \frac{6D_y^3}{D_x^3}\right)^2 = 0$

$$= \frac{1}{D_x^2}\frac{x^3y}{3} = \frac{1}{D_x}\frac{x^4y}{12} = \frac{x^5y}{60}$$

<u>Ex.9</u>: Solve $(D_x^3 - a^2D_xD_y^2)z = x$, where $a \in R$

Sol.

1) the general solution z_1

The A.E. of the given equation is

$m^3 - a^2m = 0 \;\Rightarrow\; m(m^2 - a^2) = 0$

$$\Rightarrow m(m-a)(m+a) = 0$$

$\therefore m_1 = 0$, $m_2 = a$, $m_3 = -a$ (different roots) $\therefore z_1 = \emptyset_1(y) + \emptyset_2(y + ax) + \emptyset_3(y - ax)$

where $\emptyset_1, \emptyset_2$ and $\emptyset_3$ are arbitrary functions.

2) The P.I. of the given equation is

P.I. $= z_2 = \frac{1}{D_x^3 - a^2D_xD_y^2}x$

$= \frac{1}{D_x^3\left[1 - \frac{a^2D_y^2}{D_x^2}\right]}x$

$= \frac{1}{D_x^3}\left[1 + \frac{a^2D_y^2}{D_x^2} + \left(\frac{a^2D_y^2}{D_x^2}\right)^2 + \cdots\right]x$, where $\frac{a^2D_y^2}{D_x^2} = 0$ and $\left(\frac{a^2D_y^2}{D_x^2}\right)^2 = 0$,

$$= \frac{1}{D_x^3}[x]$$

$$= \frac{1}{D_x^2}\left[\frac{x^2}{2}\right]$$

$$= \frac{1}{D_x}\left[\frac{x^3}{6}\right] = \frac{x^4}{24}$$

then, the required general solution is

$$z = z_1 + z_2 = \emptyset_1(y) + \emptyset_2(y + ax) + \emptyset_3(y - ax) + \frac{x^4}{24}$$

where $\emptyset_1, \emptyset_2$ and $\emptyset_3$ are arbitrary functions.

Case 4 **When $f(x,y)=e^{ax+by}V$ where V is a function of x and y**

The P.I. in this case is $z=\frac{1}{F(D_x,D_y)}e^{ax+by}\,V$

$$=e^{ax+by}\frac{1}{F(D_x+a\,,\,D_y+b)}\,V$$

and solving this equation depending on the type of V can get the particular integral (P.I.), as follows:

<u>Ex.10</u>: Find P.I. of the equation $D_xD_yz=e^{2x+3y}x^2y$

Sol. P.I. $=\frac{1}{D_xD_y}e^{2x+3y}x^2y, a=2, b=3$ and $V=x^2y$

$=e^{2x+3y}\frac{1}{(D_x+2)(D_y+3)}x^2y$

$$=e^{2x+3y}\frac{1}{3(D_x+2)(1+\frac{D_y}{3})}x^2y$$

$$=e^{2x+3y}\frac{1}{3(D_x+2)}\left[1-\frac{D_y}{3}+\frac{D_y^2}{9}-\cdots\right]x^2y$$

$$=e^{2x+3y}\frac{1}{3(D_x+2)}\left[x^2y-\frac{x^2}{3}\right]$$

$$=e^{2x+3y}\frac{1}{6(1+\frac{D_x}{2})}\left[x^2y-\frac{x^2}{3}\right]$$

$=\frac{1}{6}e^{2x+3y}\left[1-\frac{D_x}{2}+\frac{D_x^2}{4}-\frac{D_x^3}{8}+\cdots\right]\left[x^2y-\frac{x^2}{3}\right]$,$(\frac{D_x^3}{8}=0)$

$$=\frac{1}{6}e^{2x+3y}\left[x^2y-\frac{x^2}{3}-xy+\frac{x}{3}+\frac{y}{2}-\frac{1}{6}\right]$$

$$=e^{2x+3y}\left[\frac{1}{6}x^2y-\frac{x^2}{18}-\frac{1}{6}xy+\frac{x}{18}+\frac{y}{12}-\frac{1}{36}\right]$$

Ex.11: Find P.I. of the equation $\left(D_x^2 - D_x D_y\right)z = e^{x+y}\, xy^2$

Sol.

P.I. $= \frac{1}{D_x^2 - D_x D_y} e^{x+y}\, xy^2, a = 1, b = 1$ and $V = xy^2$

$= e^{x+y} \frac{1}{(D_x+1)(D_x - D_y)} xy^2$, since $D_x^2 - D_x D_y = D_x\left(D_x - D_y\right)$

$$= e^{x+y} \frac{1}{(D_x+1)D_x(1 - \frac{D_y}{D_x})} xy^2$$

$$= e^{x+y} \frac{1}{(D_x+1)D_x}\left[1 + \frac{D_y}{D_x} + \frac{D_y^2}{D_x^2} + \cdots\right] xy^2$$

$$= e^{x+y} \frac{1}{(D_x+1)D_x}\left[xy^2 + \frac{2xy}{D_x} + \frac{2x}{D_x^2}\right]$$

$$= e^{x+y} \frac{1}{(D_x+1)D_x}\left[xy^2 + x^2y + \frac{x^3}{3}\right]$$

$$= e^{x+y} \frac{1}{(D_x+1)}\left[\frac{x^2y^2}{2} + \frac{x^3y}{3} + \frac{x^4}{12}\right]$$

$= e^{x+y}[1 - D_x + D_x^2 - D_x^3 + D_x^4 - D_x^5 + \cdots]\left[\frac{x^2y^2}{2} + \frac{x^3y}{3} + \frac{x^4}{12}\right]$ where $D_x^5 = 0$

$$= e^{x+y}\left[\frac{x^2y^2}{2} + \frac{x^3y}{3} + \frac{x^4}{12} - xy^2 - x^2y - \frac{x^3}{3} + y^2 + 2xy + x^2 - 2y - 2x + 2\right]$$

Ex.12: Find P.I. of the equation $\left(D_x - D_y\right)^2 z = e^{x+y} \sin(x + 2y)$

Sol. P.I. $= \frac{1}{\left(D_x - D_y\right)^2} e^{x+y} \sin(x + 2y)$, $a_1 = 1, b_1 = 1$

$= e^{x+y} \frac{1}{(D_x + 1 - D_y - 1)^2} \sin(x + 2y)$

$$= e^{x+y} \frac{1}{\left(D_x - D_y\right)^2} \sin(x + 2y)$$

$$= e^{x+y} \frac{1}{D_x^2 - 2D_xD_y + D_y^2} \sin(x + 2y) \qquad , a_2 = 1, b_2 = 2$$

$$F(-a_2^2 \, , -a_2b_2 \, , -b_2^2) = -a_2^2 + 2a_2b_2 - b_2^2$$

$$F(-1 \, , -2 \, , -4) = -1 + 4 - 4 = -1 \neq 0$$

$$\therefore z = e^{x+y} . \frac{1}{-1} \sin(x + y) \quad \Longrightarrow \quad z = -e^{x+y} \sin(x + y)$$

Case 5 **When $f(x, y) = g(ax + by)$ where $F(a, b) \neq 0$**

The particular integral of H.L.P.D.E. of order n is

$$z = \frac{1}{F(a, b)} \underbrace{\int \cdots \int}_{n - times} g(ax + by) \, \underbrace{d(ax + by) \ldots d(ax + by)}_{n - times}$$

<u>Ex.13</u>: Find P.I. of $(D_x^2 + 2D_xD_y - 8D_y^2)z = \sqrt{2x + 3y}$

Sol.

$a = 2, b = 3 \quad , g(2x + 3y) = \sqrt{2x + 3y}$

$F(a, b) = a^2 + 2ab - 8b^2$

$F(2,3) = 4 + 12 - 72 = -56 \neq 0$, integrating g twice

$\therefore$ P.I. $= z = \frac{1}{-56} \iint \sqrt{2x + 3y} \, d(2x + 3y) d(2x + 3y)$

$$= \frac{1}{-56} \int \frac{2}{3} (2x + 3y)^{3/2} d(2x + 3y)$$

$$= \frac{4}{-56 \, (15)} (2x + 3y)^{5/2}$$

$$= \frac{-1}{210} (2x + 3y)^{5/2}$$

Case 6 **When $f(x,y) = g(ax+by)$ where $F(a,b) = 0$**

If $F(a,b) = 0$, then $F(D_x, D_y)$ can be written as

$$F(D_x, D_y) = (bD_x - aD_y)^n$$

and the particular solution is $\boxed{Z = \frac{x^n}{n!}\frac{g(ax+by)}{b^n}}$

Ex.14: Find P.I. of $(D_x^2 - 6D_xD_y + 9D_y^2)z = 3x + y$

Sol. $a = 3, b = 1 \quad , g(3x+y) = 3x + y$

$F(a,b) = a^2 - 6ab + 9b^2$

$$F(3,1) = 9 - 18 + 9 = 0$$

Then $F(D_x, D_x) = D_x^2 - 6D_xD_y + 9D_y^2 = (D_x - 3D_y)^2$, so $n = 2$

$\therefore$ P.I. $= z = \frac{x^2}{2!}\frac{3x+y}{1^2} = \frac{1}{2}x^2(3x+y)$

Ex.15: Find P.I. of $(D_x^2 - 4D_xD_y + 4D_y^2)z = \tan(2x+y)$

Sol. $a = 2, b = 1 \quad , g(2x+y) = \tan(2x+y)$

$F(a,b) = a^2 - 4ab + 4b^2$

$$F(2,1) = 4 - 8 + 4 = 0$$

Then $F(D_x, D_y) = D_x^2 - 4D_xD_y + 4D_y^2 = (D_x - 2D_y)^2$, so $n = 2$

$\therefore$ P.I. $= z = \frac{x^2}{2!}\frac{\tan(2x+y)}{1^2} = \frac{1}{2}x^2 tan(2x+y)$

Ex.16: Find P.I. of $(D_x^2 - D_y^2)z = \sec^2(x+y)$

Sol. $a = 1, b = 1 \quad , g(x+y) = \sec^2(x+y)$

$$F(a,b) = a^2 - b^2$$

$$F(1,1) = 1 - 1 = 0$$

Then $F(D_x, D_y) = D_x^2 - D_y^2 = (D_x - D_y)(D_x + D_y)$

$$\therefore z = \frac{1}{(D_x - D_y)(D_x + D_y)} \sec^2(x+y)$$

Let $u_1 = \frac{1}{(D_x + D_y)} \sec^2(x+y)$ by case (5) we have

$u_1 = \frac{1}{F(a,b)} \int g(ax+by) d(ax+by)$, $F(1,1) = 1 + 1 = 2$

$$= \frac{1}{2} \int \sec^2(x+y) d(x+y)$$

$$= \frac{1}{2} \tan(x+y)$$

$$\Longrightarrow z = \frac{1}{(D_x - D_y)} \frac{1}{2} \tan(x+y)$$

$F(D_x, D_y) = D_x - D_y$

$F(1,1) = 1 - 1 = 0$ where $n = 1$

$$\therefore z = \frac{x^1}{1!} \frac{1}{2} \frac{\tan(x+y)}{1}$$

$= \frac{x}{2} \tan(x+y)$ which its' the particular integral

…General Exercises …

1- $(D_x^4 - D_y^4)z = 0$

2- $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \cos(x-y) + x^2 + xy^2 + y^2$

3- $(D_x - 2D_y)z = e^{3x}(y+1)$

4- $(D_x^2 + 3D_xD_y + 2D_y^2)z = x + y$

5- $(D_x^2 - 5D_xD_y + 4D_y^2)z = \sin(4x+y)$

6- $(2D_x^2 - D_xD_y - 3D_y^2)z = \frac{5\,e^x}{e^y}$

7- $(D_x^2 - 3D_xD_y + 2D_y^2)z = e^{2x-y} + \cos(x+2y)$

8- $(D_x^2 - D_xD_y)z = \ln y$

9- $(D_x + D_y)z = \sec(x+y)$

10- $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

11- $(y^2 + z^2 - x^2)p - 2xyq = -2xz$

12- $pq + 2y(x+1)q + x(x+2)q - 2(x+1) = 0$

13- $(x^2 + 2x)p + (x+1)qy = 0$

14- $(D_x^3 - 3D_xD_y^2 + 2D_y^3)z = \frac{1}{\sqrt{3x-y}}$

15- $(D_x^3 + 2D_x^2D_y - D_xD_y^2 - 2D_y^3)z = (y+2)e^x$

16- $(4D_x^2 - 4D_xD_y + D_y^2)z = (x+2y)^{3/2}$

17- $D_xD_yz = e^{x-y}xy^2$

18- $(D_x - D_y)z = \tan(x+2y)$

19- $2(D_x^3 - 9D_x^2D_y + 27D_xD_y^2 - 27D_y^3)z = \tan^{-1}(3x+y)$

20- $(y^3x - 2x^4)\frac{\partial z}{\partial x} + (2y^4 - x^3y)\frac{\partial z}{\partial y} = x^3 - y^3$

Section(2.1):Non-homogeneous linear partial differential equations with constant coefficients

Definition: A linear partial differential equation with constant coefficients is known as non-homogeneous LPDE with constant coefficients if the order of all the partial derivatives involved in the equation are not all equal.

For example:

1) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + z = x + y$

2) $\frac{\partial^3 z}{\partial x^3} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = e^{x+y}$

Definition: A linear differential operator $F(D_x, D_y)$ is known as (reducible), if it can be written as the product of linear factors of the form $aD_x + bD_y + c$ with a, b and c as constants. $F(D_x, D_y)$ is known as (irreducible), if it is not reducible.

For example:

The operator $D_x^2 - D_y^2$ which can be written in the form $(D_x - D_y)(D_x + D_y)$ is reducible, whereas the operator $D_x^2 - D_y^3$ which cannot be decomposed into linear factors is irreducible.

<u>Note</u>: A LPDE with constant coefficient $F\big(D_x, D_y\big)z = f(x, y)$ is known as reducible, if $F(D_x, D_y)$ reducible, and is known as irreducible, if $F(D_x, D_y)$ is irreducible.

(2.1.1) Determination of Complementary Function (CF) (the general solution) of a reducible non-homo. LPDE with constant coefficients

(A) let $F(D_x, D_y) = (aD_x + bD_y + c)^k$, where a, b, c are constants and k is a natural number

then the equation $F(D_x, D_y)z = 0$ will be

$(aD_x + bD_y + c)^k z = 0$ and the solution is

$$z = e^{\frac{-c}{a}x}\emptyset(ay - bx) \qquad ; \quad a \neq 0 \quad , k = 1$$

or

$$z = e^{\frac{-c}{b}y}\emptyset(ay - bx) \qquad ; \quad b \neq 0 \quad , k = 1$$

For any $k > 1$, the solution is

$$z = e^{\frac{-c}{b}y}[\emptyset_1(ay - bx) + x\emptyset_2(ay - bx) + \cdots + x^{k-1}\emptyset_k(ay - bx)]; \ b \neq 0$$

or

$$z = e^{\frac{-c}{a}x}[\emptyset_1(ay - bx) + x\emptyset_2(ay - bx) + \cdots + x^{k-1}\emptyset_k(ay - bx)]; \ a \neq 0$$

Where $\emptyset_1, \ldots, \emptyset_n$ are arbitrary functions.

Ex.1: Solve $(2D_x - 3D_y - 5)z = 0$

Sol. The given equation is linear in $F(D_x, D_y)$

Then $a = 2, b = -3, c = -5, k = 1$

The general solution is

$$z = e^{\frac{5}{2}x}\emptyset(2y + 3x)$$

Where $\emptyset$ is an arbitrary function.

Ex.2: Solve $(D_x - 5)z = e^{x+y}$

Sol. To find the general solution of $(D_x - 5)z = 0$

We have $a = 1\, , b = 0\, , c = -5\, , k = 1$

$\therefore z_1 = e^{5x}\emptyset(y)$, where $\emptyset$ is an arbitrary function.

To find the P.I. z_2 , we have $a = 1\, , b = 1$

$$F(a,b) = a - 5 \;\; \rightarrow \;\; F(1,1) = 1 - 5 = -4 \neq 0$$

$$\therefore z_2 = \frac{1}{-4} e^{x+y}$$

Then the required general solution of the given equation is

$$z = z_1 + z_2 \quad \rightarrow \quad z = e^{5x}\emptyset(y) - \frac{1}{4} e^{x+y}$$

Ex.3: Solve $(2D_y + 5)^2 z = 0$

Sol. The given equation is reducible, then

$$a = 0\, , b = 2\, , c = 5\, , k = 2\, .$$

The general solution is

$$z = e^{\frac{-5}{2}y}[\emptyset_1(-2x) + x\emptyset_2(-2x)]$$

Where $\emptyset_1$ and $\emptyset_2$ are arbitrary functions

Ex.4: Solve $(D_x - 2D_y + 1)^4 z = 0$

Sol. We have $a = 1, b = -2, c = 1, k = 4$

then

$$z = e^{\frac{1}{2}y}[\emptyset_1(y + 2x) + x\emptyset_2(y + 2x) + x^2\emptyset_3(y + 2x) + x^3\emptyset_4(y + 2x)]$$

Where $\emptyset_1\, , \ldots , \emptyset_4$ are arbitrary functions

(B) when $F(D_x, D_y)$ can be written as the product of linear factors of the form $(aD_x + bD_y + c)$, i.e. $F(D_x, D_y)$ is reducible , then the general solution is the sum of the solutions corresponding to each factor.

<u>Ex.5</u>: solve $\underbrace{(2D_x - 3D_y + 1)}_{\text{linear}}\underbrace{(D_x + 2D_y - 2)}_{\text{linear}} z = 0$

Sol. The given equation is reducible, then we have

$$a_1 = 2 \ , b_1 = -3 \quad , c_1 = 1 \quad , k_1 = 1$$

$$z_1 = e^{\frac{-1}{2}x} \emptyset_1(2y + 3x)$$

$$a_2 = 1 \ , b_2 = 2 \quad , c_2 = -2 \quad , k_2 = 1$$

$$z_2 = e^{2x} \emptyset_2(y - 2x)$$

The general solution is

$$z = z_1 + z_2 \quad \rightarrow \quad z = e^{\frac{-1}{2}x} \emptyset_1(2y + 3x) + e^{2x} \emptyset_2(y - 2x)$$

Where $\emptyset_1 , \emptyset_2$ are two arbitrary functions.

<u>Ex.6</u>: solve $D_x(D_x + D_y + 1)(D_x + 3D_y - 2)z = 0$

Sol. We have

$$a_1 = 1 \ , b_1 = 0 \quad , c_1 = 0 \quad , k_1 = 1$$

$$a_2 = 1 \ , b_2 = 1 \quad , c_2 = 1 \quad , k_2 = 1$$

$$a_3 = 1 \ , b_3 = 3 \quad , c_3 = -2 \quad , k_3 = 1$$

Then the general solution is

$$z = \emptyset_1(y) + e^{-x} \emptyset_2(y - x) + e^{2x} \emptyset_3(y - 3x)$$

Where $\emptyset_1, \dots, \emptyset_3$ are arbitrary functions.

<u>Ex.7</u>: solve $\boldsymbol{\left(D_x^3 - D_x D_y^2 - D_x^2 + D_x D_y\right) z = 0}$

Sol. We have , $\left(D_x^3 - D_x D_y^2 - D_x^2 + D_x D_y\right) z = 0$

$$D_x\left(D_x^2 - D_y^2 - D_x + D_y\right) z = 0$$

$$D_x\left[\left(D_x - D_y\right)\left(D_x + D_y\right) - \left(D_x - D_y\right)\right] = 0$$

$$D_x\left(D_x - D_y\right)\left(D_x + D_y - 1\right) z = 0$$

Then , $a_1 = 1$, $b_1 = 0$, $c_1 = 0$, $k_1 = 1$

$$a_2 = 1 \ , b_2 = -1 \quad , c_2 = 0 \quad , k_2 = 1$$

$$a_3 = 1 \ , b_3 = 1 \quad , c_3 = -1 \quad , k_3 = 1$$

Then the general solution is

$$\mathrm{z} = \emptyset_1(y) + \emptyset_2(y + x) + e^x \emptyset_3(y - x)$$

Where $\emptyset_1, \dots, \emptyset_3$ are arbitrary functions.

(C) When $F(D_x, D_y)$ is irreducible then the complete solution is

$$z = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y}$$

Where $F(a_i, b_i) = 0$,A_i , a_i, b_i are all constants.

<u>Ex.8</u>: Solve $\left(D_x - D_y^3\right)z = 0$

Sol. The given equation is irreducible, then

$$F(a, b) = 0 \quad \rightarrow \quad F(a_i, b_i) = 0$$

$$a - b^3 = 0 \quad \rightarrow \quad a_i - {b_i}^3 = 0 \quad \rightarrow \quad a_i = {b_i}^3$$

The complete solution is

$$z = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y} = \sum_{i=1}^{\infty} A_i\, e^{{b_i}^3 x + b_i y}$$

Where A_i , b_i are constants.

<u>Ex.9</u>: Solve $\left(D_x^2 + D_x + D_y\right)z = 0$

Sol. The given equation is irreducible, then

$$F(a, b) = a^2 + a + b = 0 \quad \rightarrow {a_i}^2 + a_i + b_i = 0$$

$$\rightarrow \quad b_i = -{a_i}^2 - a_i$$

The complete solution is

$$z = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y} = \sum_{i=1}^{\infty} A_i\, e^{a_i x + (-{a_i}^2 - a_i) y}$$

Where A_i , a_i are constants.

<u>Ex.10</u>: Solve $\left(D_x - D_y^2\right)z = e^{2x+3y}$

Sol. (1) we find the complete solution of the irreducible equation

$$\left(\mathrm{D_x} - \mathrm{D_y^2}\right)\mathrm{z} = 0$$

$$F(a,b) = a - b^2 = 0 \;\rightarrow\; F(a_i,b_i) = a_i - {b_i}^2 = 0 \rightarrow a_i = {b_i}^2$$

Then

$$z_1 = \sum_{i=1}^{\infty} A_i\, e^{a_i x + b_i y} = \sum_{i=1}^{\infty} A_i\, e^{{b_i}^2 x + b_i y}$$

Where A_i , b_i are constants.

(2) The P.I. is

$$F(a,b) = a - b^2$$

$$\therefore F(2,3) = 2 - 9 = -7 \neq 0$$

$$\therefore z_2 = \frac{1}{-7} e^{2x+3y}$$

and the required complete solution is

$$z = z_1 + z_2 = \sum_{i=1}^{\infty} A_i\, e^{{b_i}^2 x + b_i y} - \frac{1}{7} e^{2x+3y}$$

(D) When $F(D_x, D_y)$ can be written as the product of reducible and irreducible factors the general solution is the sum of the solutions corresponding to each factor.

Ex.11: Solve $(D_x + 2D_y)(D_x - 2D_y + 1)(D_x - D_y^2)z = 0$

Sol:

Factor 1, $a_1 = 1$, $b_1 = 2$, $c_1 = 0$, $k_1 = 1$

Factor 2, $a_2 = 1$, $b_2 = -2$, $c_2 = 1$, $k_2 = 1$

Factor 3, $F(a,b) = a - b^2 = 0 \;\rightarrow\; a = b^2 \;\rightarrow\; a_i = b_i^2$

$$\therefore z = \emptyset_1(y - 2x) + e^{\frac{1}{2}y} \emptyset_2(y + 2x) + \sum_{i=1}^{\infty} A_i\, e^{{b_i}^2 x + b_i y}$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions and $A_i\ , b_i$ are constants.

<u>Ex.12</u>: Solve $\left(D_x^2 - D_y^2 + D_x\right)z = x^2 + 2y$

Sol: (1) The complete solution of $\left(\mathrm{D}_x^2 - \mathrm{D}_y^2 + \mathrm{D}_x\right)z = 0$ is $F(a,b) = a^2 - b^2 + a = 0 \rightarrow b = \pm\sqrt{a^2 + a} \rightarrow b_i = \pm\sqrt{a_i^2 + a_i}$

Then

$$z_1 = \sum_{i=1}^{\infty} A_i\, e^{a_i x \pm \sqrt{a_i^2 + a_i}\, y}$$

(2) The P.I. is

$$z_2 = \frac{1}{\mathrm{D}_x^2 - \mathrm{D}_y^2 + \mathrm{D}_x}(x^2 + 2y)$$

$$= \frac{1}{\mathrm{D}_x(1 + \mathrm{D}_x - \frac{\mathrm{D}_y^2}{\mathrm{D}_x})}(x^2 + 2y)$$

$$= \frac{1}{\mathrm{D}_x[1 - \left(\frac{\mathrm{D}_y^2}{\mathrm{D}_x} - \mathrm{D}_x\right)]}(x^2 + 2y)$$

$$= \frac{1}{\mathrm{D}_x}[1 + \underbrace{\frac{\mathrm{D}_y^2}{\mathrm{D}_x}}_{=0} - \mathrm{D}_x + \left(\frac{\mathrm{D}_y^2}{\mathrm{D}_x} - \mathrm{D}_x\right)^2 + \cdots](x^2 + 2y)$$

$$= \frac{1}{\mathrm{D}_x}[x^2 + 2y - 2x + 2] = \frac{x^3}{3} + 2xy - x^2 + 2x$$

The required complete solution is

$$z = z_1 + z_2 = \sum_{i=1}^{\infty} A_i\, e^{a_i x \pm \sqrt{a_i^2 + a_i y}} + \frac{x^3}{3} + 2xy - x^2 + 2x$$

<u>Ex.13:</u> Solve $\boldsymbol{(2D_x + 3D_y)(3D_x - 4D_y + 5)(3D_x - D_y^2)z = 0}$

Sol:

Factor 1, $a_1 = 2$, $b_1 = 3$, $c_1 = 0$, $k_1 = 1$

Factor 2, $a_2 = 3$, $b_2 = -4$, $c_2 = 5$, $k_2 = 1$

Factor 3, $F(a,b) = 3a - b^2 = 0 \quad \rightarrow \quad a = \frac{b^2}{3} \quad \rightarrow \quad a_i = \frac{b_i^2}{3}$

The general solution is

$$\therefore z = \emptyset_1(2y - 3x) + e^{\frac{5}{4}y}\emptyset_2(3y + 4x) + \sum_{i=1}^{\infty} A_i\, e^{\frac{b_i^2}{3}x + b_i y}$$

Where $\emptyset_1, \emptyset_2$ are arbitrary functions and A_i , b_i are constants.

Note To determine the P.I. of non-homo. PDE when

$f(x,y) = \sin(ax + by)$ *or* $\cos(ax + by)$ we put $D_x^2 = -a^2$, $D_y^2 = -b^2$, $D_x D_y = -ab$, which provided the denominator is non-zero, as follows.

Ex.14: Solve $\boldsymbol{(D_x^2 - D_y)z = \sin(x - 2y)}$

Sol: (1) The complete solution z_1 of $(D_x^2 - D_y)z = 0$ is

$$F(a,b) = a^2 - b = 0 \quad \rightarrow \boxed{a_i^2 = b_i}$$

$$z_1 = \sum_{i=1}^{\infty} A_i\, e^{a_i x + a_i^2 y}$$

(2) To find the P.I. of the given equation

$$P.I. = z_2 = \frac{1}{D_x^2 - D_y} \sin(x - 2y)$$

$a = 1, \qquad b = -2 \quad \rightarrow D_x^2 = -a^2 = -1$

$$= \frac{1}{-1-D_y} \sin(x - 2y) \quad \text{, multiplying by } \frac{1}{-1+D_y}$$

$$= \frac{-1+D_y}{1-D_y^2} \sin(x - 2y)$$

$D_y^2 = -b^2 = -4$

$$= \frac{-1+D_y}{1+4} \sin(x - 2y)$$

$$= \frac{1}{5}[-\sin(x - 2y) - 2\cos(x - 2y)]$$

Then the complete solution is

$$z = z_1 + z_2 = \sum_{i=1}^{\infty} A_i\, e^{a_i x + a_i^2 y} + \frac{1}{5}[-\sin(x - 2y) - 2\cos(x - 2y)]$$

…Exercises…

Solve the following equations:

1. $\left(D_x^2 + D_x D_y + D_y - 1\right)z = 0$
2. $(D_x + 1)\left(D_x - D_y + 1\right)z = 0$
3. $(D_x^2 + D_x D_y + D_x)z = 0$
4. $\left(D_x^2 + D_y + 4\right)z = e^{4x-y}$
5. $\left(D_x^2 + D_x D_y + D_y - 1\right)z = \sin(x + 2y)$
6. $\left(D_x - D_y - 1\right)\left(D_x - D_y - 2\right)z = x$
7. $\left(D_x^2 - D_y^2 + D_x + 3D_y - 2\right)z = x^2 y$
8. $(D_x + 3D_y - 2)^2 z = 2e^{2x}\sin(y + 3x)$

Section(2.2): Partial differential equations of order two with variable coefficients

In the present section, we propose to discuss partial differential equations of order two with variable coefficients. An equation is said to be of order two, if it involves at least one of the differential coefficients $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, but none of higher order, the quantities p and q may also inter into the equation. Thus the general form of a second order partial differential equation is

$$\boldsymbol{R(x,y)\frac{\partial^2 z}{\partial x^2} + S(x,y)\frac{\partial^2 z}{\partial x \partial y} + T(x,y)\frac{\partial^2 z}{\partial y^2} + P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} + V(x,y)z = f(x,y)}$$

...(1)

Or $\quad Rr + Ss + Tt + Pp + Qq + Vz = f \ldots \qquad (2)$

Where R, S, T, P, Q, V, f are functions of x and y only and not all R, S, T are zero.

We will discuss three cases of the equation (2):

Case 1 when one of R, S, T not equal to zero and P, Q, V are equal to zero, then the solution can be obtained by integrating both sides of the equation directly.

<u>Ex.15:</u>Solve $\boldsymbol{y\frac{\partial^2 z}{\partial x^2} + 5y - x^2 y^2 = 0}$

Sol: Given equation can be written (dividing by 'y')

$\frac{\partial^2 z}{\partial x^2} = yx^2 - 5 \ldots \qquad (3)$

Integrating (3) w.r.t. x

$\frac{\partial z}{\partial x} = \frac{yx^3}{3} - 5x + \emptyset_1(y)$… (4)

Integrating (4) w.r.t. x, then the general solution will be:

$$z = \frac{yx^4}{12} - \frac{5}{2}x^2 + x\emptyset_1(y) + \emptyset_2(y)$$

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

<u>Ex.16:</u> Solve $xy\frac{\partial^2 z}{\partial x \partial y} - y^2 x = 0$

Sol: Given equation can be written (dividing by 'xy')

$\frac{\partial^2 \mathrm{z}}{\partial \mathrm{x}\, \partial \mathrm{y}} = y$ … (5)

Integrating (5) w.r.t. x

$\frac{\partial z}{\partial y} = xy + \emptyset_1(y)$… (6)

Integrating (6) w.r.t. y, then the general solution will be:

$$z = \frac{xy^2}{2} + \int \emptyset_1(y)\partial y + \emptyset_2(x) = \frac{xy^2}{2} + \varphi(y) + \emptyset_2(x)$$

Where $\varphi$and $\emptyset_2$ are two arbitrary functions.

Case2 **When all the derivatives in the equation for one independent variable i.e the equation is of the form**

$\boldsymbol{Rr + Pp + Vz = f(x,y)}$ **or** $\boldsymbol{Tt + Qq + Vz = f(x,y)}$

Some of these coefficients may be Zeros.

These equations will be treated as an ordinary linear differential equation, a follows:

Ex.17: Solve $\boldsymbol{y\frac{\partial^2 z}{\partial y^2}+3\frac{\partial z}{\partial y}=2x+3}$

Sol: let $\frac{\partial z}{\partial y}=q \;\rightarrow\; \frac{\partial^2 z}{\partial y^2}=\frac{\partial q}{\partial y}$

Substituting in the given equation, we get

$y\frac{\partial q}{\partial y}+3q=2x+3 \quad\rightarrow\quad \frac{\partial q}{\partial y}+\frac{3}{y}q=\frac{2x+3}{y}$ ...(7)

Which it's linear diff. eq. in variables q and y , regarding x as a constant.

Integrating factor (I.F.)of (7) $=e^{\int\frac{3}{y}\partial y}=e^{3lny}=y^3$

And solution of (7) is

$$y^3 q=\int\frac{2x+3}{y}y^3\partial y+\emptyset_1(x)$$

$$y^3 q=(2x+3)\frac{y^3}{3}+\emptyset_1(x)$$

$$q=\frac{2x+3}{3}+y^{-3}\emptyset_1(x)$$

$\frac{\partial z}{\partial y}=\frac{2x+3}{3}+y^{-3}\emptyset_1(x)$, integrating w.r.t. y, then the general solution will be: $\quad z=\frac{2x+3}{3}y-\frac{1}{2y^2}\emptyset_1(x)+\emptyset_2(x)$

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

Ex.18: Solve $\boldsymbol{\frac{\partial^2 z}{\partial x^2}-2y\frac{\partial z}{\partial x}+y^2 z=(y-3)e^{2x+3y}}$

Sol: The given equation can be written as

$$D_x^2 - 2yD_x + y^2 z = (y-3)e^{2x+3y}$$

$\rightarrow (D_x - y)^2 z = (y-3)e^{2x+3y}$... (8)

The A.E. of the equation $(D_x - y)^2 z = 0$ is

$$(m-y)^2 = 0 \rightarrow m_1 = m_2 = y$$

$\therefore z_1 = \emptyset_1(y)e^{yx} + x\emptyset_2(y)e^{yx}$... (9)

Where $\emptyset_1$ and $\emptyset_2$ are two arbitrary functions.

The P.I. (z_2) is

$$z_2 = \frac{1}{(D_x - y)^2}(y-3)e^{2x+3y} = (y-3)\frac{1}{(2-y)^2}e^{2x+3y}$$

$$\therefore z = z_1 + z_2$$

$$= \emptyset_1(y)e^{yx} + x\emptyset_2(y)e^{yx} + (y-3)\frac{1}{(2-y)^2}e^{2x+3y}$$

Case3 under this type, we consider equations of the form

$$\boldsymbol{Rr + Ss + Pp = f(x,y) \rightarrow R\frac{\partial^2 z}{\partial x^2} + S\frac{\partial^2 z}{\partial x \partial y} + P\frac{\partial z}{\partial x} = f(x,y)}$$

And $\boldsymbol{Ss + Tt + Qq = f(x,y) \rightarrow S\frac{\partial^2 z}{\partial x \partial y} + T\frac{\partial^2 z}{\partial y^2} + Q\frac{\partial z}{\partial y} = f(x,y)}$

This equation can be reduced to a first order PDEs with p or q as dependent variable and x, y as independent variables. In such situations we shall apply well known Lagrange's method.

<u>Ex.19</u>: Solve $\boldsymbol{x\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} = 0}$

Sol: let $p = \frac{\partial z}{\partial x} \rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}, \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$

Substituting in the given equation , we get

$x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y} - p = 0$...(10)

Which it is in Lagrange's form, the Lagrange's auxiliary equations are

: $\frac{dx}{x} = \frac{dy}{-y} = \frac{dp}{p}$ ... (11)

Taking the first and second fractions of (11)

: $\frac{dx}{x} = \frac{dy}{-y} \rightarrow lnx = -lny + lna \rightarrow xy = a$... (12)

Taking the first and the third fractions of (11)

$\frac{dx}{x} = \frac{dp}{p} \rightarrow lnx = lnp + lnb \rightarrow \frac{x}{p} = b$... (13)

From (12) &(13) , the general solution is

$$\emptyset(a, b) = 0 \rightarrow \emptyset\left(xy, \frac{x}{p}\right) = 0 \rightarrow \frac{x}{p} = g(xy) \rightarrow p = \frac{x}{g(xy)}$$

$\rightarrow \frac{\partial z}{\partial x} = \frac{x}{g(xy)}$... (14)

Integrating (14) w.r.t. x , we get

$z = \int \frac{x}{g(xy)}\, \partial x + \varphi(y)$... (15)

Where g and φ are two arbitrary functions.

Then (15) is the required general solution of the given equation.

...Exercises...

Solve the following equations:

1)) $\ln\left(\frac{\partial^2 z}{\partial x \partial y}\right) = x + y$

2)) $\frac{\partial^2 z}{\partial y^2} - x\frac{\partial z}{\partial y} = x^2$

3)) $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = \frac{x}{y}$

4)) $y^2\frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} = 1$

Section 2.3: Euler-Cauchy PDEs reducible to equations with constant coefficients

In this section, we propose to discuss the method of solving the partial differential equation, which is also called Euler-Cauchy type partial differential equations of the form :

$$a_0 x^n \frac{\partial^n z}{\partial x^n} + a_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + a_n y^n \frac{\partial^n z}{\partial y^n} + \cdots = f(x, y) \text{...(1)}$$

i.e. all the terms of the equation of the formula $a_r x^n y^m \frac{\partial^{n+m} z}{\partial x^n \partial y^m}$ To solve this equation ,define two new variables u and v by

$x = e^u$ and $y = e^v$ so that $u = lnx$ and $v = lny$... (2)

Let $D_u = \frac{\partial}{\partial u}$ and $D_v = \frac{\partial}{\partial v}$

Now, $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{x} \cdot \frac{\partial z}{\partial u}$, using (2)

$\therefore \frac{\partial z}{\partial u} = x \cdot \frac{\partial z}{\partial x} \rightarrow$ $D_u z = x D_x z$... (3)

Again $x^2 \cdot \frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial}{\partial x} (\frac{\partial z}{\partial x})$

$= x^2 \frac{\partial}{\partial x} (\frac{1}{x} \cdot \frac{\partial z}{\partial u})$ from (3)

$$= x^2 \cdot \frac{1}{x} \cdot \frac{\partial^2 z}{\partial x \partial u} - x^2 \cdot \frac{\partial z}{\partial u} \cdot x^{-2}$$

$$= x \cdot \frac{\partial^2 z}{\partial x \partial u} - \frac{\partial z}{\partial u}$$

$$= x\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial z}{\partial u}$$

$$= x\frac{\partial}{\partial u}\left(\frac{1}{x}\cdot\frac{\partial z}{\partial u}\right) - \frac{\partial z}{\partial u}$$

$$= x\cdot\frac{1}{x}\cdot\frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u}$$

$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u}$$

$$\therefore x^2 D_x^2 z = D_u(D_u - 1)z$$

and so on similarly, we have

$yD_y z = D_v z$,$y^2 D_y^2 z = D_v(D_v - 1)z$,…

Hence

$x^n \frac{\partial^n z}{\partial x^n} z = D_u(D_u - 1)(D_u - 2) \dots (D_u - n + 1)z$…(4)

$y^m \frac{\partial^m z}{\partial y^m} z = D_v(D_v - 1)(D_v - 2) \dots (D_v - m + 1)z$…(5)

$x^n y^m \frac{\partial^{n+m} z}{\partial x^n \partial y^m} z = D_u(D_u - 1) \dots (D_u - n + 1)D_v(D_v - 1) \dots (D_v - m + 1)z$ …(6)

Substituting (4),(5),(6) in (1) to get an equation having constant coefficients can easily be solved by the methods of solving homo. and non-homo. Partial differential equations with constant coefficients, Finally , with help of (2), the solution is obtained in terms of old variables x and y.

<u>Ex.20</u>: Solve $\boldsymbol{x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}-y\frac{\partial z}{\partial y}+x\frac{\partial z}{\partial x}=0}$

Sol: let $x=e^u$, $y=e^v$ then $u=lnx$ and $v=lny$

and $\left.\begin{matrix} x\frac{\partial z}{\partial x}=D_u z \quad , \quad x^2\cdot\frac{\partial^2 z}{\partial x^2}=D_u(D_u-1)z \\ y\frac{\partial z}{\partial y}=D_v z \quad , \quad y^2\cdot\frac{\partial^2 z}{\partial y^2}=D_v(D_v-1)z \end{matrix}\right\}$...(7)

Substituting (7) in the given equation,

$$(D_u^2-D_u-D_v^2+D_v-D_v+D_u)z=0$$

$$(D_u^2-D_v^2)z=0\rightarrow(D_u-D_v)(D_u+D_v)z=0$$

The A.E. is $\underbrace{(m-1)}_{m_1=1}\underbrace{(m+1)}_{m_2=-1}=0$

Then the general solution is

$$z=\emptyset_1(v+u)+\emptyset_2(v-u)$$

$$=\emptyset_1(lny+lnx)+\emptyset_2(lny-lnx)$$

$$=\emptyset_1(lnxy)+\emptyset_2\left(ln\frac{y}{x}\right)$$

$$=h_1(xy)+h_2\left(\frac{y}{x}\right)$$

Where h_1 and h_2 are two arbitrary functions.

…Exercises…

Solve the following equations:

1)) $(x^2D_x^2 - y^2D_y^2 - yD_y + xD_x)z = xy$

2)) $(x^2D_x^2 - 2xyD_xD_y + y^2D_y^2 + yD_y + xD_x)z = 0$

3)) $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} = lnxy$

Classification of partial differential equations of second order:

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \ldots \quad (1)$$

Where A, B, C, D, E, F, G are function of x, y or constants.

The equation (1) is said to be

(i) Hyperbolic at a point (x, y)in domain D if $B^2 - 4AC > 0$.
(ii) Parabolic at a point (x, y)in domain D if $B^2 - 4AC = 0$.
(iii) Elliptic at a point (x, y)in domain D if $B^2 - 4AC < 0$.

Ex.21: Classify the following partial differential equation $\boldsymbol{2u_{xx} + 3u_{xy} = 0}$

Sol:

Comparing the given equation with (1),we get $A = 2, B = 3, C = 0$

$$B^2 - 4AC = 9 - 4(2)(0) = 9 > 0$$

Showing that the given equation is hyperbolic at all points.

Ex.22: Classify the following PDEs.

(1) $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

(2) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Sol. (1) Re-writing the given equation, we get

$$\alpha^2 u_{xx} - u_t = 0$$

Comparing with (1), we get $A = \alpha^2, B = 0, C = 0$

$$B^2 - 4AC = 0 - 4(\alpha^2)(0) = 0$$

Showing that the given equation is Parabolic at all points.

Sol. (2) Re-writing the given equation, we get

$$c^2 u_{xx} - u_{tt} = 0$$

Comparing with (1), we get $A = c^2, B = 0, C = -1$

$$B^2 - 4AC = 0 - 4(c^2)(-1) = 4c^2 > 0$$

Showing that the given equation is hyperbolic at all points.

Sol. (3) Comparing with (1), we get $A = 1, B = 0, C = 1$

$$B^2 - 4AC = 0 - 4(1)(1) = -4 < 0$$

Then the equation is an Elliptic at all points.

…Exercises…

Classify the following equations:

1)) $u_x - u_{xy} - u_y = 0$

2)) $u_{rr} - ru_{r\theta} + r^2 u_{\theta\theta} = 0 \qquad ; u(r,\theta)$

3)) $z_{xx} + z_{xy} + z_y = 2x$

4))$xyz_{xx} - (x^2 - y^2)z_{xy} - xyz_{yy} + yz_x - xz_y = 2(x^2 - y^2)$

5)) $x^2(y - 1)\frac{\partial^2 z}{\partial x^2} - x(y^2 - 1)\frac{\partial^2 z}{\partial x \partial y} + 4(y - 1)\frac{\partial^2 z}{\partial y^2} + xy\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$

6))$u_r - u_{\theta\theta} = 5$

7)) $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 2$

Chapter Three

Fourier Series

Contents

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Section(3.1): The definition of Fourier series and how to find it

In this chapter we will find that we can solve many important problems involving partial differential equations provided that we can express a given function as an infinite sum of sines and (or) cosines. These trigonometric series are called (Fourier Series).

Definition: Let f be defined on $[-\pi, \pi]$, we said

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \dots (1)$$

Is Fourier Series of f if it converges at all points of f on the interval $[-\pi, \pi]$, where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

a_0 , a_n , b_n are called Fourier coefficients.

Note f is a periodic function with period 2π since sine and cosine are periodic functions with period 2π , as shown :

$$\boldsymbol{f(x + 2\pi) = f(x)}$$

$$f(x+2\pi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos n(x+2\pi)+b_n\sin n(x+2\pi)]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos(nx+2n\pi)+b_n\sin(nx+2n\pi)]$$

Since cosine and sine are periodic functions with period 2π then

$$f(x+2\pi)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$=f(x) \qquad \text{from the definition}$$

<u>Ex. 1:</u> Find the Fourier Series for $\boldsymbol{f(x)=\begin{cases}1 & ,-\pi\le x\le 0\\ 2 & ,0<x\le\pi\end{cases}}$

Sol:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}dx+\frac{1}{\pi}\int_{0}^{\pi}2dx$$

$$=\frac{1}{\pi}x|_{-\pi}^{0}+\frac{2}{\pi}x|_{0}^{\pi}=3$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}2\cos nx\,dx$$

$$= \frac{1}{n\pi} \sin nx \,\Big|\begin{matrix} 0 \\ -\pi \end{matrix} + \frac{2}{n\pi} \sin nx \,\Big|\begin{matrix} \pi \\ 0 \end{matrix} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} \sin nx \, dx + \frac{2}{\pi} \int_{0}^{\pi} \sin nx \, dx$$

$$= \frac{-1}{n\pi} \cos nx \,\Big|\begin{matrix} 0 \\ -\pi \end{matrix} + \frac{-2}{n\pi} \cos nx \,\Big|\begin{matrix} \pi \\ 0 \end{matrix}$$

$$= \frac{-1}{n\pi} + \frac{1}{n\pi} \cos n\pi - \frac{2}{n\pi} \cos n\pi + \frac{2}{n\pi}$$

$$= \frac{1}{n\pi} - \frac{1}{n\pi} \cos n\pi$$

$$= \frac{1}{n\pi} (1 - (-1)^n)$$

Hence

$$b_n = \begin{cases} 0 & if\ n\ is\ even \\ \frac{2}{n\pi} & if\ n\ is\ odd \end{cases}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{3}{2} + \sum_{\substack{n=1 \\ \text{n is odd}}}^{\infty} \frac{2}{n\pi} \sin nx$$

$$= \frac{3}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \cdots$$

Notes

1. $\cos n\pi = (-1)^n \qquad , \sin n\pi = 0$
2. f is even $\Leftrightarrow f(x) = f(-x)$

   f is odd $\Leftrightarrow f(x) = -f(-x)$

for example:

$f(x) = x^2, \cos x\ , x^4, \ldots \qquad f$ is even

$f(x) = x, x^3, \sin x\ , \ldots \qquad f$ is odd

3. even function × even function = even function

   odd function × odd function = even function

   even function × odd function = odd function

4.

$$\int_{-c}^{c} (even\ function)dx = 2\int_{0}^{c} (even\ function)dx$$

$$\int_{-c}^{c} (odd\ function)dx = 0$$

5. When f is even then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{even}\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx \qquad from\ 4$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{even} \cos nx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx\, dx \qquad from\ 3,4$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{even} \sin nx\, dx = 0 \qquad from\ 3,4$$

When f is odd function then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{odd}\, dx = 0 \qquad\qquad from\ 4$$

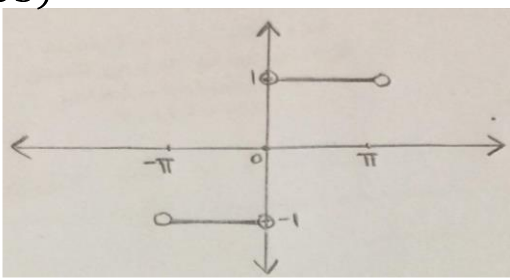
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos nx}_{odd}\, dx = 0 \qquad\qquad from\ 3,4$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin nx}_{odd}\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx \qquad\qquad from\ 3,4$$

<u>Ex. 2</u>: Find the Fourier Series for $\boldsymbol{f(x) = \begin{cases} -1 & if\ -\pi \le x < 0 \\ 1 & if\ \ 0 < x < \pi \end{cases}}$

Sol: note that f is odd, then

$$\left.\begin{array}{l} a_0 = a_n = 0 \\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \end{array}\right\} \quad from\ (note 5)$$



$$= \frac{2}{\pi}\int_{0}^{\pi} \sin nx\, dx = \frac{-2}{n\pi}\cos nx \,\Big|_{0}^{\pi}$$

$$= \frac{-2}{n\pi}[\cos n\pi - \cos 0] = \frac{-2}{n\pi}[(-1)^n - 1] \qquad from\ (note\ 1)$$

$$= \begin{cases} \frac{4}{n\pi} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

Then , $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

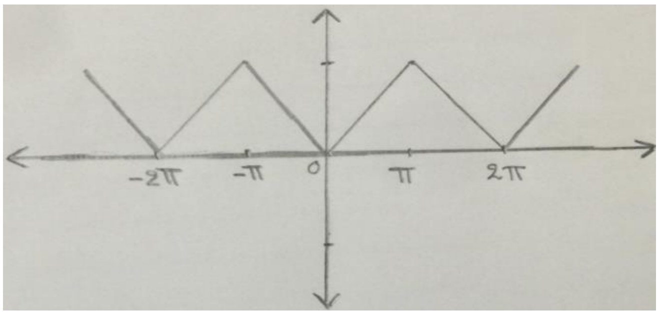
$$= \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{4}{n\pi} sin\, nx$$

Or

$$f(x)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}sin\,(2n-1)x$$

<u>Ex. 3</u>: Find the Fourier Series for $\boldsymbol{f(x)=\begin{cases}-x & if-\pi<x<0\\ x & if\quad 0\le x\le\pi\end{cases}}$

Sol. note that f is even, then



$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}x\,dx=\frac{1}{\pi}x^2\Big|_0^{\pi}=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi}x\cos nx\,dx\;=\frac{2}{\pi}[\frac{x}{n}\sin nx+\frac{1}{n^2}\cos nx]_0^{\pi}$$

$$=\frac{2}{\pi}[\frac{\pi}{n}\sin n\pi+\frac{1}{n^2}\cos n\pi-\frac{0}{n}\sin n0-\frac{1}{n^2}\cos n0]$$

$$=\frac{2}{n^2\pi}[(-1)^n-1]=\begin{cases}\frac{-4}{n^2\pi} & if\ n\ is\ odd\\ 0 & if\ n\ is\ even\end{cases}$$

$b_n=0$ since f is even .

Then , $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$$=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{-4}{(2n-1)^2\pi}cos(2n-1)x$$

Section(3.2): The Fourier convergence

When we find a Fourier series for a function f we assume that f is defined on $[-\pi,\pi]$ and periodic with period 2π, then f must satisfy $f(\pi)=f(-\pi)$ otherwise, the function becomes discontinuous at the points $\pi+2n\pi,\quad n=0,1,2,\dots$, when f discontinuous at x_0, then the series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

may not be convergent to $f(x_0)$ unless some certain conditions are satisfied. There are many conditions , if at least one of them satisfies, the Fourier series approaching f.

Here we will discuss the (Dirichlet's conditions) in the following theorem.

Dirichlet theorem:

Suppose that f and f' are piecewise continuous on the interval $-\pi\leq x<\pi$. Further suppose that f is defined outside the interval $-\pi\leq x<\pi$, So that it is periodic with period 2π then f has a Fourier series

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

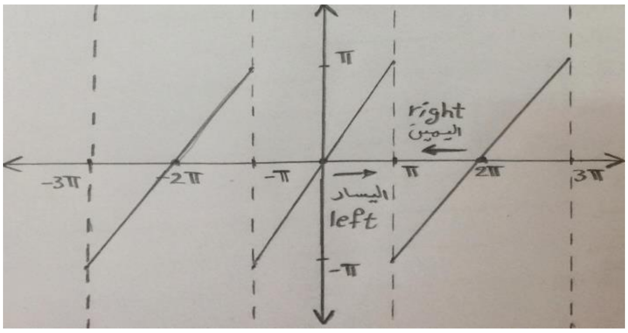
The Fourier series converges to $f(x)$ at all points where f is continuous, and to $\frac{1}{2}[\lim_{x\to x_0^+}f(x)+\lim_{x\to x_0^-}f(x)]$ at all points where f is discontinuous.

Ex. 4: Find the Fourier series for $f(x) = x$ where $x \in [-\pi, \pi]$ then find:

(i) The convergence on the interval $[-\pi, \pi]$.

(ii) The convergence of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

(iii) The approximate value of π.



Sol. f is odd function then

$$a_0 = a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx$$

$$= \frac{2}{\pi}x \cdot \frac{-1}{n}\cos nx \,\Big|_0^\pi + \frac{2}{n\pi}\int_0^\pi \cos nx\,dx$$

$$= \frac{-2}{\pi n}x\cos nx + [\frac{2}{n^2\pi}\sin nx \,\Big|_0^\pi = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$$

(i) The convergence on $[-\pi, \pi]$.

$[-\pi, \pi] = (-\pi, \pi) \cup \{-\pi, \pi\}$

The convergence on the interval $(-\pi, \pi)$ which it's continuous is

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$$

The convergence on the point $x = -\pi$ (where f is discount on it) is

$$f(-\pi) = \frac{1}{2}\left[\lim_{x\to-\pi^+} f(x) + \lim_{x\to-\pi^-} f(x)\right] = \frac{1}{2}[-\pi + \pi] = 0$$

The convergence on the point $x = \pi$ (where f is discount on it) is

$$f(\pi) = \frac{1}{2}\left[\lim_{x\to\pi^+} f(x) + \lim_{x\to\pi^-} f(x)\right] = \frac{1}{2}[-\pi + \pi] = 0$$

(ii) $f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$ ... $(*)$

Let $x = \frac{\pi}{2}$ (where f is continuous)

substituting in $(*)$, we get

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin\frac{n\pi}{2}$$

$$\frac{\pi}{4} = \underbrace{\sin\frac{\pi}{2}}_{=1} - \frac{1}{2}\underbrace{\sin\pi}_{=0} + \frac{1}{3}\underbrace{\sin\frac{3\pi}{2}}_{=-1} - \frac{1}{4}\underbrace{\sin 2\pi}_{=0} + \frac{1}{5}\underbrace{\sin\frac{5\pi}{2}}_{=1} - \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

(iii) since $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ (from (ii))

Then $\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$

let $n = 1 \to \pi = 4$

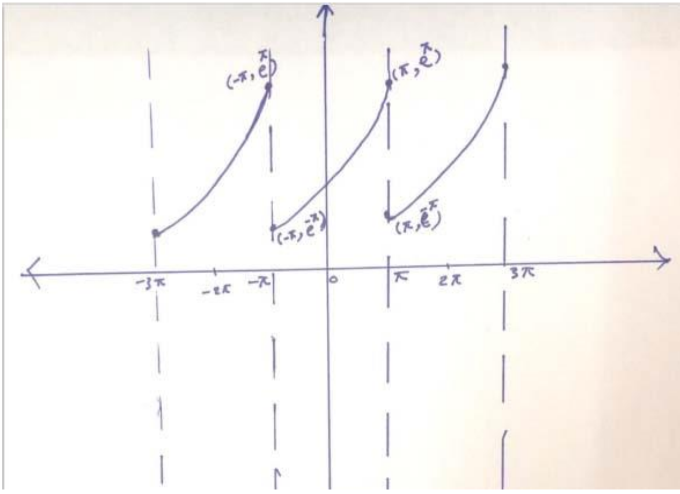
let $n = 2 \to \pi = 4 - \frac{4}{3} = \frac{8}{3} = 2.66$

let $n = 3 \rightarrow \pi = 4 - \frac{4}{3} + \frac{4}{5} = \frac{52}{15} = 3.46$

$\vdots$

So, we approaching from the approximate value of π when n increasing .

<u>Ex. 5</u>: Find the Fourier series for $f(x) = e^x$ on $[-\pi, \pi]$, then find The convergence of Fourier series to f .



Sol. f is not odd neither even

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x\, dx$$

$$= \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} dx = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx\, dx$$

$$= \frac{(-1)^n (e^{\pi} - e^{-\pi})}{\pi(1+n^2)} \text{ (by } \textstyle\int udv \text{ twice)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx\, d$$

$$= \frac{n(-1)^{n+1}(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

Then

$$f(x) = \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}\cos nx + \frac{n(-1)^{n+1}(e^{\pi}-e^{-\pi})}{\pi(1+n^2)}\sin nx\right]$$

The convergence

$[-\pi,\pi] = (-\pi,\pi) \cup \{-\pi,\pi\}$

In the interval $(-\pi,\pi)$ the Fourier series converge to e^x

at the point $x = -\pi$ the Fourier series converge to

$$f(-\pi) = \frac{1}{2}\left[\lim_{x\to-\pi^+} f(x) + \lim_{x\to-\pi^-} f(x)\right] = \frac{1}{2}[e^{-\pi} + e^{\pi}]$$

at the point $x = \pi$ the Fourier series converge to

$$f(\pi) = \frac{1}{2}\left[\lim_{x\to\pi^+} f(x) + \lim_{x\to\pi^-} f(x)\right] = \frac{1}{2}[e^{-\pi} + e^{\pi}]$$

Section 3: Extension of functions

A **The odd extension**: Let f be defined on the interval $[0,\pi]$, we will define the function $F(x)$ on the interval $[-\pi,\pi]$ such that

$$F(x)=\begin{cases} f(x) & ; x\in[0,\pi] \\ -f(-x) & ; x\in[-\pi,0) \end{cases}$$

We must prove that $F(x)=-F(-x)$ (i.e: F is odd)

let $x\in[0,\pi]\rightarrow -x\in[-\pi,0)$

$$F(-x)=-f\big(-(-x)\big)=-f(x)=-F(x)$$

$\therefore F(-x)=-F(x)$, then by multiply both sides by (-1), we obtain, $-F(-x)=F(x)$,

By the same way if $x\in[-\pi,0)$, we get

$$F(-x)=-F(x)$$

then F is odd.

hence,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{F(x)}_{odd}\,dx=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underbrace{F(x)}_{odd}\ \underbrace{\cos nx}_{even}}_{odd}\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underbrace{F(x)}_{odd}\ \underbrace{\sin nx}_{odd}}_{even}\,dx=\frac{2}{\pi}\int_{0}^{\pi}F(x)\sin nx\,dx$$

and

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ; x \in [0, \pi]$$

but $F(x) = f(x)$ on the interval $[0, \pi]$, then

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

and

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ; x \in [0, \pi]$$

This series is called ((Fourier sine series)).

<u>Ex. 6</u>: Find the Fourier sine series for the function $f(x) = cosx$ where $x \in [0, \pi]$.

Sol.

$$b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \, \sin nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} [\sin(n+1)\, x + \sin(n-1)\, x] dx$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)\, x + \sin(n-1)\, x] dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(n+1)\, x}{n+1} - \frac{\cos(n-1)\, x}{n-1} \right]_0^{\pi} \qquad ; n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(n-1) + (-1)^n(n+1) + n - 1 + n + 1}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n n - (-1)^n + (-1)^n n + (-1)^n + 2n}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n n + 2n}{n^2-1}\right]$$

$$= \frac{2n}{\pi}\left[\frac{(-1)^n + 1}{n^2-1}\right]$$

$$= \begin{cases} 0 & if\ n\ is\ odd \quad n \neq 1 \\ \dfrac{4n}{\pi(n^2-1)} & if\ n\ is\ even \end{cases}$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \ \sin x \ dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \ dx = \frac{-1}{2\pi}[cos2x]_0^{\pi}$$

$$= \frac{-1}{2\pi}[cos2\pi - \cos 0] = \frac{-1}{2\pi}[1-1] = 0$$

Then the Fourier sine series is

$$f(x) = \sum_{\substack{n=2 \\ n\ is\ even}}^{\infty} \frac{4n}{\pi(n^2-1)} \sin nx$$

B **The even extension**: Let f be defined on the interval $[0,\pi]$, we will define the function $F(x)$ on the interval $[-\pi,\pi]$, such that:

$$F(x)=\begin{cases} f(x) & ;x\in[0,\pi] \\ f(-x) & ;x\in[-\pi,0] \end{cases}$$

We must prove that F is even i.e $F(x)=F(-x)$

$$x\in[0,\pi] \quad \to F(x)=f(x)$$

let $-x\in[-\pi,0] \ \to F(-x)=f\big(-(-x)\big)=f(x)=F(x)$

$$\therefore F(-x)=F(x)$$

by the same way when $x\in[-\pi,0]$, we get $F(x)=F(-x)$

then F is even , hence the Fourier series of F is

$$a_0=\frac{2}{\pi}\int_0^{\pi} F(x)\,dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} F(x)\cos nx\,dx$$

$$b_n=0$$

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx \qquad ;x\in[0,\pi]$$

But on the interval $[0,\pi]$ the function $F(x)$ is equal to $f(x)$ then

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx \quad ; \ a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad ; \ b_n=0$$

and

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx$$

This series is called ((Fourier cosine series)).

Ex.7: Find the Fourier cosine series for the function $f(x) = \sin x \;\; ; x \in [0, \pi]$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{-2}{\pi}\cos x \,\Big|_0^{\pi} = \frac{-2}{\pi}[\cos\pi - \cos 0] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x\,]dx$$

$$= \frac{1}{\pi}\left[\frac{-1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right]_0^{\pi}$$

(by the same way in Ex.6)

$$= \begin{cases} 0 & if\ n\ is\ odd\ \ , n \neq 1 \\ \dfrac{-4}{\pi(n^2-1)} & if\ n\ is\ even \end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = 0$$

$$\therefore f(x) = \frac{2}{\pi} + \sum_{\substack{n=2 \\ n\ is\ even}}^{\infty} \frac{-4}{\pi(n^2-1)}\cos nx$$

Note

1. $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$
2. $\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$
3. $\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$

Section 4: Fourier series on the interval [-L,L]

Let f be defined on the interval $[-L, L]$, we assume that $z = \frac{\pi x}{L}$ to transform f on the interval $[-\pi, \pi]$, hence

$f(x) = F(z)$ where $-L \leq x \leq L$ & $-\pi \leq z \leq \pi$, *and*

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nz + b_n \sin nz] \qquad \dots (*)$$

s.t.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z)\, dz$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \cos nz\, dz$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(z) \sin nz\, dz$$

Replacing z from the hypothesis, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}]$$

s.t.

$$a_0 = \frac{1}{\pi} \int_{-L}^{L} f(x) \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \cdot \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \cdot \frac{\pi}{L} dx = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Ex. 8: Find the Fourier series and the convergence on $[-1,1]$ for the function $f(x)=\begin{cases} -3 & ; -1\le x\le 0 \\ 2 & ; 0<x\le 1 \end{cases}$

Sol. $[-L,L]=[-1,1]$

$$a_0=\frac{1}{L}\int_{-L}^{L} f(x)\,dx=\int_{-1}^{0}-3\,dx+\int_{0}^{1}2\,dx$$

$$=-3x\Big|_{-1}^{0}+2x\Big|_{0}^{1}=-3+2=-1$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx=\int_{-1}^{0}-3\cos n\pi x\ \,dx+\int_{0}^{1}2\cos n\pi x\ \,dx$$

$$=\frac{-3}{n\pi}\sin n\pi x\Big|_{-1}^{0}+\frac{2}{n\pi}\sin n\pi x\Big|_{0}^{1}=0$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx=\int_{-1}^{0}-3\sin n\pi x\ \,dx+\int_{0}^{1}2\sin n\pi x\ \,dx$$

$$=\frac{3}{n\pi}\cos n\pi x\Big|_{-1}^{0}-\frac{2}{n\pi}\cos n\pi x\Big|_{0}^{1}$$

$$=\frac{3}{n\pi}\cos 0-\frac{3}{n\pi}\cos n\pi-\frac{2}{n\pi}\cos n\pi+\frac{2}{n\pi}\cos 0$$

$$=\frac{5}{n\pi}-\frac{5(-1)^n}{n\pi}=\frac{5}{n\pi}(1-(-1)^n)=\begin{cases} 0 & if\ n\ is\ even \\ \dfrac{10}{n\pi} & if\ n\ is\ odd \end{cases}$$

Then

$$f(x)=\frac{-1}{2}+\sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty}\frac{10}{n\pi}\sin n\pi x$$

* the convergence on $[-1,1]$

$[-1,1] = (-1,0) \cup (0,1) \cup \{-1,0,1\}$

(i) on the interval $(-1,0)$ the Fourier series converge to -3
(ii) on the interval $(0,1)$ the Fourier series converge to 2
(iii) at the point $x = -1$ the Fourier series converge to

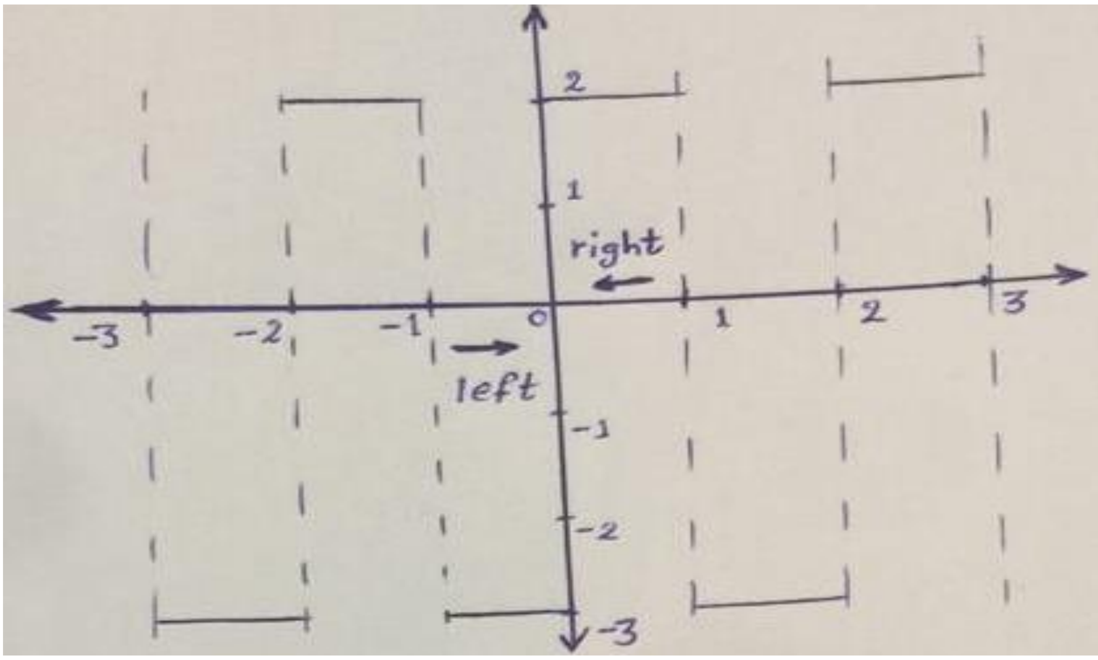
$$f(-1) = \frac{1}{2}\left[\lim_{x\to -1^+} f(x) + \lim_{x\to -1^-} f(x)\right] = \frac{1}{2}[-3+2] = \frac{-1}{2}$$

at the point $x = 0$ the Fourier series converge to

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x)\right] = \frac{1}{2}[2+(-3)] = \frac{-1}{2}$$

at the point $x = 1$ the Fourier series converge to

$$f(1) = \frac{1}{2}\left[\lim_{x\to 1^+} f(x) + \lim_{x\to 1^-} f(x)\right] = \frac{1}{2}[-3+2] = \frac{-1}{2}$$



… Exercises …

(i) Find the Fourier cosine series for $f(x) = x$ where $x \in [0, \pi]$.

(ii) For the given functions:

(a) Find the Fourier series.

(b) Find the convergence on the whole interval.

(c) Sketch the graph of the function.

1. $f(x) = x^2 \quad ; x \in [-\pi, \pi]$

2. $(x) = 1 - x^2 \quad ; x \in [-1,1]$

3. $f(x) = \begin{cases} 0 & ; -\pi \leq x < 0 \\ x^2 & ; 0 \leq x < \pi \end{cases}$

4. $f(x) = \begin{cases} x & ; -\pi \leq x < 0 \\ 0 & ; 0 \leq x \leq \pi \end{cases}$

Then find the sum of the series $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$

5. $f(x) = \begin{cases} x + \pi & ; -\pi \leq x < 0 \\ \pi - x & ; 0 \leq x < \pi \end{cases}$

(iii) Find the Fourier cosine series for $f(x) = \pi - x \ ; x \in [0, \pi]$.

(iv) Find the Fourier sine series for $f(x) = \pi - x \ ; x \in [0, \pi]$.

(v) Find the Fourier sine and Fourier cosine series for

$$f(x) = \begin{cases} 1 & ; 0 < x \leq \frac{2}{3} \\ 0 & ; \frac{2}{3} < x < 1 \end{cases}$$

Then find the sum of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \cdots$

Chapter Four

The Heat Conduction Equation

Contents

Section 1	Separation of Variables	
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Section 4	Bar with Insulated Ends	
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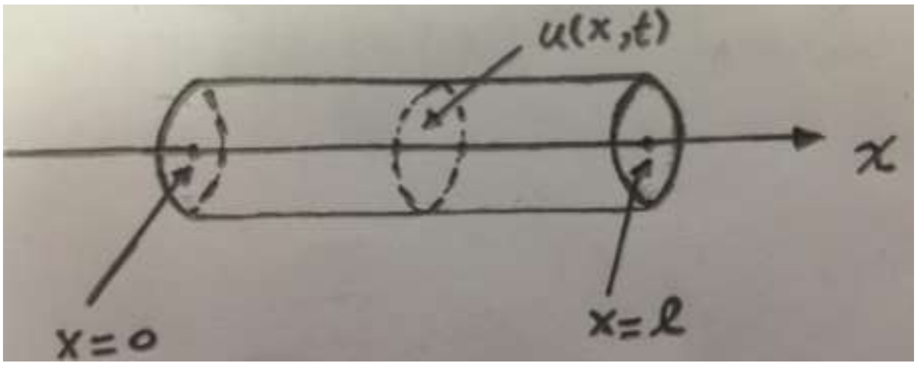
The Heat Conduction Equation

Let us consider a heat conduction problem for a straight bar of uniform cross section and homogeneous material. Let the x-axis be chosen to lie along the axis of the bar, and let $x = 0$ and $x = l$ denote the ends of the bar (as shown in the figure) suppose further that the sides of the bar are perfectly insulated so that no heat passes through them. We also assume that the cross-sectional dimensions are so small that the temperature u can be considered as constant on any given cross section. Then u is a function only of the axial coordinate x and the time t.



The variation of temperature in the bar is governed by partial differential equation called the (heat conduction equation), and has the form

$$u_t = \alpha^2 u_{xx} \quad , 0 < x < l \quad , 0 < t < \infty \qquad \ldots(1)$$

Where α^2 is a constant known as the thermal diffusivity, the parameter α^2 depends only on the material from which the bar is made, and is defined by $\alpha^2 = \frac{k}{ps}$ where k is the thermal conductivity , p is the density, and s is the specific heat of the material in the bar.

Now, to find the solution of equation (1), we start by making a basic assumption about the form of the solutions that has far-reaching, and perhaps unforeseen, consequences. The assumption is that $u(x,t)$ is a product of two other functions, one depending only on x and the other depending only on t.

This method is called (Separation of Variables).

----

1 Separation of Variables

Let $u(x,t) = X(x).T(t)$ ...(2)

Differentiating (2) w.r.t. t and x, we get

$$\frac{\partial u}{\partial t} = X(x).T'(t) \qquad ...(3)$$

and

$$\frac{\partial^2 u}{\partial x^2} = X''(x).T(t) \qquad ...(4)$$

Substituting (3) and (4) in (1), we get

$$X.T' = \alpha^2 X''.T \qquad ...(5)$$

Equation (5) is equivalent to

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = \lambda \qquad \text{(where } \lambda \text{ is a constant)}$$

Hence $T' - \alpha^2 \lambda T = 0$ ...(6)

and $X'' - \lambda X = 0$ ...(7)

where (6) and (7) are two ordinary differential equations can be solved as follows:

The (A.E.) of (6) is $m - \alpha^2\lambda = 0 \Rightarrow m = \alpha^2\lambda$

$$\therefore T(t) = c_1 e^{\alpha^2 \lambda t} \qquad ...(8)$$

Where c_1 is an arbitrary constant .

The (A.E.) of (7) is $m^2 - \lambda = 0 \Rightarrow m = \pm\sqrt{\lambda}$

$$\therefore X(x) = c_2 e^{\sqrt{\lambda}x} + c_3 e^{-\sqrt{\lambda}x} \qquad ...(9)$$

Where c_2 and c_3 are two arbitrary constants.

Substituting (8) , (9) in (2), we get

$$u(x,t) = c_1 e^{\alpha^2 \lambda t}\left[c_2 e^{\sqrt{\lambda}x} + c_3 e^{-\sqrt{\lambda}x}\right]$$

$$\Rightarrow u(x,t) = e^{\alpha^2 \lambda t}\left[A e^{\sqrt{\lambda}x} + B e^{-\sqrt{\lambda}x}\right] \qquad \ldots(10)$$

Where $A = c_1 c_2$, $B = c_1 c_3$

There are three possibilities to choose the constant λ

1. $\lambda > 0$, This is contrary to reality because the temperature increases infinitely with the passage of time.
2. $\lambda = 0$, This is also contrary to reality because the temperature will remain constant over time.
3. $\lambda < 0$, and this is the correct case because the temperature will increase slightly and be restrained with the passage of time.

Let $\lambda = -w^2 \Rightarrow \sqrt{\lambda} = wi$

Then equation (10) will be :

$$u(x,t) = e^{-\alpha^2 w^2 t}\left[A e^{iwx} + B e^{-iwx}\right]$$

from $e^{i\theta} = \cos\theta + i\sin\theta$, we get

$$u(x,t) = e^{-\alpha^2 w^2 t}[A(\cos wx + i\sin wx) + B(\cos wx - i\sin wx)]$$

$$= e^{-\alpha^2 w^2 t}[(A+B)\cos wx + (Ai - Bi)\sin wx]$$

$$u(x,t) = e^{-\alpha^2 w^2 t}[K\cos wx + L\sin wx] \qquad \ldots(11)$$

Where $K = A + B$, $L = Ai - Bi$

So equation (11) is the general solution of the heat equation.

2 General Solution of heat equation with homogeneous boundary Conditions

If both the ends of a bar of length l are at temperature zero and the initial temperature is to be prescribed function $\emptyset(x)$ in the bar. (i.e.) the boundary conditions are

$u(0,t) = 0^0$, $u(l,t) = 0^0$ (homo. Boundary conditions)

and the initial condition is $u(x,0) = \emptyset(x)$.

To find the general solution of heat equation under this conditions we will substitute the boundary and initial conditions one after the other in equation (11) as follows :

$$u(x,t) = e^{-\alpha^2 w^2 t}[K\cos wx + L\sin wx] \qquad \ldots(11)$$

Substituting the boundary condition $u(0,t) = 0$ in (11), we get

$$u(0,t) = e^{-\alpha^2 w^2 t}[K\cos 0 + L\sin 0]$$

$$0 = \underbrace{e^{-\alpha^2 w^2 t}}_{\neq 0} K \Rightarrow \boxed{K = 0} \qquad \ldots(12)$$

Butting (12) in (11), we get

$$u(x,t) = e^{-\alpha^2 w^2 t} L\sin wx \qquad \ldots(13)$$

Substituting the second boundary condition $(u(l,t) = 0)$ in (13), we get

$$u(l,t) = e^{-\alpha^2 w^2 t} L\sin wl$$

$$0 = \underbrace{e^{-\alpha^2 w^2 t}}_{\neq 0} \underbrace{L}_{\neq 0} \sin wl \Rightarrow \sin wl = 0 \Rightarrow wl = n\pi$$

$$\Rightarrow w = \frac{n\pi}{l}, n = 0, \pm 1, \pm 2, \dots$$

Then

$$u_n(x,t) = e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t} L_n \sin\frac{n\pi x}{l} \qquad \dots(14)$$

There are infinitely many functions in (14) so a general combination of them is an infinite series.

Thus we assume that

$$u(x,t) = \sum_{-\infty}^{\infty} u_n(x,t) = \sum_{-\infty}^{\infty} e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t} L_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=-\infty}^{0} e^{-\alpha^2 \frac{(n)^2\pi^2}{l^2}t} L_n \sin\frac{n\pi x}{l} + \sum_{n=0}^{\infty} e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t} L_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} e^{-\alpha^2 \frac{(-n)^2\pi^2}{l^2}t} L_{-n} \sin\frac{-n\pi x}{l} + \sum_{n=1}^{\infty} e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t} L_n \sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty} e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t} (-L_{-n}) \sin\frac{n\pi x}{l} + \sum_{n=1}^{\infty} e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t} L_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l} \qquad \dots(15)$$

where $A_n = L_n - L_{-n}$

Now, substituting the initial condition $(u(x,0) = \emptyset(x))$in equation (15), we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n e^0 \sin\frac{n\pi x}{l}$$

$$\emptyset(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

Which it is Fourier sine series, so the constants A_n are given by

$$A_n = \frac{2}{l}\int_0^l \emptyset(x)\sin\frac{n\pi x}{l}dx \; ; n = 1,2,3,\ldots$$

…(16)

Hence ,(15) is the required solution where A_n is given by (16).

<u>Ex 1:</u> Find the temperature $u(x,t)$ at any time in a metal rod (2 cm) long, homogeneous and insulated, which initially has a uniform temperature of $3x$, and it's ends are maintained at 0^0c for all $t > 0$. Then find the temperature of the middle of the rod at $t = 4$.

Sol. $l = 2 , \emptyset(x) = 3x , u(0,t) = 0 , u(2,t) = 0$

$$A_n = \frac{2}{l}\int_0^l \emptyset(x)\sin\frac{n\pi x}{l}dx = \frac{2}{2}\int_0^2 3x\sin\frac{n\pi x}{2}dx$$

$$= \left[\frac{-6x}{n\pi}\cos\frac{n\pi x}{2} + \frac{12}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2 = \frac{12(-1)^{n+1}}{n\pi}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n\pi} e^{-\alpha^2 \frac{n^2\pi^2}{4} t} \sin\frac{n\pi x}{2}$$

$$u(1,4) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n\pi} e^{-\alpha^2 \frac{n^2\pi^2}{4} 4} \sin\frac{n\pi}{2}$$

$$= \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n\pi} e^{-\alpha^2 n^2\pi^2} \sin\frac{n\pi}{2}$$

<u>Ex 2:</u> A rod of length 50 cm is homogeneous and insulated, is initially at a uniform temperature 20^0c, and it's ends are maintained at 0^0c for all $t > 0$, find the temperature $u(x,t)$.

Sol.

$l = 50\,, \emptyset(x) = 20^0\,, u(0,t) = 0\,, u(50,t) = 0$

$$A_n = \frac{2}{l}\int_0^l \emptyset(x) \sin\frac{n\pi x}{l} dx$$

$$= \frac{2}{50}\int_0^{50} 20 \sin\frac{n\pi x}{50} dx$$

$$= \frac{4}{5}\int_0^{50} \sin\frac{n\pi x}{50} dx$$

$$= \frac{-40}{n\pi} \cos\frac{n\pi x}{50} \Big|_0^{50}$$

$$= \frac{-40}{n\pi}[(-1)^n - 1] = \begin{cases} \dfrac{80}{n\pi} & if\ n\ is\ odd \\ 0 & if\ n\ is\ even \end{cases}$$

Finally, by substituting in $u(x,t)$, we get

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \sin\frac{n\pi x}{l}$$

$$= \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{80}{n\pi} e^{-\alpha^2 \frac{n^2\pi^2}{2500} t} \sin\frac{n\pi x}{50}$$

3 General Solution of heat equation with non-homogeneous boundary Conditions

Suppose now that one end of the bar is held at a constant temperature k_1 and the other is maintained at a constant temperature k_2 , then the boundary conditions are $u(0,t) = k_1$, $u(l,t) = k_2$, $t > 0$ the initial condition $u(x,0) = \emptyset(x)$ remain unchanged we can solve it by reducing it to a problem having homogeneous boundary conditions, which can then be solved as in previous case, thus we write

$$u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + U(x,t) \qquad \ldots(17)$$

We will prove that (17) is satisfy equation (1) we derive the equation (17) w.r.t. t and x , we have

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} \quad , \quad \frac{\partial u}{\partial x} = \frac{k_2 - k_1}{l} + \frac{\partial U}{\partial x} \quad \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial x^2}$$

Substituting in (1) we get :

$$\boxed{\frac{\partial U}{\partial t} = \alpha^2 \frac{\partial^2 U}{\partial x^2}} \qquad \ldots(18)$$

Substituting the boundary and initial conditions in (17), we have

$$k_1 = k_1 + 0 + U(0,t) \Rightarrow \boxed{U(0,t) = 0} \qquad \text{(from cond.1)}$$

$$k_2 = k_1 + \frac{l}{l}(k_2 - k_1) + U(l,t) \Rightarrow k_2 - k_1 - k_2 + k_1 = U(l,t)$$

$\Rightarrow \boxed{U(l,t) = 0}$ (from cond. 2)

$\emptyset(x) = k_1 + \frac{x}{l}(k_2 - k_1) + U(x,0)$ (from the initial cond.)

$$\boxed{U(x,0) = \emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)}$$

Hence the new equation $U(x,t)$ represent a heat conduction equation with homo. boundary conditions and initial condition

$$U(0,t) = 0\,, U(l,t) = 0\,, U(x,0) = \emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)$$

Then the general solution can be found as follows:

$$\boxed{U(x,t) = \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \sin \frac{n\pi}{l} x}$$

…(19) (from (15))

Where

$$\boxed{A_n = \frac{2}{l} \int_0^l \left[\emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)\right] \sin \frac{n\pi}{l} x \, dx}$$

…(20)

Substituting (19) in (17), we get

$$\boxed{u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \sin \frac{n\pi}{l} x}$$

…(21)

So (21) is the general solution with A_n shown in (20).

Ex. 3: **A rod of length 1, is initially at a uniform temperature x, the end $x = 0$ is heated to 2^0 and the end $x = 1$ is heated to 3^0, Find the temperature distribution in the rod at any time t.**

Sol.

$$u(0,t) = 2\ ,u(1,t) = 3\ \ ,u(x,0) = \emptyset(x) = x$$

$$A_n = \frac{2}{l}\int_0^l \left[\emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)\right] \sin\frac{n\pi}{l}x\,dx$$

$$= \frac{2}{1}\int_0^1 \left[x - 2 - \frac{x}{1}(3-2)\right] \sin\frac{n\pi}{1}x\,dx$$

$$= 2\int_0^1 [x - 2 - x] \sin n\pi x\,dx$$

$$= -4\int_0^1 \sin n\pi x\,dx$$

$$= \frac{4}{n\pi}\cos n\pi x \,\Big|_0^1 = \frac{4}{n\pi}((-1)^n - 1) = \begin{cases} \dfrac{-8}{n\pi} & n is\ odd \\ 0 & n\ is\ even \end{cases}$$

$$u(x,t) = 2 + \frac{x}{1}(3-2) + \sum_{n=1,3,5,\dots}^{\infty} \frac{-8}{n\pi} e^{-\alpha^2\frac{n^2\pi^2}{1}t} \sin\frac{n\pi}{1}x$$

$$= 2 + x + \sum_{\substack{n=1\\ n\ is\ odd}}^{\infty} \frac{-8}{n\pi} e^{-\alpha^2 n^2\pi^2 t} \sin n\pi x$$

<u>Ex. 4:</u> A copper rod of length 50 cm, is initially at a uniform temperature $\frac{x}{2}$, the end $x = 0$ is heated to 10^0c and the end $x = 50$ is heated to 35^0c, Find the temperature distribution in the rod at any time.

Sol.

$$l = 50\,, u(0,t) = 10^0 \;\; , u(50,t) = 35^0 \;\; , \emptyset(x) = \frac{x}{2}$$

$$A_n = \frac{2}{l}\int_0^l \left[\emptyset(x) - k_1 - \frac{x}{l}(k_2 - k_1)\right] \sin\frac{n\pi}{l} x\, dx$$

$$= \frac{2}{50}\int_0^{50} \left[\frac{x}{2} - 10 - \frac{x}{50}(35 - 10)\right] \sin\frac{n\pi}{50} x\, dx$$

$$= \frac{1}{25}\int_0^{50} \left[\frac{x}{2} - 10 - \frac{x}{2}\right] \sin\frac{n\pi}{50} x\, dx \; = \frac{-10}{25}\int_0^{50} \sin\frac{n\pi}{50} x\, dx$$

$$= \frac{20}{n\pi}\cos\frac{n\pi}{50} x \,\Big|_0^{50} = \frac{20}{n\pi}((-1)^n - 1) = \begin{cases} \dfrac{-40}{n\pi} & n\, is\ odd \\ 0 & n\ is\ even \end{cases}$$

$$u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + \sum_{n=1}^{\infty} A_n e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \sin\frac{n\pi}{l} x$$

$$u(x,t) = 10 + \frac{x}{50}(35 - 10) + \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{-40}{n\pi} e^{-\alpha^2 \frac{n^2\pi^2}{2500} t} \sin\frac{n\pi}{50} x$$

4 Bar with Insulated Ends

A slightly different problem occurs if the ends of the bar are insulated so that there is no passage of heat through them. Thus in this case of no heat flow the boundary conditions are

$$u_x(0,t) = 0\,, u_x(l,t) = 0\,, t > 0$$

<u>Ex :</u> Solve the equation $u_t = \alpha^2 u_{xx}$,$0 < x < l$,$0 \leq t < \infty$ that satisfies the conditions $u_x(0,t) = 0$, $u_x(l,t) = 0$, $u(x,0) = \emptyset(x)$

Sol. Using the equation

$$u(x,t) = e^{-\alpha^2 w^2 t}[K\cos wx + L\sin wx] \qquad \ldots(1)$$

Differentiating (1) for x

$$u_x(x,t) = e^{-\alpha^2 w^2 t}[-Kw\sin wx + Lw\cos wx] \qquad \ldots(2)$$

Butting the condition (1) in (2)

$$u_x(0,t) = e^{-\alpha^2 w^2 t}[-Kw\sin 0 + Lw\cos 0]$$

$$0 = e^{-\alpha^2 w^2 t}Lw \Rightarrow \boxed{L = 0} \qquad \ldots(3)$$

Butting (3) in (1)

$$u(x,t) = e^{-\alpha^2 w^2 t}K\cos wx \qquad \ldots(4)$$

Differentiating (4) for x

$$u_x(x,t) = -e^{-\alpha^2 w^2 t}Kw\sin wx \qquad \ldots(5)$$

Substituting condition 2 in (5), we get

$$u_x(l,t) = -e^{-\alpha^2 w^2 t}Kw\sin wl$$

$$0 = -e^{-\alpha^2 w^2 t} K w \sin wl \Rightarrow \sin wl = 0$$

$$\Rightarrow wl = n\pi \quad , n = 0, \pm 1, \pm 2, \ldots \qquad \Rightarrow w = \frac{n\pi}{l}$$

Substituting this in equation (4), we get

$$u(x,t) = e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} K \cos\frac{n\pi}{l} x \qquad \ldots(6)$$

Since n has infinite values, there are infinitely many functions in (6) so a general combination of them is an infinite series, thus

$$u_0(x,t) = K_0$$

$$u_n(x,t) = e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} K_n \cos\frac{n\pi}{l} x$$

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \; e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \cos\frac{n\pi}{l} x \qquad \ldots(7)$$

Where $K_0 = \frac{c_0}{2}$ $, c_n = K_n + K_{-n}$

substituting the third condition in (7), then

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \; e^0 \cos\frac{n\pi}{l} x$$

$$\emptyset(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi}{l} x$$

Which it is Fourier cosine series, so the constants c_0 and c_n are given by

$$c_0 = \frac{2}{l} \int_0^l \emptyset(x)\, dx$$

$$c_n = \frac{2}{l}\int_0^l \emptyset(x)\cos\frac{n\pi}{l}x\,dx \quad ,n = 1,2,3,\ldots \qquad \ldots(8)$$

Hence (8) is the required solution.

Ex. : Find the solution of the heat problem in a bar of length 2

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \qquad t > 0$$

with initial heat distribution 2x and no loss at either end (no heat flux in the x-direction at either end)

$u_x(0,t) = 0\,, u_x(2,t) = 0.$

Sol.

$l = 2\,, \emptyset(x) = 2x, u_x(0,t) = 0\,, u_x(2,t) = 0$

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n\; e^{-\alpha^2\frac{n^2\pi^2}{l^2}t}\cos\frac{n\pi}{l}x$$

where

$$c_0 = \frac{2}{l}\int_0^l \emptyset(x)\,dx \;\Rightarrow\; c_0 = \frac{2}{2}\int_0^2 2x\,dx = 2\frac{x^2}{2}\Big|_0^2 = 4$$

$$c_n = \frac{2}{2}\int_0^2 2x\cos\frac{n\pi}{2}x\,dx = 2\int_0^2 x\cos\frac{n\pi}{2}x\,dx \qquad \text{(by u dv)}$$

$$= \frac{8}{n^2\pi^2}((-1)^n - 1) = \begin{cases} \dfrac{-16}{n^2\pi^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$u(x,t) = \frac{4}{2} + \sum_{\substack{n=1 \\ n \text{ is odd}}}^{\infty} \frac{-16}{n^2\pi^2}\; e^{-\alpha^2\frac{n^2\pi^2}{4}t}\cos\frac{n\pi}{2}x$$

5 Bar with mixed boundary conditions

In this problems the boundary conditions of the heat equation of the forms

a) $u(0,t) = 0\,, u_x(l,t) = 0\,,\ t > 0$

b) $u_x(0,t) = 0\,, u(l,t) = 0\,,\ t > 0$

and the initial condition $u(x,0) = \emptyset(x)$.

Case A:

In this case the boundary condition of the heat equation is considered of the form $u(0,t) = 0\,, u_x(l,t) = 0\,,\ t > 0$ and the initial condition $u(x,0) = \emptyset(x)$.

Ex : **Solve the equation $u_t = \alpha^2 u_{xx}\ , 0 < x < l\,, 0 \le t < \infty$ that satisfies the conditions $u(0,t) = 0,\ \ u_x(l,t) = 0\,, u(x,0) = \emptyset(x)$**

Sol. Using the equation

$$u(x,t) = e^{-\alpha^2 w^2 t}[K\cos wx + L\sin wx] \qquad \dots(1)$$

Butting the condition (1) in (1)

$$u(0,t) = e^{-\alpha^2 w^2 t}[K\cos 0 + L\sin 0]$$

$$0 = e^{-\alpha^2 w^2 t}K \Rightarrow \boxed{K = 0} \qquad \dots(2)$$

Butting (2) in (1)

$$u(x,t) = e^{-\alpha^2 w^2 t}L\sin wx \qquad \dots(3)$$

Differentiating (3) for x

$$u_x(x,t) = e^{-\alpha^2 w^2 t}L\, w\cos wx \qquad \dots(4)$$

Substituting condition 2 in (4), we get

$$u_x(l,t) = e^{-\alpha^2 w^2 t} Lw \cos wl \qquad \ldots(5)$$

$$0 = e^{-\alpha^2 w^2 t} Lw \cos wl \Rightarrow \cos wl = 0$$

$$\Rightarrow wl = \frac{(2n+1)\pi}{2} \quad , n = 0,1,2,\ldots \qquad \Rightarrow w = \frac{(2n+1)\pi}{2l}$$

Substituting this in equation (3), we get

$$u(x,t) = e^{-\alpha^2 \frac{(2n+1)^2 \pi^2}{4l^2} t} L \sin \frac{(2n+1)\pi}{2l} x \qquad \ldots(6)$$

Since n has infinite values, there are infinitely many functions in (6) so a general combination of them is an infinite series, thus

$$u_n(x,t) = e^{-\alpha^2 \frac{(2n+1)^2 \pi^2}{4l^2} t} L_n \sin \frac{(2n+1)\pi}{2l} x$$

$$u(x,t) = \sum_{n=0}^{\infty} L_n \; e^{-\alpha^2 \frac{(2n+1)^2 \pi^2}{4l^2} t} \sin \frac{(2n+1)\pi}{2l} x \qquad \ldots(7)$$

substituting the third condition in (7), then

$$u(x,0) = \sum_{n=0}^{\infty} L_n \; e^0 \sin \frac{(2n+1)\pi}{2l} x$$

$$\emptyset(x) = \sum_{n=0}^{\infty} L_n \sin \frac{(2n+1)\pi}{2l} x$$

Which it is Fourier sine series, so the constant L_n are given by

$$L_n = \frac{2}{l} \int_0^l \emptyset(x) \sin \frac{(2n+1)\pi}{2l} x \, dx \quad , n = 0,1,2,\ldots \qquad \ldots(8)$$

Hence (8) is the required solution.

Ex: Find the solution of the heat problem $u_t = u_{xx}$, $0 < x < 5$,

$0 < t < \infty$, $u(0,t) = 0$, $u_x(5,t) = 0$, $u(x,0) = 5$.

Sol:

$$u(x,t) = \sum_{n=0}^{\infty} L_n \; e^{-\alpha^2 \frac{(2n+1)^2\pi^2}{4l^2} t} \sin \frac{(2n+1)\pi}{2l} x$$

where

$$L_n = \frac{2}{l}\int_0^l \emptyset(x) \sin \frac{(2n+1)\pi}{2l} x \, dx$$

$$L_n = \frac{2}{5}\int_0^5 5 \sin \frac{(2n+1)\pi}{2*5} x \, dx$$

$$= 2\int_0^5 \sin \frac{(2n+1)\pi}{10} x \, dx$$

$$= \frac{-20}{(2n+1)\pi} \cos\left(\frac{(2n+1)\pi}{10}\right) x \Big|_0^5$$

$$= \frac{-20}{(2n+1)\pi}\left[\cos\left(\frac{(2n+1)\pi}{2}\right) - \cos(0)\right]$$

$$= \frac{20}{(2n+1)\pi}$$

Then

$$u(x,t) = \sum_{n=0}^{\infty} \frac{20}{(2n+1)\pi} \; e^{-\alpha^2 \frac{(2n+1)^2\pi^2}{4*25} t} \sin \frac{(2n+1)\pi}{10} x$$

Case B:

In this case the boundary condition of the heat equation is considered of the form $u_x(0,t)=0\,,u(l,t)=0\,,\ t>0$ and the initial condition $u(x,0)=\emptyset(x)$.

<u>**Ex :**</u> **Solve the equation $u_t=\alpha^2 u_{xx}\ ,0<x<l\,,0\le t<\infty$ that satisfies the conditions $u_x(0,t)=0,\ \ u(l,t)=0\,,u(x,0)=\emptyset(x)$**

Sol. Using the equation

$$u(x,t)=e^{-\alpha^2w^2t}[K\cos wx+L\sin wx] \qquad \ldots(1)$$

Differentiating (1) for x

$$u_x(x,t)=e^{-\alpha^2w^2t}[-Kw\sin wx+Lw\cos wx] \qquad \ldots(2)$$

Butting the condition (1) in (2)

$$u_x(0,t)=e^{-\alpha^2w^2t}[-Kw\sin 0+Lw\cos 0]$$

$$0=e^{-\alpha^2w^2t}Lw\Rightarrow \boxed{L=0} \qquad \ldots(3)$$

Butting (3) in (1)

$$u(x,t)=e^{-\alpha^2w^2t}K\cos wx \qquad \ldots(4)$$

Substituting the condition (2) in (4)

$$u(l,t)=e^{-\alpha^2w^2t}K\cos wl \qquad \ldots(5)$$

$$0=e^{-\alpha^2w^2t}K\cos wl\Rightarrow \cos wl=0$$

$$\Rightarrow wl=\frac{(2n+1)\pi}{2}\quad ,n=0,1,2,\ldots \qquad \Rightarrow w=\frac{(2n+1)\pi}{2l}$$

Substituting this in equation (4), we get

$$u(x,t)=e^{-\alpha^2\frac{(2n+1)^2\pi^2}{4l^2}t}K\cos\frac{(2n+1)\pi}{2l}x \qquad \ldots(6)$$

Since n has infinite values, there are infinitely many functions in (6) so a general combination of them is an infinite series, thus

$$u_n(x,t) = e^{-\alpha^2\frac{(2n+1)^2\pi^2}{4l^2}t} K_n \cos\frac{(2n+1)\pi}{2l}x$$

$$u(x,t) = \sum_{n=0}^{\infty} K_n\ e^{-\alpha^2\frac{(2n+1)^2\pi^2}{4l^2}t} \cos\frac{(2n+1)\pi}{2l}x \qquad \ldots(7)$$

substituting the third condition in (7), then

$$u(x,0) = \sum_{n=0}^{\infty} K_n\ e^0 \cos\frac{(2n+1)\pi}{2l}x$$

$$\emptyset(x) = \sum_{n=0}^{\infty} K_n \cos\frac{(2n+1)\pi}{2l}x$$

Which it is Fourier cosine series, so the constant K_n are given by

$$K_n = \frac{2}{l}\int_0^l \emptyset(x)\cos\frac{(2n+1)\pi}{2l}x\,dx \ ,n = 0,1,2,\ldots \qquad \ldots(8)$$

Hence (8) is the required solution.

<u>Ex:</u> Find the solution of the heat problem $u_t = u_{xx}$, $0 < x < 2$,

$0 < t < \infty$, $\quad u_x(0,t) = 0$, $u(2,t) = 0$, $\quad u(x,0) = 4x$.

Sol:

$$u(x,t) = \sum_{n=0}^{\infty} K_n \; e^{-\alpha^2 \frac{(2n+1)^2\pi^2}{4l^2} t} \cos\frac{(2n+1)\pi}{2l} x$$

where

$$K_n = \frac{2}{l}\int_0^l \emptyset(x)\cos\frac{(2n+1)\pi}{2l} x\, dx$$

$$K_n = \frac{2}{2}\int_0^2 4x\cos\frac{(2n+1)\pi}{2*2} x\, dx$$

$$= 4\int_0^2 x\, cos\frac{(2n+1)\pi}{4} x\, dx \qquad \text{(by u dv)}$$

$$= \frac{32}{(2n+1)\pi}(-1)^n - \frac{64}{(2n+1)^2\pi^2}$$

Then

$$u(x,t) =$$

$$\sum_{n=0}^{\infty}\left[\frac{32}{(2n+1)\pi}(-1)^n - \frac{64}{(2n+1)^2\pi^2}\right] e^{-\alpha^2\frac{(2n+1)^2\pi^2}{4*4}t}\cos\frac{(2n+1)\pi}{4} x$$

… Exercises …

1. A rod of length (10 cm), is initially at a uniform temperature $2x$, and it's ends are maintained at 0^0c , find the temperature $u(x,t)$ at any time.

2. Find the solution of the heat problem $u_t = 100u_{xx}$, $0 < x < 1$, $0 < t < \infty$, $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 5^0$, $0 \le x \le 1$.

3. Find the solution of the heat problem $u_t = \alpha^2 u_{xx}$, $0 < x < 2$, $0 < t < \infty$, $u_x(0,t) = 0$, $u_x(2,t) = 0, u(x,0) = 4x, 0 \le x \le 2$

4. Find the temperature $u(x,t)$ in a metal rod of length (25 cm) that is insulated on the ends as well as on the sides and whose initial temperature distribution is $u(x,0) = x$ for $0 < x < 25$.

5. A rod of length (30 cm) , is initially at a uniform temperature $(60 - 2x)$, the end $x = 0$ is heated 20^0c and the end $x = 30$ is heated to 50^0c . find the temperature distribution in the rod at any time.

6. A rod of length 1 unit, is initially at a uniform temperature , the temperature of one ends is equal to zero and the rate of change of temperature in the other end is equal to zero too. find the temperature distribution in the rod at any time.

7. Find the solution of the heat problem $u_t = u_{xx}$, $0 < x < 3$, $0 < t < \infty$, $u(0,t) = 0$, $u_x(3,t) = 0$, $u(x,0) = 3x$.

Chapter Five

One Dimensional Wave Equation

Section(5.1): The Wave Equation: Vibration of an Elastic string

A second partial differential equation occurring frequent in applied mathematics is the wave equation. Some form of this equation, or a generalization of it, almost inevitably arises in any mathematical analysis of phenomena involving the propagation of waves, electromagnetic waves, and seismic waves are all based on this equation.

Perhaps the easiest situation to visualize occurs in the investigation of mechanical vibrations. Suppose that an elastic string of length l is tightly stretched between two supports at the same horizontal level, so that the x-axis lies along the string (see figure 1). The elastic string may be thought of as a violin string, a guy wire, or possibly an electric power line.

Suppose that the string is set in motion (by plucking, for example) so that it vibrates in a vertical plane and let $u(x,t)$ denote the vertical displacement experienced by the string at the point x at time t. If damping effects, such as air resistance, are neglected, and if the amplitude of the motion is not too large, then $u(x,t)$ satisfies the partial differential equation:

$$u_{tt} = c^2 u_{xx} \qquad \dots (1)$$

In the domain $0 < x < l,\ 0 < t \leq \infty$. Equation (1) is known as the (wave equation), where the constant c^2 is given by $c^2 = \frac{T}{p}$

where T is the tension of the string and p is the mass per length of the string material.

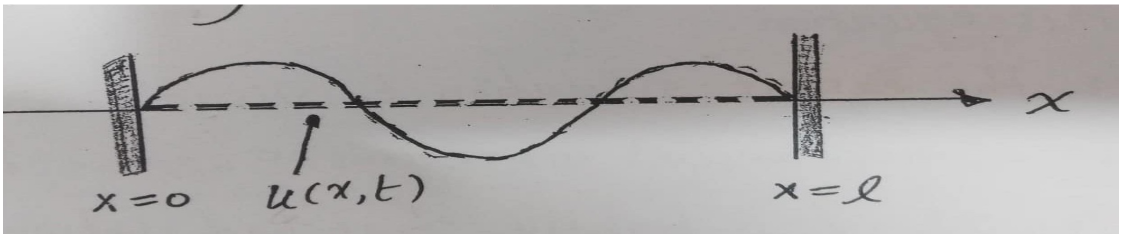


Figure .1

By the same way that in the heat equation we will solve equation (1) by the separation of variables method.

Let $u(x,t) = X(x).T(t)$ ... ... (2)

Deriving Eq.(2) twice w.r.t x and t and substituting in (1), we get

$$\frac{T''(t)}{c^2\,T(t)} = \frac{X''(x)}{X(x)}$$

Equating this equation to a constant, say λ:

$$\frac{T''(t)}{c^2\,T(t)} = \frac{X''(x)}{X(x)} = \lambda \qquad \text{... ... (3)}$$

Then, we get

$$T'' - \lambda c^2\,T = 0 \qquad \text{... .... (4)}$$

$$X'' - \lambda X = 0 \qquad \text{... .... (5)}$$

Solving this two ordinary differential equations, we obtain

$$T(t) = c_1 e^{c\sqrt{\lambda}t} + c_2 e^{-c\sqrt{\lambda}t} \qquad \text{... ... (6)}$$

$$X(x) = c_3 e^{\sqrt{\lambda}x} + c_4 e^{-\sqrt{\lambda}x} \qquad \text{... ... (7)}$$

Substituting (6), (7) in (2), we get

$$u(x,t) = \left(c_1 e^{c\sqrt{\lambda}t} + c_2 e^{-c\sqrt{\lambda}t}\right)\left(c_3 e^{\sqrt{\lambda}x} + c_4 e^{-\sqrt{\lambda}x}\right) \qquad \text{... ... (8)}$$

Where c_1, c_2, c_3 and c_4 are constants

There are three cases to choose λ:

1- $\lambda > 0$, this leads to an elastic string will vibrate without stopping, and this is contrary to reality.

2- $\lambda = 0$, this leads $u(x,t)$ is constant and this is contrary to reality too.

3- $\lambda < 0$, this is the right situation, let $\lambda = -w^2 \Rightarrow \sqrt{\lambda} = wi$

Substituting in (8), we get

$u(x,t) = [k_1 \cos cwt + k_2 \sin cwt][k_3 \cos wx + k_4 \sin wx] \ldots (9)$

Where $k_1 = c_1 + c_2, k_2 = c_1 i - c_2 i, k_3 = c_3 + c_4, k_4 = c_3 i - c_4 i$

The equation (9) is the general solution of Eq.(1).

Section (5.2): General solution of one- dimensional wave equation satisfying the given boundary and initial conditions.

5.2.1 General Solution of One- Dimensional Wave Equation with Homogeneous Dirichlet Boundary Conditions

Suppose that we have an elastic string of length l, its ends are fixed at $x = 0$ and $x = l$, then we have the two boundary conditions

$u(0,t) = 0, u(l,t) = 0 \qquad \ldots\ldots (10)$

The form of the motion of the string will depend on the initial deflection (deflection at t=0) and on the initial velocity (velocity at t=0). Denoting the initial deflection by $f(x)$ and the initial velocity by $g(x)$, we arrive at two initial conditions

$u(x,0) = f(x), u_t(x,0) = g(x), \quad 0 \le x \le l \qquad \ldots\ldots (11)$

Our problem now is to find a solution of (1) satisfying the conditions (10), (11).

Substituting the condition $u(0,t)$ in (9), we get

$u(0,t) = [k_1 \cos cwt + k_2 \sin cwt][k_3 \cos 0 + k_4 \sin 0]$

$0 = [k_1 \cos cwt + k_2 \sin cwt]k_3 \Rightarrow k_3 = 0$

Substituting in (9), we obtain

$$u(x,t) = [k_1 \cos cwt + k_2 \sin cwt]\, k_4 \sin wx \quad \ldots\ldots(12)$$

Substituting the condition $u(l,t) = 0$ in (12), we get

$$u(l,t) = [k_1 \cos cwt + k_2 \sin cwt]\, k_4 \sin wl$$

$$0 = \underbrace{[k_1 \cos cwt + k_2 \sin cwt]}_{\neq 0}\, \underbrace{k_4}_{\neq 0} \sin wl$$

$\therefore \sin wl = 0 \longrightarrow wl = n\pi; \quad n = 1,2,3, \ldots.$

$$\boxed{w = \frac{n\pi}{l}}$$

Substituting in (12), we get

$$u(x,t) = \left[k_1 \cos\frac{cn\pi t}{l} + k_2 \sin\frac{cn\pi t}{l}\right] k_4 \sin\frac{n\pi x}{l}$$

$$\Longrightarrow u(x,t) = \left[R_1 \cos\frac{cn\pi t}{l} + R_2 \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l} \quad \ldots\ldots(13)$$

Where $R_1 = k_1 k_4, R_2 = k_2 k_4$

Since *n* has infinite values, then there are non-zero solutions

$u_n(x,t)$ of (13)

$$u_n(x,t) = \left[R_{1n} \cos\frac{cn\pi t}{l} + R_{2n} \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l}$$

We consider more general solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left[R_{1n} \cos\frac{cn\pi t}{l} + R_{2n} \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l} \quad \ldots\ldots(14)$$

Substituting the initial condition $u(x,0) = f(x)$ in (14), we get

$$\mathrm{u}(\mathrm{x},0) = \sum_{n=1}^{\infty} [R_{1n} \cos 0 + R_{2n} \sin 0] \sin\frac{n\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} R_{1n} \sin\frac{n\pi x}{l}$$

Which is Fourier sine series, then

$$R_{1n} = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx \ldots\ldots (15)$$

Differentiating (14) partially w.r.t *t* we get:

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[\frac{-cn\pi}{l} R_{1n} \sin\frac{cn\pi t}{l} + \frac{cn\pi}{l} R_{2n} \cos\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l}$$

Substituting the initial velocity condition $u_t(x,0) = g(x)$, we get

$$u_t(x,0) = \sum_{n=1}^{\infty} \left[\frac{-cn\pi}{l} R_{1n} \sin 0 + \frac{cn\pi}{l} R_{2n} \cos 0\right] \sin\frac{n\pi x}{l}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{l} R_{2n} \sin\frac{n\pi x}{l}$$

Which is Fourier sine series, then

$$\frac{cn\pi}{l} R_{2n} = \frac{2}{l}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx \longrightarrow R_{2n} = \frac{2}{cn\pi}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx \quad (16)$$

Hence the required solution is given by (14) where

R_{1n} and R_{2n} are given in (15) and (16).

Ex. 1: An elastic string of length (2 cm) has its ends *x=0* and *x=2* fixed with no initial displacement. The string is released with initial velocity equal *x*. Find the displacement function $u(x,t)$.

Sol.:

$l = 2, f(x) = 0, g(x) = x$

Then $R_{1n} = 0$ because $f(x) = 0$

$$R_{2n} = \frac{2}{cn\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{cn\pi}\int_0^2 x\sin\frac{n\pi x}{2}\,dx = \frac{8\,(-1)^{n+1}}{c\,n^2\pi^2}$$

$$u(x,t) = \sum_{n=1}^{\infty} R_{2n}\quad sin\frac{cn\pi t}{l}\,\sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty}\frac{8\,(-1)^{n+1}}{c\,n^2\pi^2}\quad sin\frac{cn\pi t}{2}\,\sin\frac{n\pi x}{2}$$

Ex. 2: Solve the wave equation $u_{tt} = u_{xx},\ 0 \le x \le 1,\ 0 \le t < \infty$ under the following conditions:

$$\mathbf{u(0,t) = 0,\ u(1,t) = 0, u(x,0) = 3, u_t\,(x,0) = 5.}$$

Sol.: $l = 1$ then from equation (14), we have

$$u(x,t) = \sum_{n=1}^{\infty}\left[R_{1n}\cos\frac{cn\pi t}{l} + R_{2n}\sin\frac{cn\pi t}{l}\right]\sin\frac{n\pi x}{l}$$

$$R_{1n} = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$R_{1n} = \frac{2}{1}\int_0^1 3\ \sin\frac{n\pi x}{1}\ dx = \frac{-6}{n\pi}\cos n\pi x|_0^1 = \frac{-6}{n\pi}[(-1)^n - 1]$$

$$R_{2n} = \frac{2}{cn\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\ dx$$

$$R_{2n} = \frac{2}{cn\pi}\int_0^1 5\sin\frac{n\pi x}{1}\ dx$$

$$= \frac{-10}{cn^2\pi^2}\cos n\pi x|_0^1 \quad = \frac{-10}{cn^2\pi^2}[(-1)^n - 1]$$

$$\therefore u(x,t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty}\left[\frac{-6}{n\pi}[(-1)^n - 1]\cos cn\pi t - \frac{10}{cn^2\pi^2}[(-1)^n - 1]\sin cn\pi t\right]\sin n\pi x$$

5.2.2 General Solution of One- Dimensional Wave Equation with Non-Homogeneous Dirichlet Boundary Conditions

Consider the initial-value problem for the wave equation on an interval with non-homogeneous Dirichlet boundary Conditions

$$u(0,t) = k_1 \, , u(l,t) = k_2 \tag{1}$$

The form of the motion of the string will depend on the initial deflection (deflection at *t*=0) and on the initial velocity (velocity at *t*=0). Denoting the initial deflection by $f(x)$ and the initial velocity by $g(x)$, we arrive at two initial conditions

$$u(x,0) = f(x) \, , u_t(x,0) = g(x), \quad 0 \leq x \leq l \tag{2}$$

Our problem now is to find a solution of the wave function $u_{tt} = c^2 u_{xx}$ satisfying the conditions (1), (2).

In this case, we can use the transformation

$$u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + U(x,t) \tag{3}$$

We will prove that (3) is satisfy the wave equation with the homogeneous boundary conditions

We derive the equation (3) w.r.t. t and x twice, we have

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} \Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 U}{\partial t^2}, \quad \frac{\partial u}{\partial x} = \frac{k_2 - k_1}{l} + \frac{\partial U}{\partial x} \quad \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 U}{\partial x^2}$$

Substituting in $u_{tt} = c^2 u_{xx}$ we get :

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$$

Substituting the boundary and initial conditions in (3), we have

$$k_1 = k_1 + 0 + U(0,t) \Rightarrow \quad U(0,t) = 0$$

$$k_2 = k_1 + \frac{l}{l}(k_2 - k_1) + U(l,t) \Rightarrow k_2 - k_1 - k_2 + k_1 = U(l,t)$$

$$\therefore U(l,t) = 0$$

$$f(x) = k_1 + \frac{x}{l}(k_2 - k_1) + U(x,0)$$

$$\therefore U(x,0) = f(x) - k_1 - \frac{x}{l}(k_2 - k_1)$$

$U_t(x,t) = u_t(x,t)$ then $U_t(x,0) = u_t(x,0) = g(x)$

Hence the new equation $U(x,t)$ represent a wave conduction equation with homogeneous Dirichlet boundary conditions and initial conditions

$$U(0,t) = 0\,, U(l,t) = 0\,,$$

$$U(x,0) = f(x) - k_1 - \frac{x}{l}(k_2 - k_1),\ \ U_t(\mathrm{x},0) = g(x)$$

Then, the general solution can be found as follows:

$$U(x,t) = \sum_{n=1}^{\infty} \left[R_{1n} \cos\frac{cn\pi t}{l} + R_{2n} \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l} \tag{4}$$

where

$$R_{1n} = \frac{2}{l}\int_0^l (f(x) - k_1 - \frac{x}{l}(k_2 - k_1)) \sin\frac{n\pi x}{l}\, dx \tag{5}$$

and

$$R_{2n} = \frac{2}{cn\pi}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx \tag{6}$$

Substituting (4) in (3), we get

$$u(x,t) = k_1 + \frac{x}{l}(k_2 - k_1) + \sum_{n=1}^{\infty}\left[R_{1n}\cos\frac{cn\pi t}{l} + R_{2n}\sin\frac{cn\pi t}{l}\right]\sin\frac{n\pi x}{l} \quad (7)$$

So (7) is the general solution with R_{1n} and R_{2n} given in (5) and (6).

Example 1: Solve the wave equation $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$ under the following conditions:

$u(0,t) = 1, \qquad u(1,t) = 2$

$u(x,0) = 1 + x, \qquad u_t(x,0) = \pi \sin\pi\ x$

Sol:

$$R_{1n} = \frac{2}{l}\int_0^l (f(x) - k_1 - \frac{x}{l}(k_2 - k_1))\sin\frac{n\pi x}{l}\,dx$$

$$R_{1n} = \frac{2}{1}\int_0^1 ((1+x) - 1 - \frac{x}{1}(2-1))\sin\frac{n\pi x}{1}\,dx = 0$$

$$R_{2n} = \frac{2}{cn\pi}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx$$

$$R_{2n} = \frac{2}{n\pi}\int_0^1 \pi\sin\pi x\sin\frac{n\pi x}{1}\,dx =$$

$$= \frac{2}{n}\int_0^1 \frac{1}{2}[\cos(1-n)\pi x - \cos(1+n)\pi x]dx$$

$$= \frac{1}{n}\left[\frac{\sin(1-n)\pi x}{\pi(1-n)} - \frac{\sin(1+n)\pi x}{\pi(1+n)}\right]_0^1 = 0 \qquad ; n \neq 1$$

When $n = 1$

$$R_{21} = \frac{2}{\pi}\int_0^1 \pi \sin \pi x \sin \pi x \; dx =$$

$$= 2\int_0^1 \sin^2 \pi x \, dx$$

$$= 2\int_0^1 \frac{1}{2}(1 - \cos 2\pi x)\, dx = x|_0^1 - \frac{1}{2\pi}\sin 2\pi x\Big|_0^1 = 1$$

$$u(x,t) = 1 + \frac{x}{1}(2-1) + \sin\frac{\pi t}{1} \sin\frac{\pi x}{1} = 1 + x + \sin \pi t \; \sin \pi x$$

Example 2: Solve the wave equation $u_{tt} = u_{xx}, \;\; 0 < x < \pi, \;\; t > 0$ under the following conditions:

$\mathrm{u}(0,\mathrm{t}) = 1, \qquad \mathrm{u}(\pi,\mathrm{t}) = 1 + \pi$

$\mathrm{u}(\mathrm{x},0) = 1 + \mathrm{x}, \qquad \mathrm{u_t}(\mathrm{x},0) = \sin x$

Sol:

$$R_{1n} = \frac{2}{l}\int_0^l (f(x) - k_1 - \frac{x}{l}(k_2 - k_1)) \sin\frac{n\pi x}{l}\; dx$$

$$R_{1n} = \frac{2}{\pi}\int_0^\pi ((1+\mathrm{x}) - 1 - \frac{x}{\pi}(1 + \pi - 1)) \sin\frac{n\pi x}{\pi}\; dx = 0$$

$$R_{2n} = \frac{2}{cn\pi}\int_0^l g(x) \sin\frac{n\pi x}{l}\; dx$$

$$R_{2n} = \frac{2}{n\pi}\int_0^\pi \sin x \sin\frac{n\pi x}{\pi}\; dx =$$

$$= \frac{2}{n\pi}\int_0^\pi \frac{1}{2}[\cos\,(1-n)\,x - \cos\,(1+n)\,x] dx$$

$$= \frac{1}{n\pi}\left[\frac{\sin(1-n)\ x}{1-n} - \frac{\sin(1+n)\,x}{1+n}\right]_0^{\pi} = 0 \qquad ; n \neq 1$$

When $n = 1$

$$R_{21} = \frac{2}{\pi}\int_0^{\pi} \sin x \sin x \ dx =$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(1-\cos 2x)\, dx = \frac{1}{\pi}\left[x|_0^{\pi} - \frac{1}{2}\sin 2x\Big|_0^{\pi}\right] = 1$$

Then $R_{21} = 1$

$$u(x,t) = 1 + \frac{x}{\pi}(1+\pi-1) + \sin t \ \sin x = 1 + x + \sin t \ \sin x$$

Section 5.3: The D'Alembert Solution of the Wave Equation

In the case of the free vibration of an infinite string, the required function $u(x, t)$ must satisfy the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

With the initial conditions:

$$u(x, 0) = f(x), u_t\,(x, 0) = g(x) \qquad (2)$$

Where $f(x)$ and $g(x)$ must be specified in the interval $(-\infty, \infty)$ since the string is infinite.

The general solution of (1) can in fact be found, and in such a form that conditions (2) can easily be satisfied.

For this, we transform (1) to the new independent variables:

$$\xi = x + ct, \;\; \eta = x - ct \qquad (3)$$

These variables are called the (canonical coordinates).

On taking u as depending on x and t indirectly via ξ and η

We can express the derivatives with respect to the first variables in term of the derivatives with respect to the new variables:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial u}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial x}}_{=1} \quad \text{(from (3))}$$

$$= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\therefore \boxed{u_x = u_\xi + u_\eta} \qquad (4)$$

Also, $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \cdot \underbrace{\frac{\partial \xi}{\partial t}}_{=c} + \frac{\partial u}{\partial \eta} \cdot \underbrace{\frac{\partial \eta}{\partial t}}_{=-c}$ (from (3))

$$= c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$$

$$\therefore \boxed{u_t = c(\,u_\xi - u_\eta)} \qquad (5)$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}.\frac{\partial u}{\partial x}$$

$$= \frac{\partial}{\partial x}.\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \qquad \text{(from (4))}$$

$$= \frac{\partial}{\partial \xi}.\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \frac{\partial}{\partial \eta}.\frac{\partial \mathrm{u}}{\partial \mathrm{x}}$$

$$= \frac{\partial}{\partial \xi}.\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}.\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\therefore \boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}} \qquad (6)$$

By the same way, we get

$$\boxed{u_{tt} = c^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right)} \qquad (7)$$

Substituting (6) and (7) in (1), we get

$$c^2\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right) = c^2\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right)$$

$$c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta} - c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} - c^2 u_{\eta\eta} = 0$$

$$-4c^2 u_{\xi\eta} = 0 \xRightarrow{\div -4c^2} \boxed{u_{\xi\eta} = 0} \qquad (8)$$

Integrating (8) w.r.t ξ and η, we get $\quad u_\eta = \phi_1(\eta)$

$$u = \int \phi_1(\eta)\,\partial\eta + \phi_2(\xi)$$

$u(\xi,\eta) = \Phi(\eta) + \Psi(\xi)$

Where $\Phi(\eta) = \int \phi_1(\eta)\, \partial\eta$ and $\Psi(\xi) = \phi_2(\xi)$

Φ and Ψ are two arbitrary functions.

Now, returning to the old variables x and t, we get

$$\boxed{u(x,t) = \Phi(x-ct) + \Psi(x+ct)} \qquad (9)$$

Substituting the initial condition $u(x,0) = f(x)$ in (9)

We get $u(x,0) = \Phi(x) + \Psi(x)$

$$\therefore \boxed{f(x) = \Phi(x) + \Psi(x)} \qquad (10)$$

Differentiating (9) w.r.t (t),

$u_t(x,t) = \Phi'(x-ct)(-c) + c\Psi'(x+ct)$

Substituting the second initial condition $u_t(x,0) = g(x)$

$g(x) = -c\Phi'(x) + c\Psi'(x)$

Integrating this equation from 0 to x:

$$k + \int_0^x g(z)\, dz = -c\Phi(x) + c\Psi(x)$$

$$\Longrightarrow \boxed{\frac{k}{c} + \frac{1}{c}\int_0^x g(z)\, dz = -\Phi(x) + \Psi(x)} \qquad (11)$$

From (10) and (11), we can easily find $\Phi(x)$ and $\Psi(x)$, as follows:

$$\Phi(x) = \frac{1}{2}f(x) - \frac{k}{2c} - \frac{1}{2c}\int_0^x g(z)\, dz$$

$$\Psi(x) = \frac{1}{2}f(x) + \frac{k}{2c} + \frac{1}{2c}\int_0^x g(z)\, dz$$

Then replacing x by $(x-ct)$ in Φ and by $(x+ct)$ in Ψ and substituting in (9)

$$u(x,t)=\frac{1}{2}f\,(x-ct)-\frac{k}{2c}-\frac{1}{2c}\int_{0}^{x-ct} g(z)\,dz+\frac{1}{2}f\,(x+ct)+\frac{k}{2c}$$
$$+\frac{1}{2c}\int_{0}^{x+ct} g(z)\,dz$$

$$\Longrightarrow \boxed{u(x,t)=\frac{1}{2}[f\,(x-ct)+f\,(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz} \qquad (12)$$

This is D'Alembert 's solution to (1) subject to (2), on the interval $(-\infty,\infty)$.

<u>Ex.1:</u> Solve the equation $\mathbf{u_{tt}=c^2u_{xx}}$, $-\infty<\mathbf{x}<\infty$, $\mathbf{0\le t<\infty}$ under the following conditions: $\mathbf{u(x,0)=\sin x}$, $\mathbf{u_t\,(x,0)=0}$

sol: from D'Alembert solution

$$u(x,t)=\frac{1}{2}[f\,(x+ct)+f\,(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

Since $f(x)=\sin x$ and $g(x)=0$, then

$u(x,t)=\frac{1}{2}[\sin\,(x+ct)+\sin\,(x-ct)]$

Ex.2: Solve the equation $u_{tt} = c^2 u_{xx}$, $-\infty < x < \infty$, $0 \le t < \infty$ under the following conditions: $u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$, $u_t(x,0) = 0$

sol: from D'Alembert solution

$$u(x,t) = \frac{1}{2}[f\,(x+ct) + f\,(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

$$= \frac{1}{2}[1+1] + 0 = 1$$

Ex.3: A string is set in motion its equilibrium position with an initial velocity $u_t(x,0) = \sin x$ Find the displacement $u(x,t)$ of the string.

sol:

$u(x,0) = f\,(x) = 0$ (since the string is an equi. position)

$u_t(x,0) = \sin x$

$$u(x,t) = \frac{1}{2}[f\,(x-ct) + f\,(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

$$= \frac{1}{2}[0+0] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin z\,dz$$

$$= \frac{1}{2c}[cos\,(x-ct) - cos\,(x+ct)]$$

… Exercises …

1- An elastic string of length (5) has its ends x=0 and x=5 fixed is initially in the position equal $\frac{x}{2}$. The string is set in motion with initial velocity equal $\frac{1}{2}$. Find the displacement function $u(x,t)$ for all t.

2- An elastic string of length (2 cm) has its ends x=0 and x=2 fixed with no initial displacement. The string is set in motion with initial velocity equal 3. Find the displacement $u(x,t)$ for all t.

3- An elastic string of length (4 cm) has its ends x=0 and x=4 fixed. It is released from rest in the position 2x. Find the displacement of the string $u(x,t)$.

4- An elastic string of length (2 cm) has its ends x=0 and x=2 fixed. It is released from rest in the position x^2. Find the displacement of the string at any time.

5- Solve the wave equation $u_{tt} = u_{xx}$, $0 \le x \le 1$, $0 \le t < \infty$ under the following conditions:

$u_x(0,t) = 0, u(1,t) = 0, u(x,0) = f(x), u_t(x,0) = 0.$

Chapter six

Laplace' s Equation

Contents

Section 1	Laplace's Equation in Two Dimensions	

Section(6.1): 1- Laplace's Equation in Two Dimensions

In Chapter 4 we learned about the PDEs that control the heat flow in one dimensional spaces given by

$$u_t = \alpha^2 u_{xx}$$

where α^2 is a constant known as thermal diffusion, and we know that the heat equation in the two-dimensional spaces is given by

$$u_t = \alpha^2(u_{xx} + u_{yy}). \qquad \ldots(*)$$

If the temperature u reaches a steady state, that is, when u does not depend on time t and depends only on the space variables, then the time derivative u_t vanishes as $t \to \infty$. In view of this, we substitute $u_t = 0$ into (*), hence we obtain the Laplace's equation in two dimensions given by

$$u_{xx} + u_{yy} = 0$$

One of the most important of all partial differential equations occurring in applied mathematics in that associated with the name of Laplace, in two dimensions

$$u_{xx} + u_{yy} = 0 \qquad \ldots\ldots(1)$$

Laplace's equation appears in many branches of mathematical physics, for example in a steady- state heat problems (i.e. the problems which the temperature does not depend on time), as well as in steady- state electrical problems.

We denote to Laplace's equation by $\nabla^2 u = 0$ where ∇^2 is Laplace's operator.

2- General Solution of Two- Dimensional Laplace's Equation

To solve equation (1), we assume that (by separation of variables)

$$u(x,y) = X(x).Y(y) \qquad \dots\dots(2)$$

Where X and Y are functions of x and y, respectively.

From(2), $\frac{\partial^2 u}{\partial x^2} = X''(x).Y(y)$ and $\frac{\partial^2 u}{\partial y^2} = X(x).Y''(y)$

Hence (2) reduces to

$$X''(x).Y(y) + X(x).Y''(y) = 0 \Longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} \qquad \dots..(3)$$

Since the left hand side of (3) depends only on x and the right hand side depend only on y, both sides of (3) must be equal to same constant, say μ. This leads to two ordinary differential equations.

$$X'' - \mu X = 0 \text{ and } Y'' + \mu Y = 0 \qquad \dots\dots.(4)$$

Whose solutions depends only on the value of μ. Three cases arise:

Case 1- When $\mu = 0$, then reduces to $X'' = 0$ and $Y'' = 0$

Solving these, $X = Ax + B$and $Y = Cy + D$, then a solution of (1) is

$$\boxed{u(x,y) = (Ax + B)(Cy + D)} \qquad \dots..(5)$$

When $A = 0$ and $B=0$ or $C = 0$ and $D = 0$ then $u(x,y) = 0$ and this will be a trivial solution.

Case 2- When $\mu = \lambda^2$ i.e. positive. Here $\lambda \neq 0$, then (4) reduces to

$$X'' - \lambda^2 X = 0 \text{ and } Y'' + \lambda^2 Y = 0$$

Solving these, we get

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ and $Y(y) = Ccos\lambda y + Dsin\, \lambda y$

Then a solution of (1) is

$$\boxed{u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(Ccos\lambda y + Dsin\, \lambda y)} \qquad \dots.(6)$$

Where A, B, C,D are constant

Case 3- When $\mu = -\lambda^2$ i.e. negative. Here $\lambda \neq 0$, then (4) reduces to

$X'' + \lambda^2 X = 0$ and $Y'' - \lambda^2 Y = 0$

Solving these, we get

$X(x) = A\cos\lambda x + B\sin\lambda x$ and $Y(y) = Ce^{\lambda y} + De^{-\lambda y}$

Then a solution of (1) is

$$\boxed{u(x,y) = (A\cos\lambda x + B\sin\lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right)} \qquad \ldots.(7)$$

Where A, B, C, and D are arbitrary constants

3- Dirichlet problem in a rectangle

Suppose that we have a rectangular metal plate isolated ends not depend on time, as follows

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leq x \leq a, \qquad 0 \leq y \leq b$$

With boundary conditions:

$$u(0,y) = f_1(y), u(a,y) = f_2(y), u(x,0) = g_1(x), u(x,b) = g_2(x)$$

here we will study the situation where one of the boundary conditions is a function of x and the other conditions are equal to zero, as shown in the following example.

Ex.1: **Find the solution $\mathbf{u(x,y)}$ of Laplace's equation in the rectangle $\mathbf{0 < x < a, \quad 0 < y < b}$ also satisfying the boundary conditions:**

$$\mathbf{u(0,y) = 0 \,, u(a,y) = 0 \,, u(x,0) = 0 \,, u(x,b) = f(x)}$$

Sol: From equation (7)

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right) \qquad \ldots\ldots(7)$$

Substituting the condition $u(0,y) = 0$ in (7)

$$u(0,y) = (A\cos 0 + B\sin 0)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$0 = A\underbrace{\left(Ce^{\lambda y} + De^{-\lambda y}\right)}_{\neq 0} \implies \boxed{A = 0}$$

Substituting in (7)

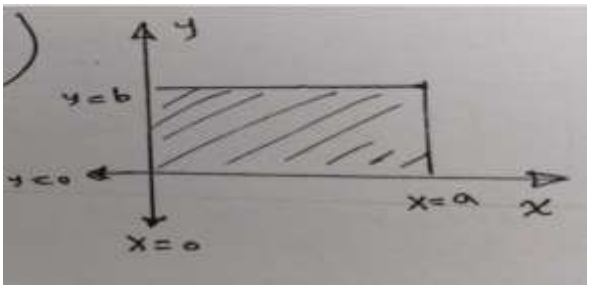
$$u(x,y) = B\sin\lambda x\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = \sin\lambda x\left(Ee^{\lambda y} + Fe^{-\lambda y}\right) \qquad \ldots\ldots(8)$$

Substituting the condition $u(x,0) = 0$

$$u(x,0) = \sin\lambda x\,(Ee^{0} + Fe^{0})$$

$$0 = \underbrace{\sin\lambda x}_{\neq 0}\,(E+F) \implies E + F = 0 \implies$$



$\boxed{F = -E}$ Substituting in (8)

$$u(x,y) = \sin\lambda x\left(Ee^{\lambda y} - Ee^{-\lambda y}\right)$$

$$\implies u(x,y) = E\sin\lambda x\left(e^{\lambda y} - e^{-\lambda y}\right)$$

$\ldots\ldots.(9)$

Substituting the condition $u(a,y) = 0$ in (9)

$$\underbrace{u(a,y)}_{=0} = \underbrace{E}_{\neq 0}\ \sin\lambda a\ \underbrace{\left(e^{\lambda y} - e^{-\lambda y}\right)}_{\neq 0} \implies \sin\lambda a = 0$$

$$\therefore \lambda a = n\pi,\ n = 1,2,3,\ldots.. \implies \lambda = \frac{n\pi}{a}$$

Substituting in (9), hence non zero solutions $u_n(x, y)$ of (9) are given by

$$u_n(x, y) = E_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a}$$

For more general solution, we take

$$u(x, y) = \sum_{n=1}^{\infty} u_n\,(x, y)$$

$$\Rightarrow u(x, y) = \sum_{n=1}^{\infty} E_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)\sin\frac{n\pi x}{a} \qquad \ldots\ldots(10)$$

Substituting the condition $u(x, b) = f(x)$ in (10)

$$\underbrace{u(x, b)}_{=f(x)} = \sum_{n=1}^{\infty} E_n\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)\sin\frac{n\pi x}{a}$$

Which is Fourier sine series of $f(x)$, hence we get

$$E_n\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right) = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

$$\Rightarrow E_n = \frac{2}{a\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx \qquad \ldots\ldots(11)$$

Hence (10) is the required solution where E_n is given in (11)

Now, if $f(x) = x$, $a = 2, b = 1$ then $E_n = \frac{2}{2\left(e^{\frac{n\pi}{2}} - e^{-\frac{n\pi}{2}}\right)}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$

$$E_n = \frac{4\,(-1)^{n+1}}{n\pi\left(e^{\frac{n\pi}{2}} - e^{-\frac{n\pi}{2}}\right)} \quad \Rightarrow u(x, y) = \sum_{n=1}^{\infty} \frac{4\,(-1)^{n+1}}{n\pi\left(e^{\frac{n\pi}{2}} - e^{-\frac{n\pi}{2}}\right)}\left(e^{\frac{n\pi y}{2}} - e^{-\frac{n\pi y}{2}}\right)\sin\frac{n\pi x}{2}$$

Ex.2: Find the steady state temperature distribution in a rectangular plate of sides a and b isolated at the lateral surface and satisfying the boundary conditions:

$\mathrm{u}(0, \mathrm{y}) = \mathrm{u}(\mathrm{a}, \mathrm{y}) = 0$ for $0 \leq y \leq b$, and $\mathrm{u}(\mathrm{x}, \mathrm{b}) = 0$ and , $\mathrm{u}(\mathrm{x}, 0) = \mathrm{x}(a - x)$ for $0 \leq x \leq a$

sol: The boundary conditions are

$\mathrm{u}(0, \mathrm{y}) = 0$, $\mathrm{u}(\mathrm{x}, \mathrm{b}) = 0$, $\mathrm{u}(\mathrm{a}, \mathrm{y}) = 0$, $\mathrm{u}(\mathrm{x}, 0) = \mathrm{x}(a - x)$, then we begin with equation (7)

$$u(x, y) = (A cos\lambda x + B sin\, \lambda x)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

where A, B, C, and D are arbitrary constants

Substituting the first condition, we get

$$0 = A\left(Ce^{\lambda y} + De^{-\lambda y}\right) \Longrightarrow \boxed{A = 0}$$

Substituting in (7) , we get

$$u(x, y) = sin\lambda x\left(Ee^{\lambda y} + Fe^{-\lambda y}\right) \qquad \ldots\ldots(12)$$

where $E = BC, \quad F = BD$

Substituting the second condition in (16), we get

$$\underbrace{u(x, b)}_{=0} = \underbrace{sin\lambda x}_{\neq 0}\left(Ee^{\lambda b} + Fe^{-\lambda b}\right) \Longrightarrow Ee^{\lambda b} + Fe^{-\lambda b} = 0$$

$$\Longrightarrow \boxed{F = -Ee^{2\lambda b}}$$

Substituting in (12), we get

$$u(x, y) = E\ sin\lambda x\ \left(e^{\lambda y} - e^{2\lambda b}\ e^{-\lambda y}\right) \qquad \ldots\ldots(13)$$

Substituting the third condition in (13),

$$0 = \underbrace{E}_{\neq 0}\ sin\lambda a\ \underbrace{\left(e^{\lambda y} - e^{2\lambda b}\ e^{-\lambda y}\right)}_{\neq 0} \Rightarrow sin\lambda a = 0$$

$$\therefore\ \lambda a = n\pi,\ \ n = 1,2,3, \ldots\ldots \quad \Rightarrow \lambda = \frac{n\pi}{a}$$

Putting in (13), hence non zero solutions $u_n(x, y)$ are given by

$$u_n(x, y) = E_n\ \sin\frac{n\pi x}{a}\ \left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}}\ e^{-\frac{n\pi y}{a}}\right)$$

For more general solution, we take the sum of $u_n\ (x, y)$

$$u(x, y) = \sum_{n=1}^{\infty} E_n\ \sin\frac{n\pi x}{a}\ \left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}}\ e^{-\frac{n\pi y}{a}}\right) \qquad \ldots.(14)$$

Substituting the fourth condition in (14)

$$u(x, 0) = \sum_{n=1}^{\infty} E_n\ \sin\frac{n\pi x}{a}\ \left(e^{0} - e^{\frac{2n\pi b}{a}}\ e^{0}\right)$$

$$x(a - x) = \sum_{n=1}^{\infty} E_n\ \left(1 - e^{\frac{2n\pi b}{a}}\right)\sin\frac{n\pi x}{a}$$

Which is the Fourier sine series, then E_n is given by

$$E_n = \frac{2}{a\left(1 - e^{\frac{2n\pi b}{a}}\right)}\int_0^a x(a - x)\sin\frac{n\pi x}{a}\ dx$$

$$= \frac{2}{a\left(1-e^{\frac{2n\pi b}{a}}\right)} \int_0^a (ax - x^2) \sin\frac{n\pi x}{a}\, dx$$

$$= \frac{2}{a\left(1-e^{\frac{2n\pi b}{a}}\right)} \left[\int_0^a ax \sin\frac{n\pi x}{a}\, dx - \int_0^a x^2 \sin\frac{n\pi x}{a}\, dx\right]$$

$$= \frac{4a^2}{n^3\pi^3\left(1-e^{\frac{2n\pi b}{a}}\right)}[1-(-1)^n] = \begin{cases} 0 & , if\ n\ is\ even \\ \frac{8a^2}{n^3\pi^3\left(1-e^{\frac{2n\pi b}{a}}\right)}, & if\ n\ is\ odd \end{cases}$$

$$u(x,y) = \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{8a^2}{n^3\pi^3\left(1-e^{\frac{2n\pi b}{a}}\right)} \sin\frac{n\pi x}{a} \left(e^{\frac{n\pi y}{a}} - e^{\frac{2n\pi b}{a}} e^{-\frac{n\pi y}{a}}\right) \qquad \ldots.(15)$$

Ex.3: Find the steady state temperature distribution in a rectangular plate of sides <u>*a*</u> and b isolated at the lateral surface and satisfying the boundary conditions:

$u(0,y) = f(y)$, $u(a,y) = 0$ for $0 \le x \le a$, and $u(x,0) = 0$ and ,$u(x,b) = 0$ for $0 \le y \le b$.

sol: The boundary conditions are

$u(x,0) = 0, u(a,y) = 0, u(x,b) = 0$, $u(0,y) = f(y)$, then we begin with equation (6)

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(Ccos\lambda y + Dsin\, \lambda y) \ldots\ldots\ldots\ldots.(*)$$

where A, B, C, and D are arbitrary constants.

Substituting the first condition, we get

$$u(x,0) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(Ccos0 + Dsin\, 0)$$

$$0 = \underbrace{\left(Ae^{\lambda x} + Be^{-\lambda x}\right)}_{\neq 0} C \quad \Longrightarrow \boxed{C = 0}$$

Substituting in (*), we get

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)Dsin\lambda y$$

$$u(x,y) = \left(Ee^{\lambda x} + Fe^{-\lambda x}\right)sin\lambda y \qquad \ldots\ldots(1)$$

where $E = AD, \;\; F = BD$.

Substituting the second condition in (1), we get

$$\underbrace{u(a,y)}_{=0} = \underbrace{sin\lambda y}_{\neq 0} \left(Ee^{\lambda a} + Fe^{-\lambda a}\right) \Longrightarrow Ee^{\lambda a} + Fe^{-\lambda a} = 0$$

$\Rightarrow \boxed{F = -Ee^{2\lambda a}}$

Substituting in (1), we get

$$u(x,y) = E\ sin\lambda y\ \left(e^{\lambda x} - e^{2\lambda a}\ e^{-\lambda x}\right) \qquad \ldots..(2)$$

Substituting the third condition in (2),

$$0 = \underbrace{E}_{\neq 0}\ \ sin\lambda b\ \ \underbrace{\left(e^{\lambda x} - e^{2\lambda a}\ e^{-\lambda x}\right)}_{\neq 0} \Rightarrow sin\lambda b = 0$$

$$\therefore \lambda b = n\pi,\ \ n = 1,2,3, \ldots.. \quad \Rightarrow \lambda = \frac{n\pi}{b}$$

Putting in (2), hence non zero solutions $u_n(x,y)$ are given by

$$u_n(x,y) = E_n\ \sin\frac{n\pi y}{b}\ \left(e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}}\ e^{-\frac{n\pi x}{b}}\right)$$

For more general solution, we take the sum of $u_n\ (x,y)$

$$u(x,y) = \sum_{n=1}^{\infty} E_n\ \sin\frac{n\pi y}{b}\ \left(e^{\frac{n\pi x}{b}} - e^{\frac{2n\pi a}{b}}\ e^{-\frac{n\pi x}{b}}\right) \qquad \ldots.(3)$$

Substituting the fourth condition in (3)

$$u(0,y) = \sum_{n=1}^{\infty} E_n\ \sin\frac{n\pi y}{b}\ \left(e^0 - e^{\frac{2n\pi a}{b}}\ e^0\right)$$

$$f(y) = \sum_{n=1}^{\infty} E_n\left(1 - e^{\frac{2n\pi a}{b}}\right)\ \sin\frac{n\pi y}{b}$$

Which is the Fourier sine series, then E_n is given by

$$E_n = \frac{2}{b\left(1 - e^{\frac{2n\pi a}{b}}\right)} \int_0^b f(y) \sin \frac{n\pi y}{b} \; dy$$

Now**, if $\boldsymbol{f(y) = y + 1}$, $\boldsymbol{b = 1}$, $\boldsymbol{a = 2}$.**

$$\Rightarrow E_n = \frac{2}{(1 - e^{4n\pi})} \int_0^1 (y + 1) \sin n\pi y \;\; dy$$

$$= \frac{2}{(1 - e^{4n\pi})} \left[\frac{-y}{n\pi} \cos n\pi y \Big|_0^1 + \frac{1}{n^2\pi^2} \sin n\pi y \Big|_0^1 - \frac{1}{n\pi} \cos n\pi y \Big|_0^1 \right]$$

$$= \frac{2}{(1 - e^{4n\pi})} \left[\frac{-1}{n\pi} (-1)^n - \frac{1}{n\pi} (-1)^n + \frac{1}{n\pi} \right]$$

$$= \frac{2}{(1 - e^{4n\pi})} \left[\frac{-2}{n\pi} (-1)^n + \frac{1}{n\pi} \right].$$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{(1 - e^{4n\pi})} \left[\frac{-2}{n\pi} (-1)^n + \frac{1}{n\pi} \right] \sin n\pi y \; \left(e^{n\pi x} - e^{4n\pi} \, e^{-n\pi x}\right)$$

Ex.4: Find the steady state temperature distribution in a rectangular plate of sides *a* and *b* isolated at the lateral surface and satisfying the boundary conditions:

$u(0, y) = 0$, $u(a, y) = f(y)$ for $0 \le x \le a$, and $u(x, 0) = 0$ and , $u(x, b) = 0$ for $0 \le y \le b$.

sol: The boundary conditions are

$u(x,0) = 0,\ u(0,y) = 0, u(x,b) = 0,\ \ u(a,y) = f(y),$ then we , with equation (6)

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(Ccos\lambda y + Dsin\,\lambda y)$$

where A, B, C, and D are arbitrary constants

Substituting the first condition, we get

$$u(x,0) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(Ccos0 + Dsin\,0)$$

$$0 = \underbrace{\left(Ae^{\lambda x} + Be^{-\lambda x}\right)}_{\neq 0} C \quad \Longrightarrow \boxed{\mathrm{C} = 0}$$

Substituting in (6) , we get

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)Dsin\lambda y$$

$$u(x,y) = \left(Ee^{\lambda x} + Fe^{-\lambda x}\right)sin\lambda y \qquad \ldots\ldots(4)$$

where $E = AD,\ \ F = BD.$

Substituting the second condition in (4), we get

$$\underbrace{u(0,y)}_{=0} = \underbrace{sin\lambda y}_{\neq 0}\ (Ee^{0} + Fe^{0}) \Longrightarrow E + F = 0$$

$$u(x,y) = E\ sin\lambda y\left(e^{\lambda x} - e^{-\lambda x}\right) \qquad \ldots\ldots(5)$$

Substituting the third condition in (5),

$$0 = \underbrace{E}_{\neq 0}\ sin\lambda b\ \underbrace{\left(e^{\lambda x} - e^{-\lambda x}\right)}_{\neq 0} \Longrightarrow sin\lambda b = 0$$

$$\therefore \lambda b = n\pi,\ n = 1,2,3,\ldots.. \quad \Longrightarrow \lambda = \frac{n\pi}{b}$$

Putting in (5), hence non zero solutions $u_n(x,y)$ are given by

$$u_n(x,y) = E_n \sin\frac{n\pi y}{b}\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right)$$

For more general solution, we take the sum of $u_n\,(x,y)$

$$u(x,y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi y}{b}\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right) \qquad \text{....}(6)$$

Substituting the fourth condition in (6)

$$u(a,y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi y}{b}\left(e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}\right)$$

$$f(y) = \sum_{n=1}^{\infty} E_n \left(e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}\right)\sin\frac{n\pi y}{b}$$

Which is the Fourier sine series, then E_n is given by

$$E_n = \frac{2}{b\left(e^{\frac{n\pi a}{b}} - e^{-\frac{n\pi a}{b}}\right)}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

Now**, if $\boldsymbol{f(y) = 4}$, $\boldsymbol{b = 4}$, $\boldsymbol{a = 1}$.**

$$E_n = \frac{2}{4\left(e^{\frac{n\pi}{4}} - e^{-\frac{n\pi}{4}}\right)}\int_0^4 4\sin\frac{n\pi y}{4}\,dy$$

$$= \frac{-8}{n\pi\left(e^{\frac{n\pi}{4}} - e^{-\frac{n\pi}{4}}\right)}[(-1)^n - 1] = \begin{cases} 0 & ,if\ n\ is\ even \\ \dfrac{16}{n\pi\left(e^{\frac{n\pi}{4}} - e^{-\frac{n\pi}{4}}\right)}, & if\ n\ is\ odd \end{cases}$$

$$u(x,y) = \sum_{\substack{n=1 \\ n\ is\ odd}}^{\infty} \frac{16}{n\pi\left(e^{\frac{n\pi}{4}} - e^{-\frac{n\pi}{4}}\right)}\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right)\sin\frac{n\pi y}{b}$$

Now, we will study the situation where one of the boundary conditions is a function of y and the other conditions are equal to zero, as shown in the following example.

Ex.5: Find the solution $u(x, y)$ of Laplace's equation in the semi-infinite plate $0 < x < \infty$, $0 < y < b$ also satisfying the boundary conditions:

$$u(0, y) = f(y)\,, u(\infty, y) = 0\,, u(x, 0) = 0\,, u(x, b) = 0$$

Sol: Rearrange conditions

1- $u(x, 0) = 0$, 2- $u(\infty, y) = 0$, 3- $u(x, b) = 0$, 4- $u(0, y) = f(y)$

From equation (6)

$$u(x, y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(C\cos\lambda y + D\sin\lambda y)$$

Substituting the first condition, we get

$$u(x, 0) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)(C\cos 0 + D\sin 0)$$

$$0 = \underbrace{\left(Ae^{\lambda x} + Be^{-\lambda x}\right)}_{\neq 0} C \quad \Rightarrow \boxed{C = 0}$$

Butting in equation (6)

$$u(x, y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)D\sin\lambda y$$

$$u(x, y) = \left(Ee^{\lambda x} + Fe^{-\lambda x}\right)\sin\lambda y \qquad \ldots\ldots(1)$$

where $E = AD$, $F = BD$.

Substituting the second condition in (1)

$$u(\infty, y) = \lim_{x\to\infty}\left(Ee^{\lambda x} + Fe^{-\lambda x}\right)sin\lambda y$$

$$0 = \left(Esin\lambda y \lim_{x\to\infty} e^{\lambda x} + Fsin\lambda y \underbrace{\lim_{x\to\infty} e^{-\lambda x}}_{=0}\right)$$

$$\therefore 0 = E \underbrace{sin\lambda y}_{\neq 0} \underbrace{\lim_{x\to\infty} e^{\lambda x}}_{\neq 0} \Longrightarrow \boxed{E = 0}$$

Putting in (1)

$$u(x, y) = F\, e^{-\lambda x}\, sin\lambda y \qquad \dots\dots(2)$$

Substituting the third condition in (2)

$$u(x, b) = F\, e^{-\lambda x}\, sin\lambda b$$

$$0 = \underbrace{F}_{\neq 0}\ \underbrace{e^{-\lambda x}}_{\neq 0}\ sin\lambda b \Longrightarrow sin\lambda b = 0$$

$$\therefore \lambda b = n\pi,\ n = 1,2,3,\dots.. \quad \Longrightarrow \lambda = \frac{n\pi}{b}$$

Substituting in (2), hence non zero solutions $u_n(x, y)$ are given by

$$u_n(x, y) = F_n e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b}$$

For more general solution, we take the sum of $u_n\,(x, y)$

$$u(x, y) = \sum_{n=1}^{\infty} F_n e^{-\frac{n\pi x}{b}} \sin\frac{n\pi y}{b} \qquad \dots.(3)$$

Substituting the fourth condition in (3) $u(0, y) = \sum_{n=1}^{\infty} F_n e^{0} \sin\frac{n\pi y}{b}$

$$f(y)=\sum_{n=1}^{\infty} F_n \ \sin\frac{n\pi y}{b}$$

Which is the Fourier sine series, then F_n is the Fourier coefficient in the form

$$F_n=\frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy \qquad \ldots..(4)$$

Then the equation (3) is the required solution with F_n that given in (4)

Now, if $f(y)=2y,\ \ b=3.\ \Rightarrow F_n=\frac{2}{3}\int_0^3 2y\ \sin\frac{n\pi y}{3}\,dy$

$F_n=\frac{12\,(-1)^{n+1}}{n\,\pi} \quad \Rightarrow u(x,y)=\sum_{n=1}^{\infty}\frac{12\,(-1)^{n+1}}{n\,\pi}e^{-\frac{n\pi x}{3}}\sin\frac{n\pi y}{3}.$

Ex.6: Find the solution $\mathbf{u(x,y)}$ of Laplace's equation in the semi-infinite plate $\mathbf{0<x<a,\ \ 0<y<\infty}$ also satisfying the boundary conditions:

$u(0,y)=0,\ u(a,y)=0, u(x,0)=f(x)$ and , $u(x,\infty)=0$

sol: The boundary conditions are

$u(0,y)=0, u(x,\infty)=0, u(a,y)=0,\ u(x,0)=f(x)$, then we use Eq.(7):

$u(x,y)=(Acos\lambda x+Bsin\,\lambda x)\left(Ce^{\lambda y}+De^{-\lambda y}\right)$.........(*)

Substituting the condition $u(0,y)=0$ in (*)

$u(0,y)=(Acos0+Bsin\,0)\left(Ce^{\lambda y}+De^{-\lambda y}\right)$

$$0 = A\underbrace{\left(Ce^{\lambda y} + De^{-\lambda y}\right)}_{\neq 0} \;\; \Longrightarrow \boxed{A = 0}$$

Substituting in (*), we obtain:

$$u(x,y) = Bsin\lambda x\,\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = sin\lambda x\,\left(Ee^{\lambda y} + Fe^{-\lambda y}\right) \qquad \ldots\ldots(1)$$

where $E = BC, \quad F = BD.$

Substituting the second condition in (1)

$$u(x,\infty) = \lim_{y\to\infty}\left(Ee^{\lambda y} + Fe^{-\lambda y}\right)sin\lambda x$$

$$0 = \left(Esin\lambda x \lim_{y\to\infty} e^{\lambda y} + Fsin\lambda x \underbrace{\lim_{y\to\infty} e^{-\lambda y}}_{=0}\right)$$

$$\therefore\ 0 = E\,\underbrace{sin\lambda x}_{\neq 0}\;\underbrace{\lim_{y\to\infty} e^{\lambda y}}_{\neq 0} \Longrightarrow \boxed{E = 0}$$

Putting in (1)

$$u(x,y) = F\,e^{-\lambda y}\,sin\lambda x \qquad \ldots\ldots(2)$$

Substituting the third condition in (2)

$$u(a,y) = F\,e^{-\lambda y}\,sin\lambda a$$

$$0 = \underbrace{F}_{\neq 0}\;\underbrace{e^{-\lambda y}}_{\neq 0}\;sin\lambda a \Longrightarrow sin\lambda a = 0$$

$$\therefore\ \lambda a = n\pi,\;\; n = 1,2,3,\ldots.. \quad \Longrightarrow \lambda = \frac{n\pi}{a}$$

Substituting in (2), hence non zero solutions $u_n(x, y)$ are given by

$$u_n(x, y) = F_n e^{-\frac{n\pi y}{a}} \sin\frac{n\pi x}{a}$$

For more general solution, we take the sum of $u_n\,(x, y)$

$$u(x, y) = \sum_{n=1}^{\infty} F_n e^{-\frac{n\pi y}{a}} \sin\frac{n\pi x}{a} \qquad \ldots.(3)$$

Substituting the fourth condition in (3)

$$u(x, 0) = \sum_{n=1}^{\infty} F_n e^0 \sin\frac{n\pi x}{a}$$

$$f(x) = \sum_{n=1}^{\infty} F_n \ \sin\frac{n\pi x}{a}$$

Which is the Fourier sine series, then F_n is the Fourier coefficient in the form

$$F_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx \qquad \ldots..(4)$$

Then the equation (3) is the required solution with F_n that given in (4)

Now, if $\boldsymbol{f(x) = x,\ \ b = 1, a = 1.}$

$$\Rightarrow F_n = 2\int_0^1 x\sin n\pi x\,dx = 2\left[\frac{-x}{n\pi}\cos n\pi x\Big|_0^1 + \frac{1}{n^2\pi^2}\sin n\pi y\Big|_0^1\right]$$

$$= \frac{2\,(-1)^{n+1}}{n\,\pi}$$

$$\Rightarrow u(x, y) = \sum_{n=1}^{\infty} \frac{2\,(-1)^{n+1}}{n\,\pi} e^{-n\pi y}\sin n\pi x.$$