- In this section we examine equations of the form $dy/dt = f(y)$, called **autonomous** equations, where the independent variable t does not appear explicitly.

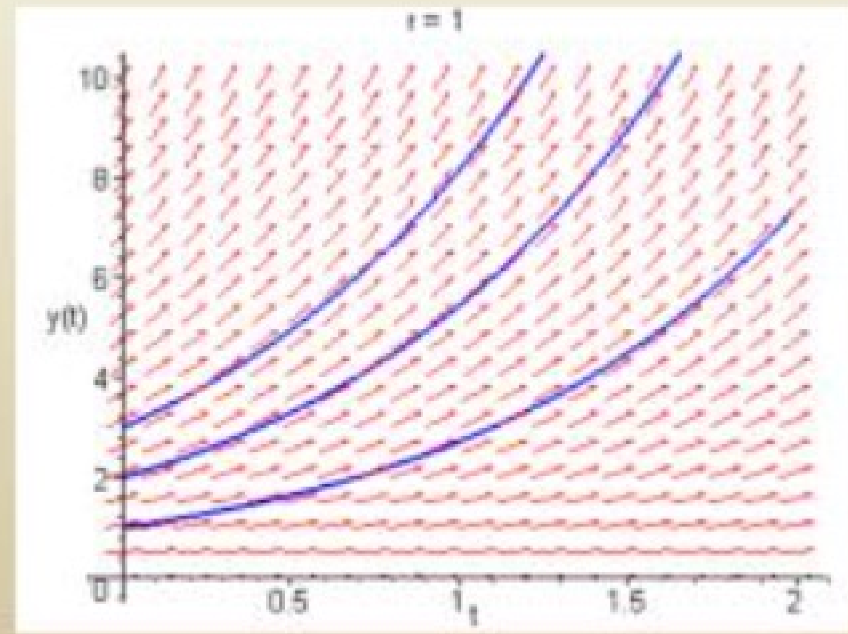
- The main purpose of this section is to learn how geometric methods can be used to obtain qualitative information directly from differential equation without solving it.

- Example (Exponential Growth):

- Solution:

$$\frac{dy}{dt} = ry, \quad r > 0$$

$$y = y_0 e^{rt}$$



- An exponential model $y' = ry$, with solution $y = e^{rt}$, predicts unlimited growth, with rate $r > 0$ independent of population.

- Assuming instead that growth rate depends on population size, replace r by a function $h(y)$ to obtain $dy/dt = h(y)y$.

- We want to choose growth rate $h(y)$ so that
  - $h(y) \cong r$ when y is small,
  - $h(y)$ decreases as y grows larger, and
  - $h(y) < 0$ when y is sufficiently large, (i.e. Population decreases)

  The simplest such function is $h(y) = r - ay$, where $a > 0$.

- Our differential equation then becomes $$\frac{dy}{dt} = (r - ay)\,y, \quad r, a > 0$$

- This equation is known as the Verhulst, or **logistic**, equation.

- The logistic equation from the previous slide is

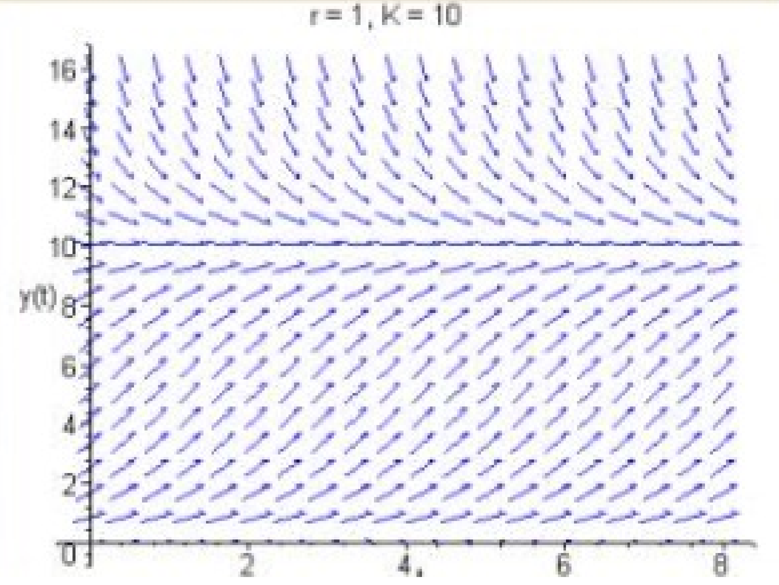
$$\frac{dy}{dt} = (r - ay)\,y, \quad r, a > 0$$

- This equation is often rewritten in the equivalent form

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y,$$

where $K = r/a$. The constant r is called the **intrinsic growth rate**, and as we will see, K represents the **carrying capacity** of the population.

- A direction field for the logistic equation with $r = 1$ and $K = 10$ is given here.



- Our logistic equation is

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \quad r,\ K > 0$$

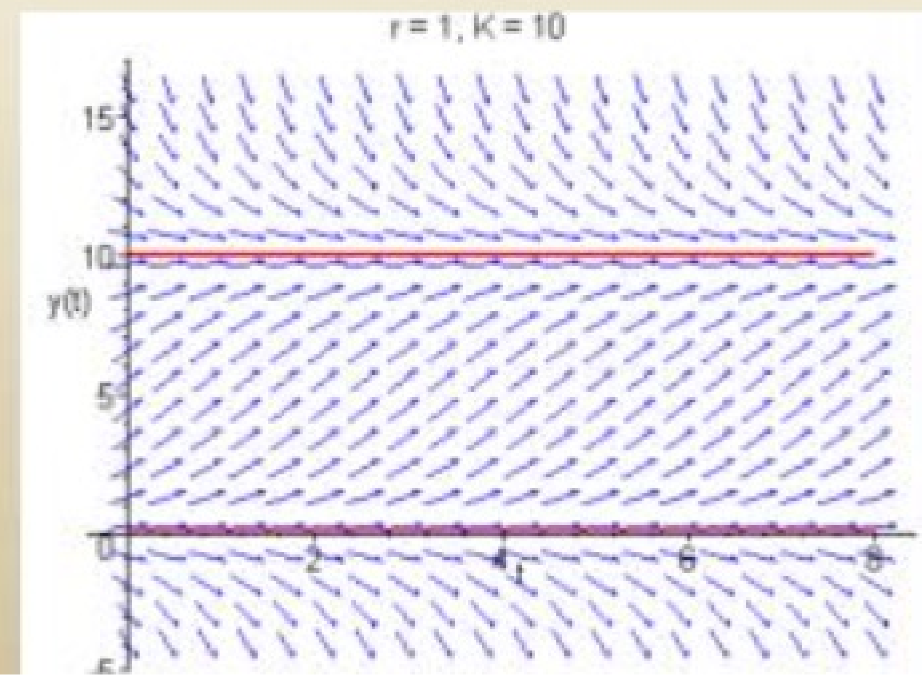
- Two equilibrium solutions are clearly present:

$$y = \phi_1(t) = 0, \quad y = \phi_2(t) = K$$

- In direction field below, with $r = 1$, $K = 10$, note behavior of solutions near equilibrium solutions:

$y = 0$ is **unstable**,

$y = 10$ is **asymptotically stable**.

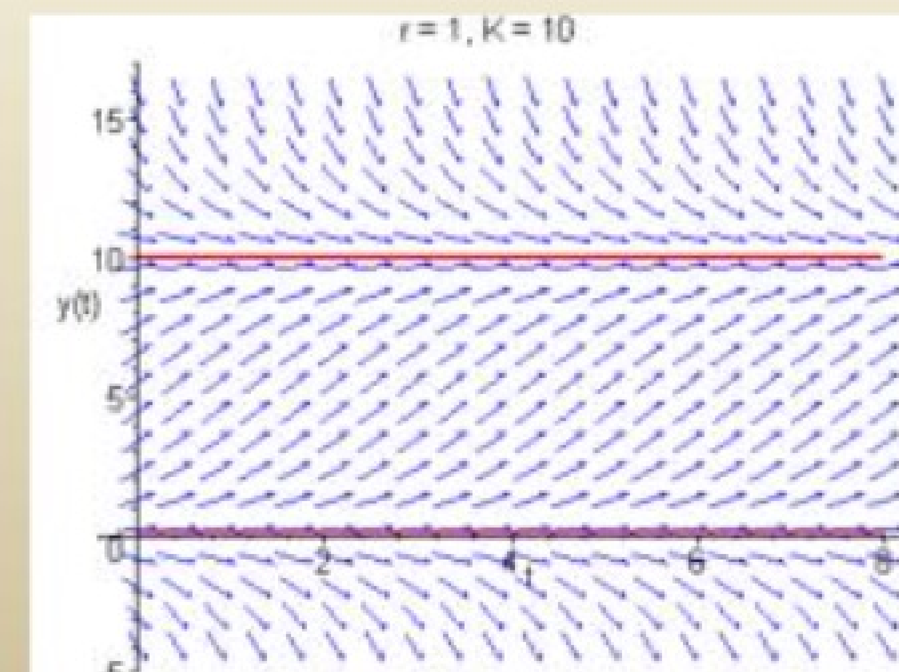


- Equilibrium solutions of a general first order autonomous equation $y' = f(y)$ can be found by locating roots of $f(y) = 0$.

- These roots of $f(y)$ are called **critical points**.

- For example, the critical points of the logistic equation $$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$ are $y = 0$ and $y = K$.

- Thus critical points are constant functions (equilibrium solutions) in this setting.

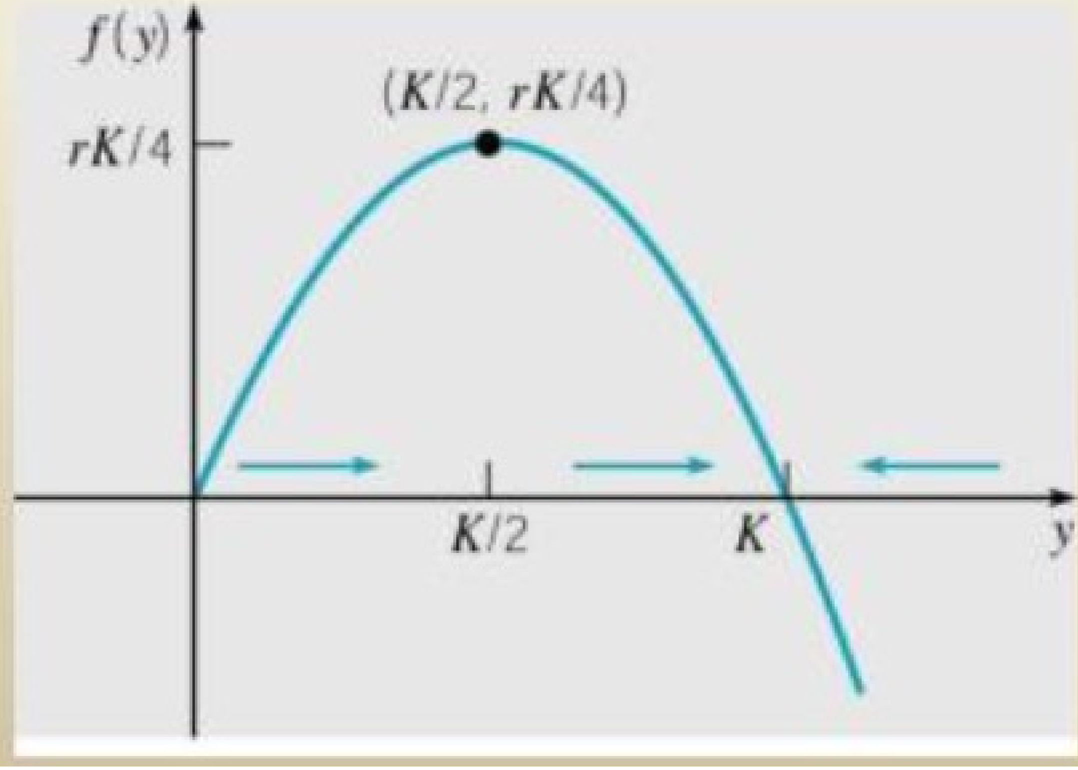


(Question) How do we find the vertex of f(y) and its graph?

- To better understand the nature of solutions to autonomous equations, we start by graphing $f(y)$ vs y.

- In the case of logistic growth, that means graphing the following function and analyzing its graph using calculus.

$$f(y) = r\left(1 - \frac{y}{K}\right)y = \left(\frac{r}{K}\right)(K - y)y$$



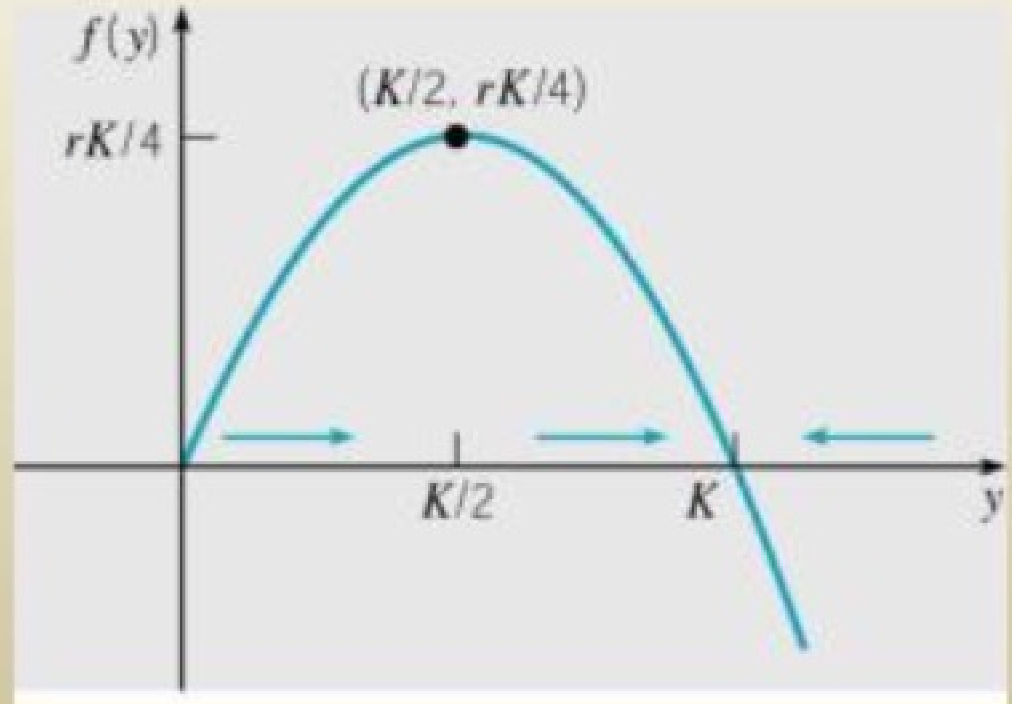
- The intercepts of f occur at $y = 0$ and $y = K$, corresponding to the critical points of logistic equation.
- The vertex of the parabola is $(K/2, rK/4)$, as shown below.

$$f(y) = r\left(1 - \frac{y}{K}\right)y$$

$$f'(y) = r\left[\left(-\frac{1}{K}\right)y + \left(1 - \frac{y}{K}\right)\right]$$

$$= -\frac{r}{K}[2y - K] \stackrel{set}{=} 0 \Rightarrow y = \frac{K}{2}$$

$$f\left(\frac{K}{2}\right) = r\left(1 - \frac{K}{2K}\right)\left(\frac{K}{2}\right) = \frac{rK}{4}$$

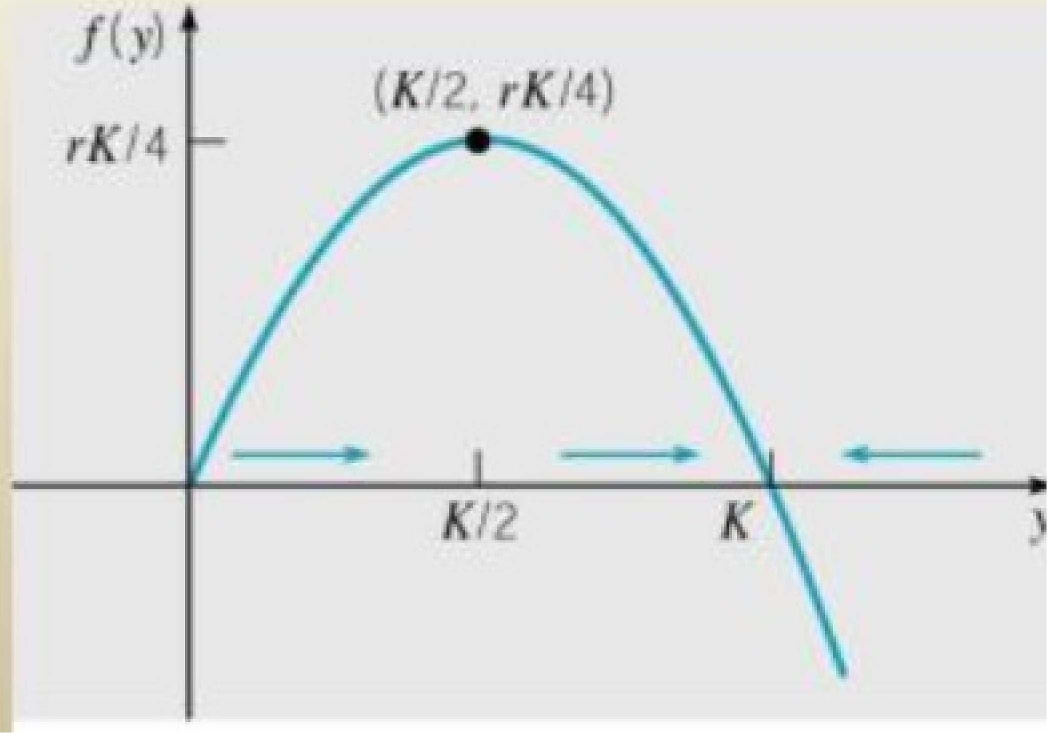


- Note $dy/dt > 0$ for $0 < y < K$, so y is an increasing function of t there (indicate with right arrows along y-axis on $0 < y < K$).

- Similarly, y is a decreasing function of t for $y > K$ (indicate with left arrows along y-axis on $y > K$).

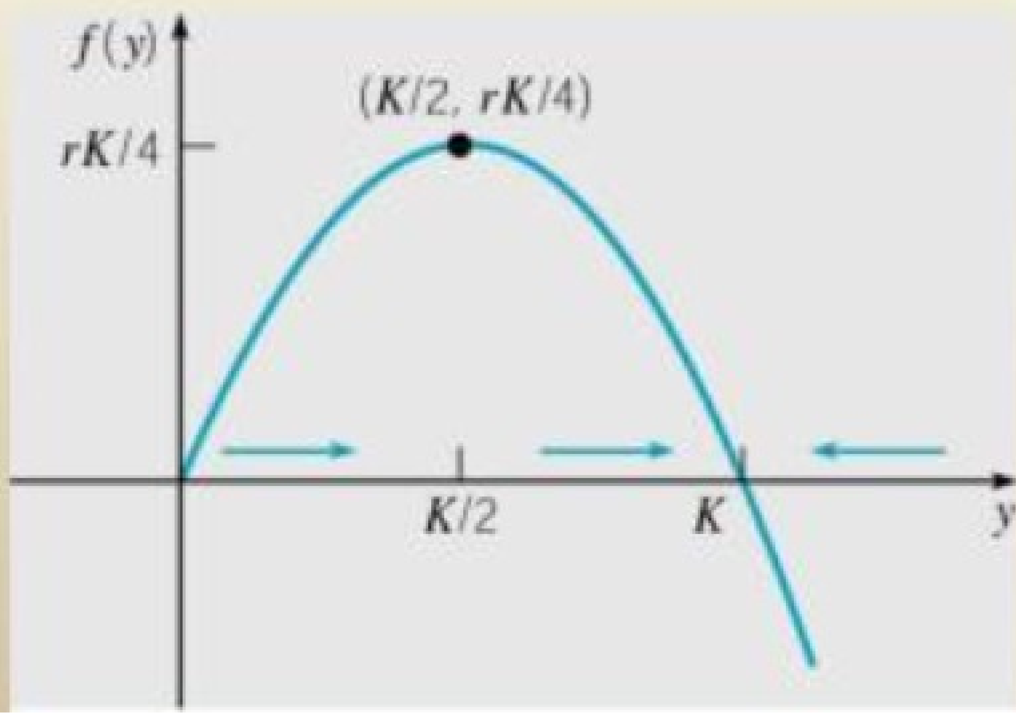
- In this context the y-axis is often called the **phase line**.

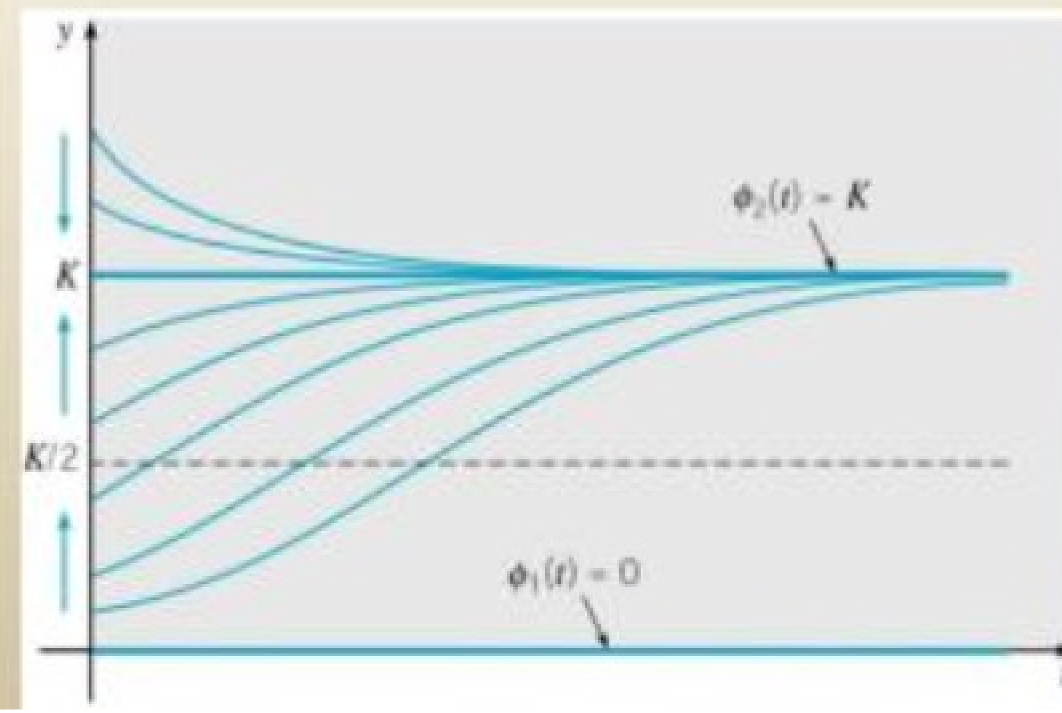
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \quad r > 0$$



- Note $dy/dt \cong 0$ when $y \cong 0$ or $y \cong K$, so y is relatively flat there, and y gets steep as y moves away from 0 or K.

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

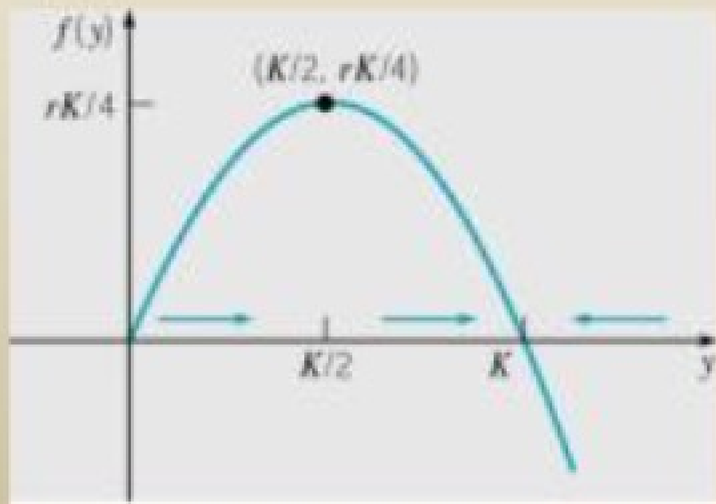


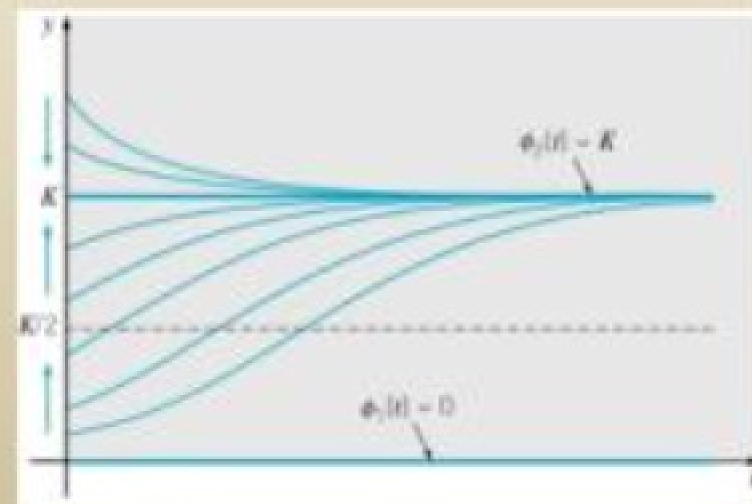


- Next, to examine concavity of $y(t)$, we find y'':

$$\frac{dy}{dt} = f(y) \quad \Rightarrow \quad \frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)f(y)$$

- Thus the graph of y is concave up when f and f' have same sign, which occurs when $0 < y < K/2$ and $y > K$.
- The graph of y is concave down when f and f' have opposite signs, which occurs when $K/2 < y < K$.
- Inflection point occurs at intersection of y and line $y = K/2$.





(Example 1) Consider the logistic equation: K = 10

$$\frac{dy}{dt} = r\left(1 - \frac{y}{10}\right)y$$

(1) Find equilibrium solutions

(2) Find a general solution of the ODE

- Hint: Use partial fractions

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \qquad y(0) = y_0$$

- Provided $y \neq 0$ and $y \neq K$, we can rewrite the logistic ODE:

$$\frac{dy}{(1 - y/K)y} = rdt$$

- Expanding the left side using partial fractions,

$$\frac{1}{(1 - y/K)y} = \frac{A}{1 - y/K} + \frac{B}{y} \;\Rightarrow\; 1 = Ay + B(1 - y/K) \Rightarrow B = 1,\ A = y/K$$

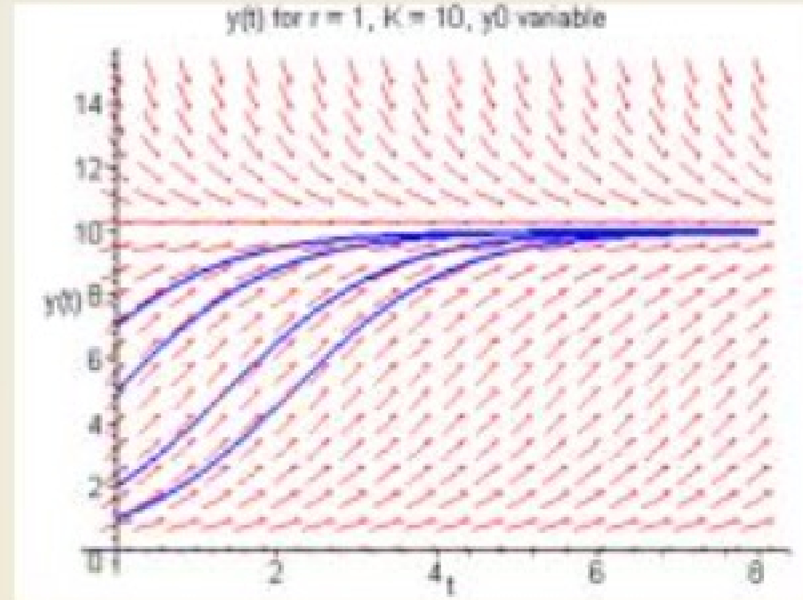
- Thus the logistic equation can be rewritten as

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right)dy = rdt$$

- Integrating the above result, we obtain

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

- We have:



$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

- If $0 < y_0 < K$, then $0 < y < K$ and hence

$$\ln y - \ln\left(1 - \frac{y}{K}\right) = rt + C$$

- Rewriting, using properties of logs:

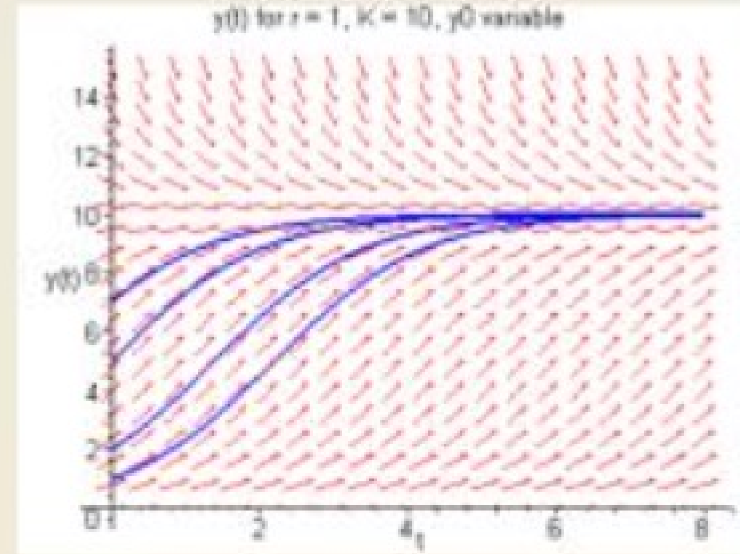
$$\ln\left[\frac{y}{1 - y/K}\right] = rt + C \iff \frac{y}{1 - y/K} = e^{rt+C} \iff \frac{y}{1 - y/K} = ce^{rt}$$

$$\text{or } \; y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}, \quad \text{where } y_0 = y(0)$$

- We have:

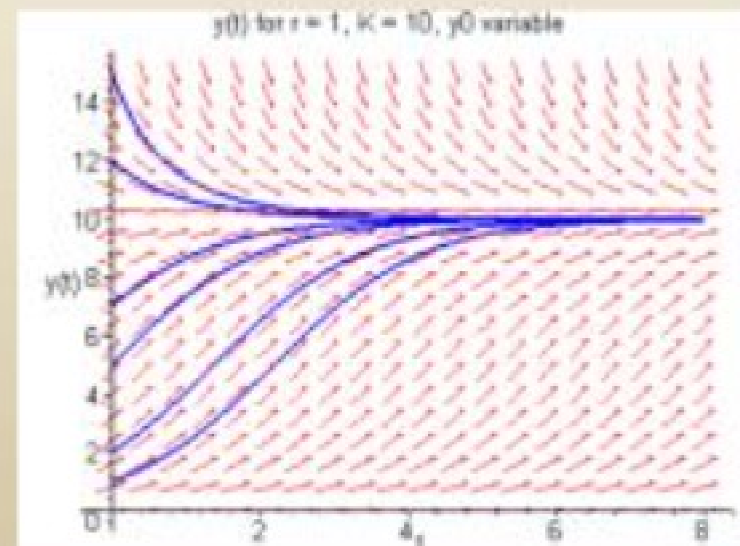
$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

for $0 < y_0 < K$.



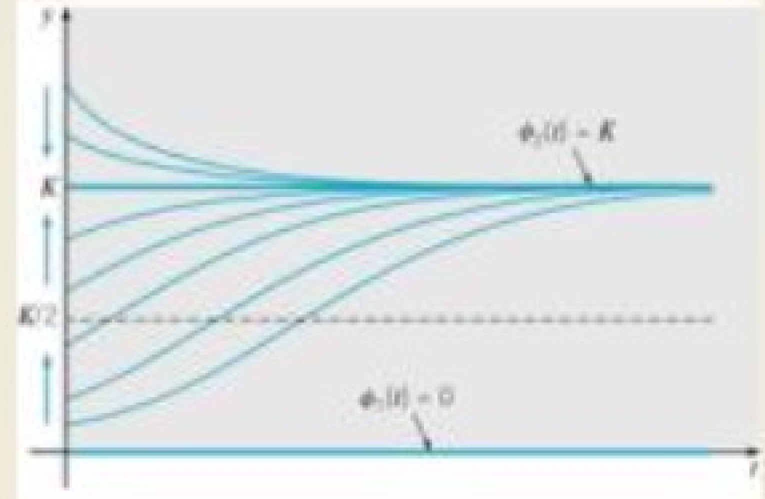
- It can be shown that solution is also valid for $y_0 > K$. Also, this solution contains equilibrium solutions $y = 0$ and $y = K$.
- Hence solution to logistic equation is

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$



- The solution to logistic ODE is

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

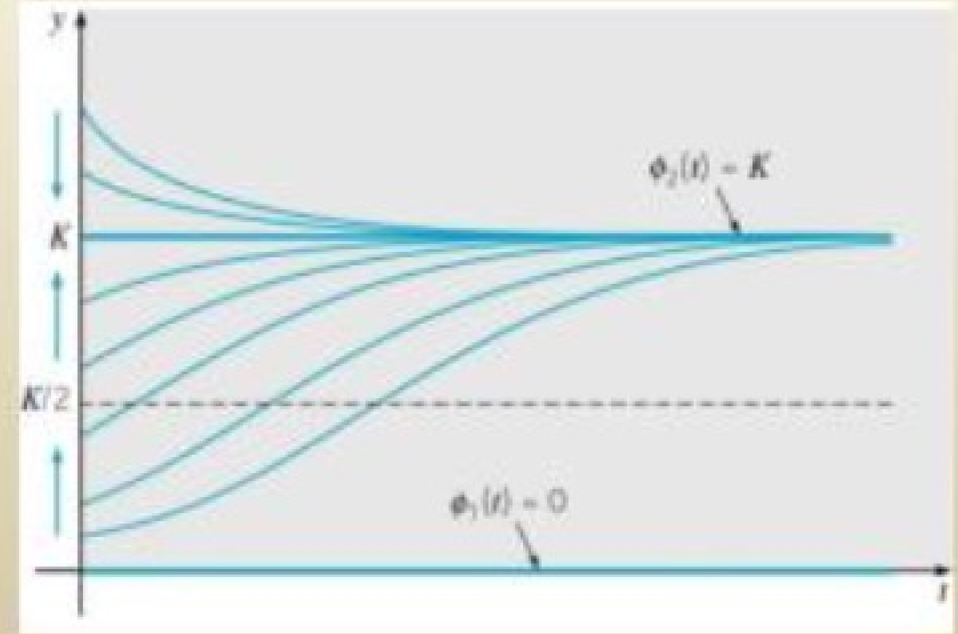


- We use limits to confirm asymptotic behavior of solution:

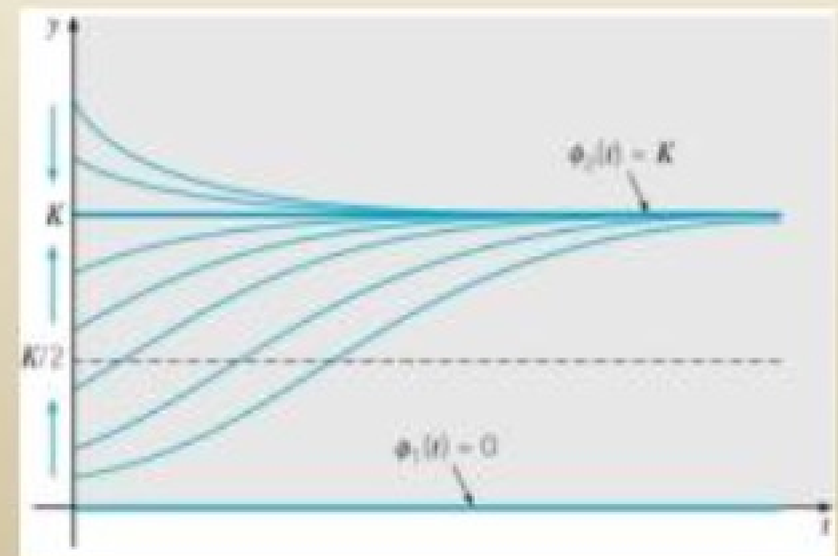
$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} = \lim_{t\to\infty} \frac{y_0 K}{y_0} = K$$

- Thus we can conclude that the equilibrium solution $y(t) = K$ is **asymptotically stable**, while equilibrium solution $y(t) = 0$ is **unstable**.
- The only way to guarantee solution remains near zero is to make $y_0 = 0$.

- Combining the information on the previous slides, we have:
  - Graph of y increasing when $0 < y < K$.
  - Graph of y decreasing when $y > K$.
  - Slope of y approximately zero when $y \cong 0$ or $y \cong K$.
  - Graph of y concave up when $0 < y < K/2$ and $y > K$.
  - Graph of y concave down when $K/2 < y < K$.
  - Inflection point when $y = K/2$.
- Using this information, we can sketch solution curves y for different initial conditions.



- Using only the information present in the differential equation and without solving it, we obtained qualitative information about the solution y.
- For example, we know where the graph of y is the steepest, and hence where y changes most rapidly. Also, y tends asymptotically to the line $y = K$, for large t.
- The value of K is known as the **carrying capacity**, or **saturation level**, for the species.
- Note how solution behavior differs from that of exponential equation, and thus the decisive effect of nonlinear term in logistic equation.



Example 2: Pacific Halibut (1 of 2)

- Let the halibut population in the pacific ocean satisfy the logistic equation with $r = 0.71$/year, $K = 80.5 \times 10^6$ kg and $y_0 = 0.25K$.

  Let y be biomass (in kg) of halibut population at time t.

(a) Set up an ODE and solve the equation $\dfrac{dy}{dt} = r\left(1 - \dfrac{y}{K}\right)y, \quad y(0) = y_0$

(b) Find the population 2 years later : $y = \dfrac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

(c) Find the time τ such that $y(\tau) = 0.75K$.

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Ex 2: Pacific Halibut (1 of 2)

- Let y be biomass (in kg) of halibut population at time t, with r = 0.71/year and $K = 80.5 \times 10^6$ kg. If $y_0 = 0.25K$, find

  (a) biomass 2 years later

  (b) the time τ such that $y(\tau) = 0.75K$.

(a) For convenience, scale equation:

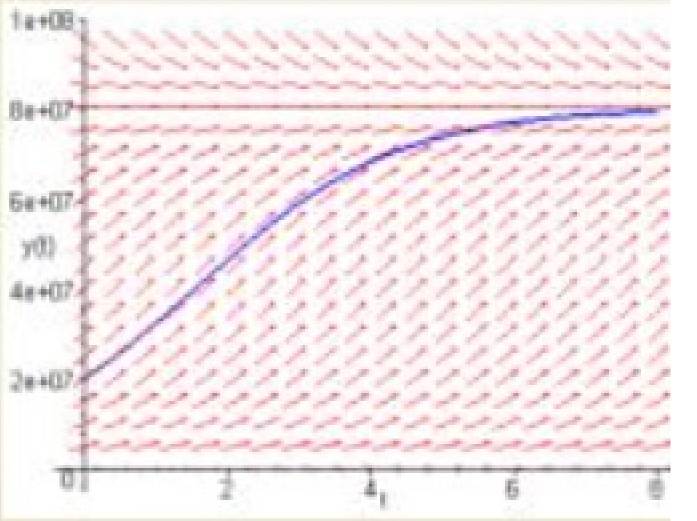
$$\frac{y}{K} = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rt}}$$

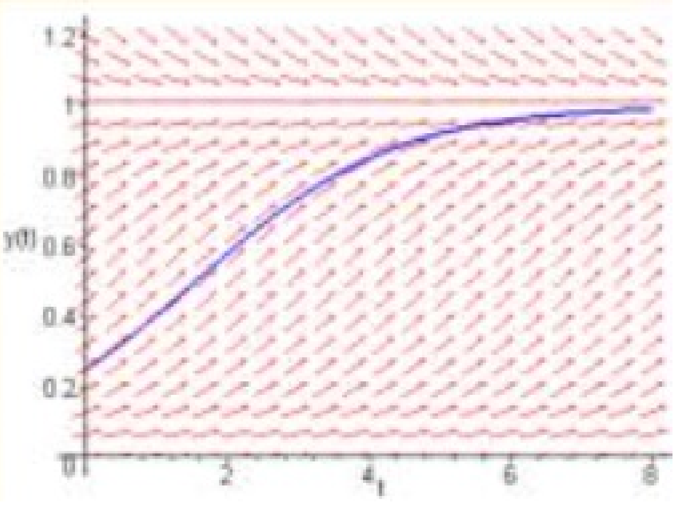
Then

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + 0.75e^{-(0.71)(2)}} \approx 0.5797$$

and hence

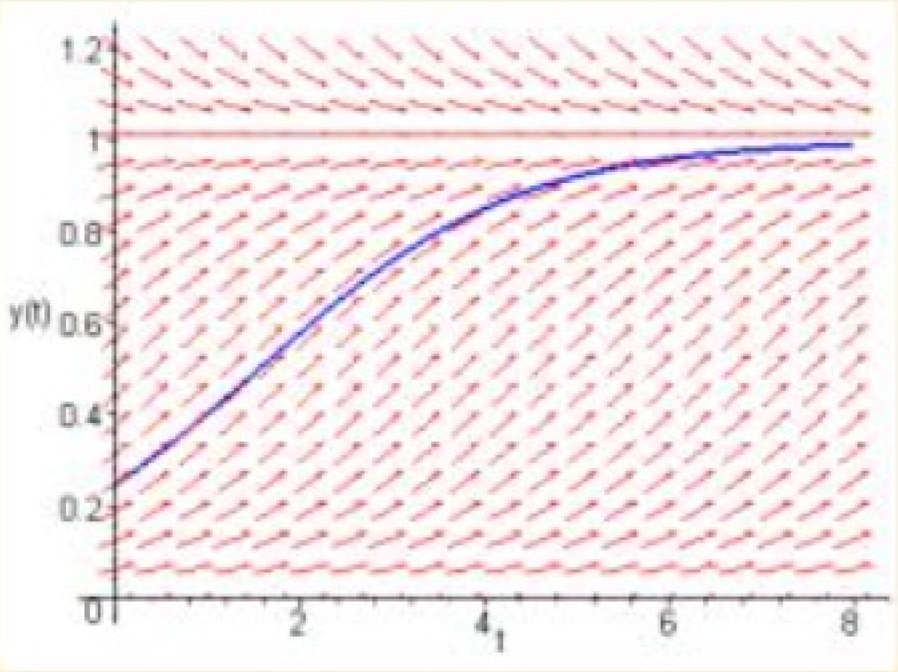
$$y(2) \approx 0.5797K \approx 46.7 \times 10^6 \text{ kg}$$





(b) Find time τ for which y(τ) = 0.75K.

$$\frac{y}{K} = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rt}}$$



$$0.75 = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-r\tau}}$$

$$0.75\left[\frac{y_0}{K} + \left(1 - \frac{y_0}{K}\right)e^{-r\tau}\right] = \frac{y_0}{K}$$

$$0.75\, y_0/K + 0.75(1 - y_0/K)e^{-r\tau} = y_0/K$$

$$e^{-r\tau} = \frac{0.25 y_0/K}{0.75(1 - y_0/K)} = \frac{y_0/K}{3(1 - y_0/K)}$$

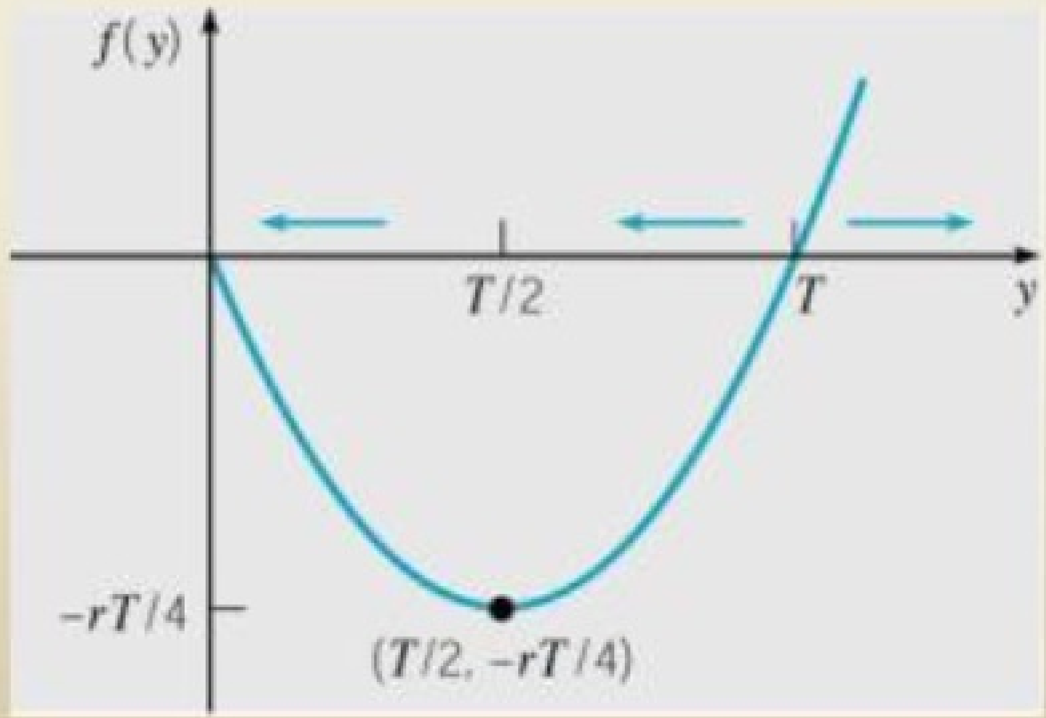
$$\tau = \frac{-1}{0.71}\ln\left(\frac{0.25}{3(0.75)}\right) \approx 3.095 \text{ years}$$

Critical Threshold Equation

- Consider the following modification of the logistic ODE:

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad r > 0$$

- The graph of the right hand side $f(y)$ is given below.



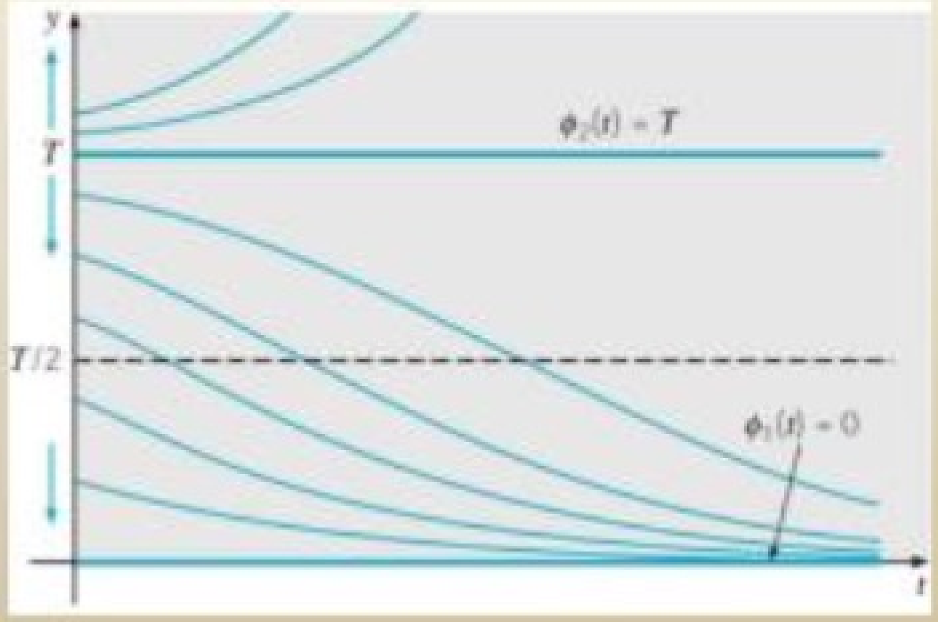
Critical Threshold Equation: Qualitative Analysis and Solution (2 of 2)

- Performing an analysis similar to that of the logistic case, we obtain a graph of solution curves shown below.
- T is a **threshold value** for y_0, in that population dies off or grows unbounded, depending on which side of T the initial value y_0 is.
- See also laminar flow discussion in text.
- It can be shown that the solution to the threshold equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad r > 0$$

is

$$y = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

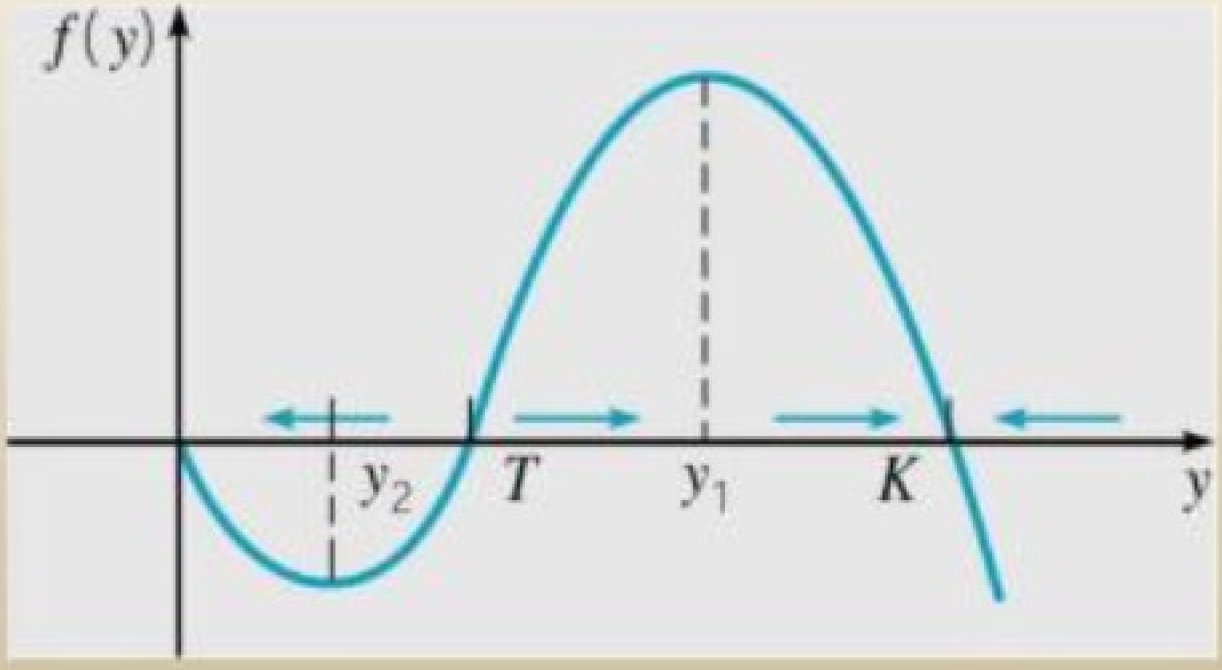


Logistic Growth with a Threshold

- In order to avoid unbounded growth for $y > T$ as in previous setting, consider the following modification of the logistic equation:

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y, \quad r>0 \text{ and } 0<T<K$$

- The graph of the right hand side $f(y)$ is given below.



Logistic Growth with a Threshold (2 of 2)

- Performing an analysis similar to that of the logistic case, we obtain a graph of solution curves shown below right.
- T is threshold value for y_0, in that population dies off or grows towards K, depending on which side of T y_0 is.
- K is the carrying capacity level.
- Note: $y = 0$ and $y = K$ are stable equilibrium solutions, and $y = T$ is an unstable equilibrium solution.