Chapter Two
Models for Single Species

Continuous Population

In many cases, mathematical modeling is applied to understand population growth dynamics for animal and human population. For example, modeling the way fish populations grow, and accounting for the effect of fishing is essential to the fishing industry, as we cannot afford to deplete this resource. Another use of modeling is to understand the manner in which human populations grow: in the world, in individual countries, in towns and in organizations.

Model 1- Exponential Growth

We can consider this problem as a compartmental model, with the compartment being the 'world', 'town', etc. as in the sketch:

This compartmental sketch leads to a word equation describing a changing population

$$\left\{\begin{matrix}\text{Rate of}\\ \text{Change of}\\ \text{Population size}\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate}\\ \text{of}\\ \text{births}\end{matrix}\right\} - \left\{\begin{matrix}\text{Rate}\\ \text{of}\\ \text{deaths}\end{matrix}\right\} \qquad \dots (1)$$

Model assumptions

- We assume that the populations are sufficiently large so that we ignore random difference between individuals.
- We assume that the births and deaths are continuous in time.
- We assume that per-capita birth and death rates are constant.
- We ignore immigration and emigration.

Then we assume

$$\begin{Bmatrix}\text{Rate}\\ \text{of}\\ \text{births}\end{Bmatrix} = \beta P(t), \qquad \begin{Bmatrix}\text{Rate}\\ \text{of}\\ \text{deaths}\end{Bmatrix} = \alpha P(t) \qquad \dots(2)$$

Substituting (2) in (1) we obtain

$$\frac{dP}{dt} = \beta P - \alpha P \qquad \dots(3)$$

To solve this equation

Let $r = \beta - \alpha$ and then

$$\frac{dP}{dt} = rP.$$

We call r the growth rate or the reproduction rate

- when $r > 0$ this is a model describing exponential growth.
- when $r < 0$ the process is exponential decay.

$$\therefore \frac{dP}{P} = r\,dt \implies \ln P = r\,t + c$$

$$\therefore P(t) = A\,e^{rt}$$

and by $P(0) = P_0$

then

$$P(t) = P_0\,e^{rt} \qquad \dots(4)$$

Model validation

Taking the 1990 world population values $r = 0.017$ and $P_0 = 5.3$ billion, we apply equation (3.4)

- To find the population in 1995

$$P(5) = 5.3\, e^{(0.017)(5)}$$
$$= 5.77 \text{ billion.}$$

- To find or predicts the population in 2090

$$P(100) = 5.3\, e^{(0.017)(100)}$$
$$= 29.01 \text{ billion.}$$

Model 2:Exponential with Harvesting:

Harvesting from population models may result from hunting or capturing individuals, resulting in a population drop. A population will expand without limit if it is experiencing exponential growth. For illustration, we looked at several harvesting techniques that may be used to manage a rabbit population. Population growth will slow as the population size rises since an ecosystem cannot support unlimited species. The Foundational Equation of A model of exponential (natural with harvesting) is

$$\frac{dP}{dt} = Pr - m \;\ldots\ldots\ldots..(1)$$

Using the initial condition $P = P_0$ To solve this equation

$dP = (rP - m)dt$ by separating variables

$\frac{dP}{rP-m} = dt \ldots\ldots\ldots\ldots\ldots..(2)$ by integral

$$\int \frac{dP}{rP - m} = \int dt$$

$$\frac{1}{r}\ln|rP - m| = t + c \ldots\ldots(3)$$

where c is the constant of integration

$$P(0) = P_0$$

$c = \frac{1}{r}\ln |rP_0 - m|$ ....(4) by substituting an equation (4) in (3)

$\frac{1}{r}\ln|rP-m| = t + \frac{1}{r}\ln|rP_0 - m|$ multiplied by r

$\ln|rP-m| = rt + \ln|rP_0 - m|$

$\ln|rP-m| - \ln|rP_0 - m| = rt$

$$\ln\left|\frac{rP-m}{rP_0-m}\right| = rt$$

$$\frac{rP-m}{rP_0-m} = e^{rt}$$

$rP - P = (rP_0 - m)\,e^{rt}$

$rP = m + + (rP_0 - m)e^{rt}$ by dividing by r

$P(t) = \frac{m}{r} + \left(P_0 - \frac{m}{r}\right)e^{rt}$ is the analytical solution

If $P_0 r > m$, P(t) is increase

If $P_0 r < m$ P (t) is the decline

If $P_0\,r = m$ P(t) is fixed and stable

See Figure 2.3 Exponential harvesting with different values of m

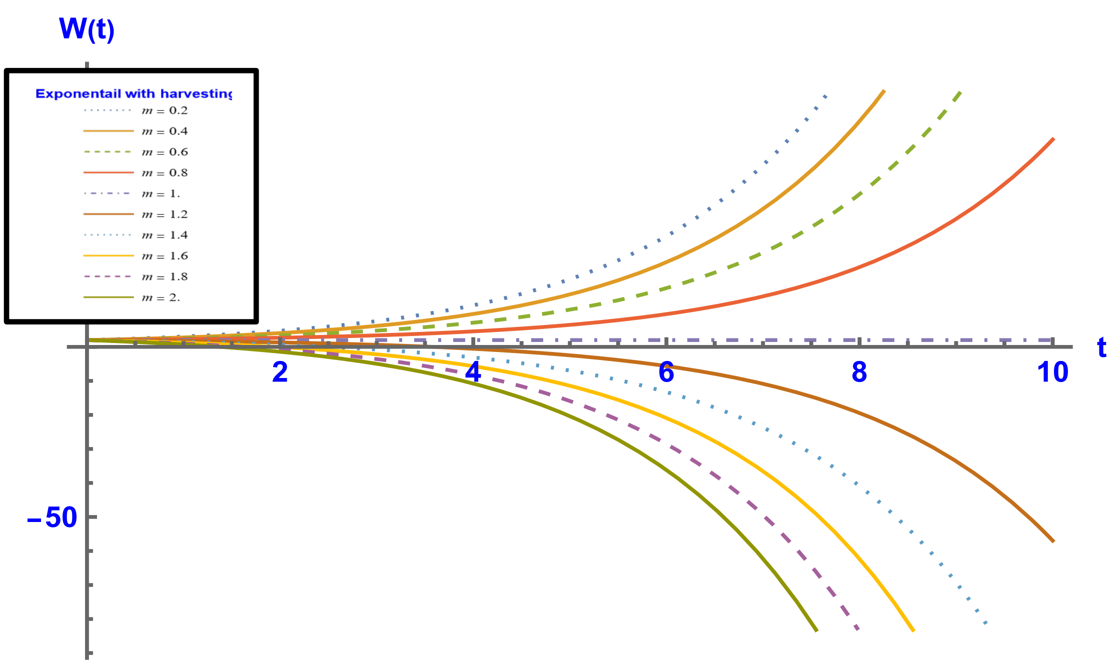

Figure 2.1 Exponential harvesting with $r = 0.5$, w0 = 2

Exponential with harvesting stability:

Stability of Exponential (natural)with harvesting stability model

We make the equation (1) equal to zero

$\frac{dP}{dt} = rP - m$ equal to zero

$$\frac{dP}{dt} = 0$$

$$rP - m = 0$$

$P_e = \frac{m}{r}$ is a fixed point

$$\frac{dP}{dt} = rP - m$$

$$\frac{d^2P}{d^2t} = r$$

Then the point $P_e = \frac{m}{r}$ is unstable point

Model 3- Logistic Growth (or Density-Dependent Growth)

Populations cannot continue growing exponentially over time due to limited resources and/or competition for these with other species.

Instead of assuming a constant death rate, we allow the death rates to increase as the population increases

$$\left\{\begin{matrix}\text{Rate}\\ \text{of}\\ \text{births}\end{matrix}\right\} = \alpha P(t) + \gamma P^2(t) \qquad \ldots (5)$$

Normal rate of deaths

Rate of death by crowding

Then the eq. (3) becomes

$$\frac{dP}{dt} = \beta P - \alpha P - \gamma P^2.$$

by $r = \beta - \alpha$

$$\frac{dP}{dt} = rP - \gamma P^2 \qquad \text{... (6)}$$

with $K = \frac{r}{\gamma}$, the differential equation (6) becomes

↑

Carrying capacity

$$\frac{dP}{dt} = rP - \frac{r}{K}P^2$$

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \qquad \text{... (7)}$$

This model leads to a nonlinear differential equation called logistic equation (some time called density-dependent model). We consider only $r > 0$ and $K > 0$ to ensure positive population values.

Interpretation of the parameters

We can write a general differential equation for population growth as

$$\frac{dP}{dt} = R(P)P$$

where $R(P) = r\left(1 - \frac{P}{K}\right)$.

Note that $R(P)$, a linear function of P, tends to zero as the population approaches to carrying capacity K, while as the population size tends to zero, $R(P)$ approaches r, as the figure.

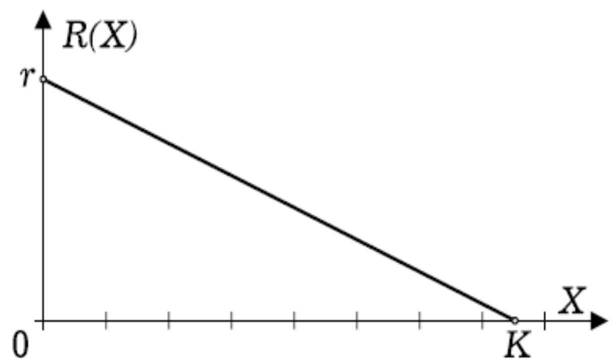

Figure 2.2 Interpretation of the parameters

Analytic solution

Example

Solve the logistic differential equation initial value problem

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),\ P(0) = P_0\ .$$

Solution:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right) = \frac{rP(k-P)}{k}$$

$$\frac{k}{P(k-P)}\ \frac{dP}{dt} = r$$

Assuming that $P \neq 0$ and $P \neq K$. Integrating gives

$$\int \frac{k}{P(k-P)}\,dP = \int r\,dt$$

For the integral on the left-hand side (LHS), we need to use partial fractions

$$\frac{k}{P(k-P)} = \frac{a}{P} + \frac{b}{k-P} = \frac{a(k-P)+bP}{P(k-P)}$$

Solving for the constants a and b gives

$$ak = k$$

$$(-a+b)P = 0$$

which implies that $a = b = 1$, and then

$$\frac{k}{P(k-P)} = \frac{1}{P} + \frac{1}{k-P}$$

Now

$$\int \frac{k}{P(k-P)} dP = \int \frac{1}{X}\, dP + \int \frac{1}{k-X} dP = \int r\, dt$$

$$\ln|P| - \ln|k-P| = rt + c$$

$$\left|\frac{P}{k-P}\right| = c_1 e^{rt} \qquad ; c_1 = e^c$$

Assuming $0 < P < k$, then

$$P = c_1 e^{rt}(k-P)$$

Using the initial condition $P(0) \ = \ P_0$

$$P_0 = c_1 e^0 (k - P_0)$$

$$c_1 = \frac{P_0}{k-P_0}$$

$$\therefore P = \left(\frac{P_0}{k-P_0}\right) e^{rt}(k-P)$$

$$\therefore P(t) = \frac{k}{1 + m\, e^{-rt}}$$

where $m = \frac{k-P_0}{P_0}$.

Alternatively, in the case where $0 < k < P$,

$$P(t) = c_1 e^{rt}(P - k).$$

with $P(0) = P_0$

we get

$$P(t) = \frac{k}{1 + m\, e^{-rt}} \qquad \dots (8)$$

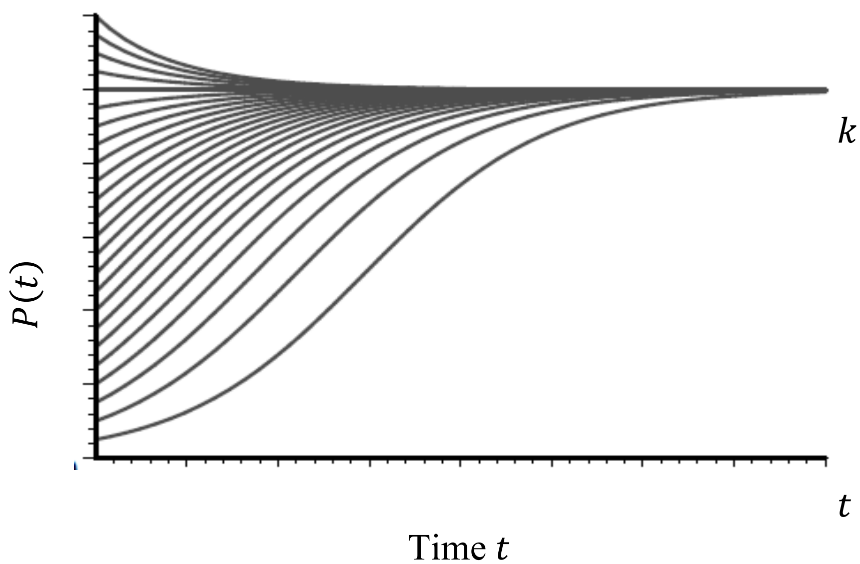

Figure 2.3 Logistic model

Equilibrium Solutions and Stability

If we observe the levelling of a population over time, this implies that the rate of change of the population approaches 0, that is, $X' \to 0$. Any value of X that gives a zero rate of change is called an equilibrium point (solution).

Equilibrium solutions are constant solutions where, here, the rate of increase (births) exactly balances the rate of decrease (deaths). Equilibrium solutions satisfy

$$\frac{dP}{dt} = 0, \quad \Rightarrow \quad rP\left(1 - \frac{P}{k}\right) = 0 \qquad \dots (9)$$

There are two possible equilibrium solutions, $P_e = 0$ and $P_e = K$, that satisfy equation (3). We are interested in which of these are stable. For stable solutions, this means that if we start near the equilibrium solution then we are attracted towards it. The condition for local stability is $f'(P_e) < 0$, where f is the RHS of the differential equation.

Here $f(P) = rP\left(1 - \frac{P}{K}\right) \longrightarrow f'(P) = r - \frac{2rP}{k}$

and so $f'(0) = r > 0$ and $f'(K) = -r < 0$, for all positive values of r.

The equilibrium solution $P = 0$ is always unstable and the equilibrium solution $P = K$ is always stable.

Model 4-Logistic growth with Harvesting model

The effect of harvesting a population on a regular or constant basis is extremely important to many industries. One example is the fishing industry. Will a high harvesting rate destroy the population? Will a low harvesting rate destroy the viability of the industry?

Formulating the equation

Including a constant harvesting rate in our logistic model gives

$$\begin{Bmatrix}\text{rate of change in}\\ \text{population}\end{Bmatrix} = \begin{Bmatrix}\text{rate of}\\ \text{births}\end{Bmatrix} - \begin{Bmatrix}\text{normal}\\ \text{rate of}\\ \text{deaths}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{death by}\\ \text{crowding}\end{Bmatrix} - \begin{Bmatrix}\text{rate of}\\ \text{death by}\\ \text{harvesting}\end{Bmatrix} \quad \dots (10)$$

Assuming the harvesting rate to be constant, equation (10) translates to

$$\frac{dX}{dt} = rP\left(1 - \frac{P}{k}\right) - h \quad \dots (11)$$

Here h is included as the constant rate of harvesting

Solving the differential equation

First, we can write (11) in factored form

$$\frac{dP}{dt} = -\frac{r}{k}(P^2 - kP + kh)$$

with r, k and h positive constant

Example

Let $r = 1, k = 10, h = \frac{9}{10}$ and $P(0) = P_0$.

Solution:

$$\frac{dP}{dt} = -\frac{1}{10}\left(P^2 - 10P + 10\left(\frac{9}{10}\right)\right) = -\frac{1}{10}(P^2 - 10P + 9)$$

$$\frac{dP}{dt} = -\frac{1}{10}(P-1)(P-9)$$

$$\int \frac{1}{(P-1)(P-9)}dX = -\int \frac{1}{10}dt$$

and using partial fractions,

$$\frac{1}{8}\int \left(\frac{1}{(P-9)} - \frac{1}{(P-1)}\right)dX = -\int \frac{1}{10}dt$$

$$\ln\left|\frac{P-9}{P-1}\right|^{\frac{1}{8}} = -\frac{1}{10}t + c,$$

Then rearranging with constant $b = e^{8c}$

$$\left|\frac{P-9}{P-1}\right|^{\frac{1}{8}} = be^{-\frac{4t}{5}}$$

By $P(0) = x_0 \longrightarrow b = \left|\frac{P_0-9}{P_0-1}\right|$

and the explicit solution is

$$P(t) = \frac{9 - b\,e^{-4t/5}}{1 - b\,e^{-4t/5}}$$

Stability

We have the following cases to consider when sketching a graph.

- If $P_0 < 1$, then $P' < 0$ and the population declines.
- If $1 < P_0 < 9$, then $P' > 0$ and the population increases.
- If $P_0 > 9$, then $P' < 0$ and the population declines.
- If $P_0 = 1$ or $P_0 = 9$, then the population does not change

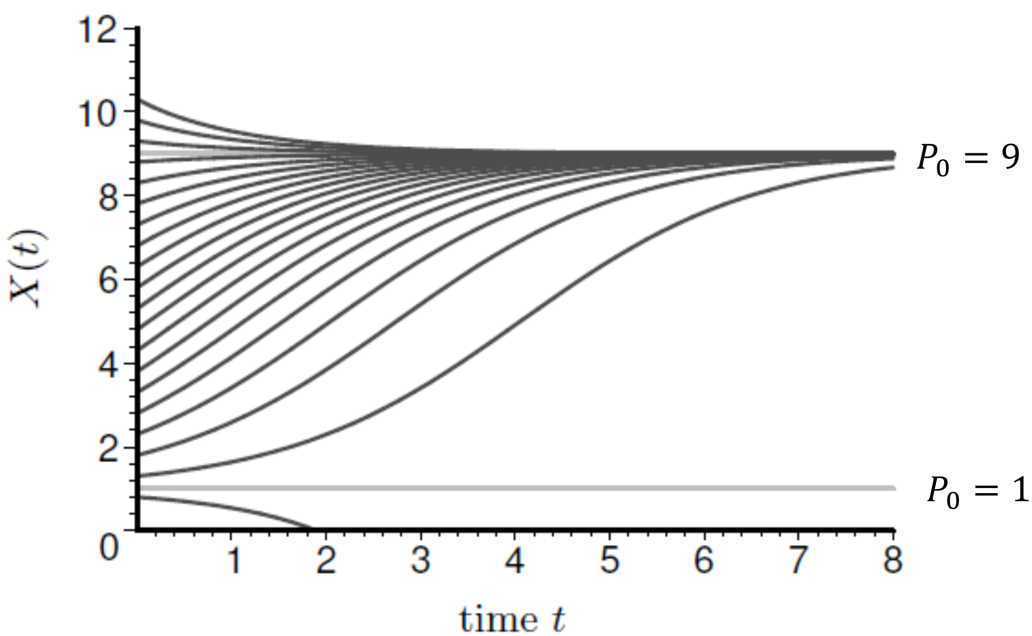

Figure 2.4 The Logistic Equation with Harvesting

Model 5 – Gompertz Growth model

The model of Gompertz growth is characterized by an exponential decrease when the population reaches its peak. Both Gompertz and logistic models produce comparable curves. The logistic mode develops slower than the Gompertz model when M is low. Gompertz curves or functions are mathematical models for time series named after Benjamin Gompertz (1779–1865). A sigmoid function characterizes growth as slowest at the beginning and conclusion of a period.

The Foundational Equation of A model Gompertz growth model is:

$$\frac{dP}{dt} = rp \ln\left(\frac{M}{P}\right) \quad .....(1)$$

where r is constant and M is Carrying capacity

Using the initial condition $P(0) = P_0$ To solve Eq. 1

For the integral on both sides, we need to use partial fractions

$$\int \frac{dP}{P \ln\left(\frac{M}{P}\right)} = \int r dt \quad ....(2)$$

$$\text{let } u = \ln\left(\frac{M}{P}\right) \quad ....(3), \qquad du = \frac{1}{\frac{M}{P}} * \frac{-M}{P^2} dP$$

$$dP = \frac{P}{M} * \frac{-M}{P^2} dP \text{ leads to } du = \frac{-1}{P} dP \quad .....(4)$$

By substitute equation (3),(4) in (2)

$$-\int \frac{1}{u} du = \int r dt$$

$$\int \frac{1}{u} du = -\int r dt$$

$$\ln|u| = -(r\,t + K_1)$$

$$\ln|u| = -r\,t - K_1 \quad ..(5), \quad -K_1 = K_2 \quad ....(6)$$

By substituting equation (6) in (5)

$$\ln|u| = -rt + K_2$$

$$|u| = e^{-rt+K_2} \quad ...(7), \quad e^{K_2} = K_3 \quad ....(8)$$

By substitute equation (8) in (7)

$u = \mp K_3 e^{-rt}$ ….(9) , $k_4 = \mp K_3$ ….(10)

By substituting equation (10) in (9)

$$u = k_4 e^{-rt}$$

$$\ln\left(\frac{M}{P}\right) = k_4 e^{-rt}$$

By $P(0) = P_0$ we get $\ln\left(\frac{M}{P_0}\right) = k_4 e^0$

$$k_4 = \ln\left(\frac{M}{P_0}\right)$$

$$\ln\left(\frac{P}{w}\right) = k_4 e^{-rt}$$

$$\frac{M}{P} = e^{k_4 e^{-rt}}$$

$P = \frac{M}{e^{k_4 e^{-rt}}}$ leads to $P = \frac{M}{e^{\ln\left(\frac{M}{P_0}\right) e^{-rt}}}$

$P\,(t) = M(e^{-(\ln\frac{M}{P_0} e^{-rt})})$ is The analytical solution

See Figure 2.11 the growth model with different values of r

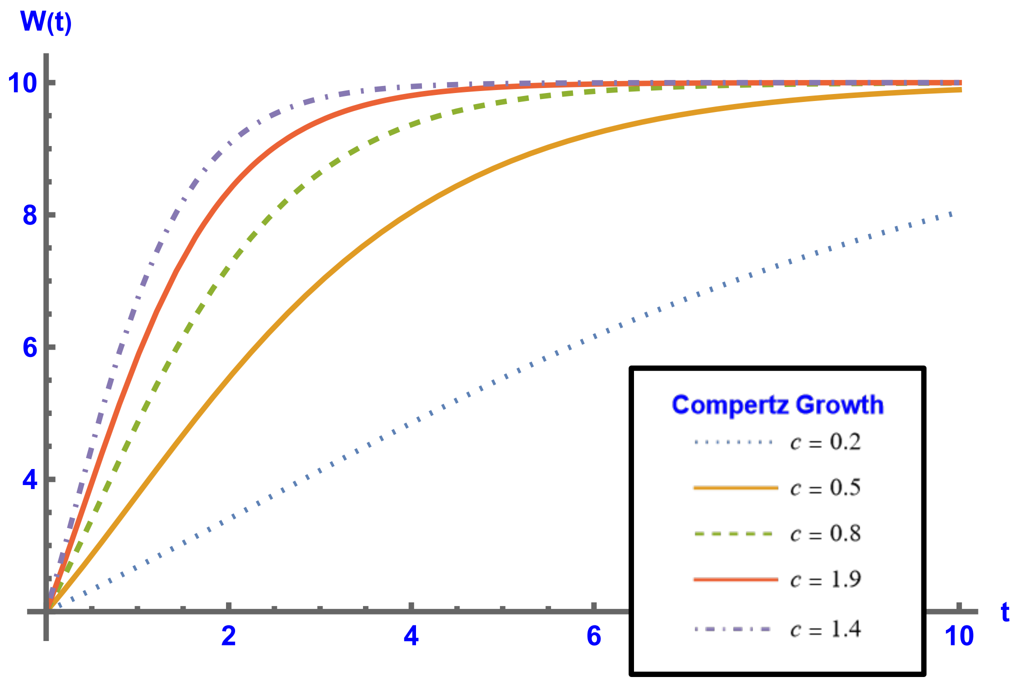

Figure 2.5: The Gompertz Growth model with $P_0 = 2, M = 10$

Model 6 Allee Effects Model

Allee effects are a dynamic phenomenon that affects population dynamics, including extinction and invasion. These sources are often quoted, but their strength and ubiquity have not been rigorously assessed. Allee's impacts on wild animal populations are reviewed from 91 research. We focus on empirical signatures used or might be used to detect Allee effects, data types in which they are evident, empirical support for critical densities in natural populations, and taxa differences in Allee effects and primary causal mechanisms. We found clear instances in Mollusca, Arthropoda, and Chordata, including three vertebrate groups, most often resulting from mating restriction in invertebrate's predator–prey interactions in vertebrates. Most population-level dynamic implications of Allee effects (e.g., an unstable critical density associated with strong Allee effects) depend on distinguishing component and demographic Allee effects in data. Still, more than half of the studies failed to do so. Thus, we find conclusive evidence for Allee effects due to various mechanisms in natural populations of 59 animal species. Still, we lack data on the strength and commonness of Allee effects across species and populations and a critical density for most populations. We recommend population-scale experiments and methodologies linking component and demographic effects to augment observational investigations (Kramer et al., 2009).

The Foundational Equation of A model Allee effects Growth Model is:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

Where w is population size, and r is the rate of increasing

While M is the Carrying capacity and m is the threshold point