Mathematical Modelling

(النمذجة الرياضية)

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Chapter One

The Scope of Mathematical Modeling

<u>Definition (1):</u> A model is an object or concept that is used to represent something else. It is reality scaled and converted to a form we can comprehend.

For example, a model airplane, made of wood, plastic, and glue, is a model of the real aeroplane. Another example is the idea that, in politics, public opinion is like a pendulum because it changes periodically from left to right-wing ideas then back again in a way which reminds us of a pendulum swinging back and forth. In our terminology, we would say that a pendulum is a model for public opinion.

A model airplane and pendulum are physical objects; so, they are not a mathematical model.

What Is a Mathematical Model?

The following is a possible informal definition of a mathematical model:

<u>Definition (2):</u> A mathematical model is a translation of a real-world problem into mathematics notation by forming a mathematics problem corresponding to the real-world problem. Then mathematics tools, ideas, concepts, and techniques are utilized to solve the mathematics problem. The obtained solution is translated back into the real-world.

Or

A mathematical model is a model whose parts are mathematical concepts, such as constants, variables, functions, equations, inequalities, etc.

Example (1): To find out how an aircraft will behave in flight; we could make a physical model of the aircraft and test it under various weather conditions. There are a great many things one might want to know: Is the plane stable in the air? How fast can it go? How steeply can it climb? Etc. To focus our discussion, let's consider the question of how great the lift force on the plane who it is takes off.

The lift force is the force pushing up on the wings. This force is largely what determines how steeply the plane can climb.

If we did experiments with a physical model, we could find out almost anything we want to know about it.

For example, we could discover that the lift force was dependent on how fast the plane was moving.

By flying the plane at different speeds, we could make a table of values relating lift force to velocity and a graph of this table of values that might look like figure (1).

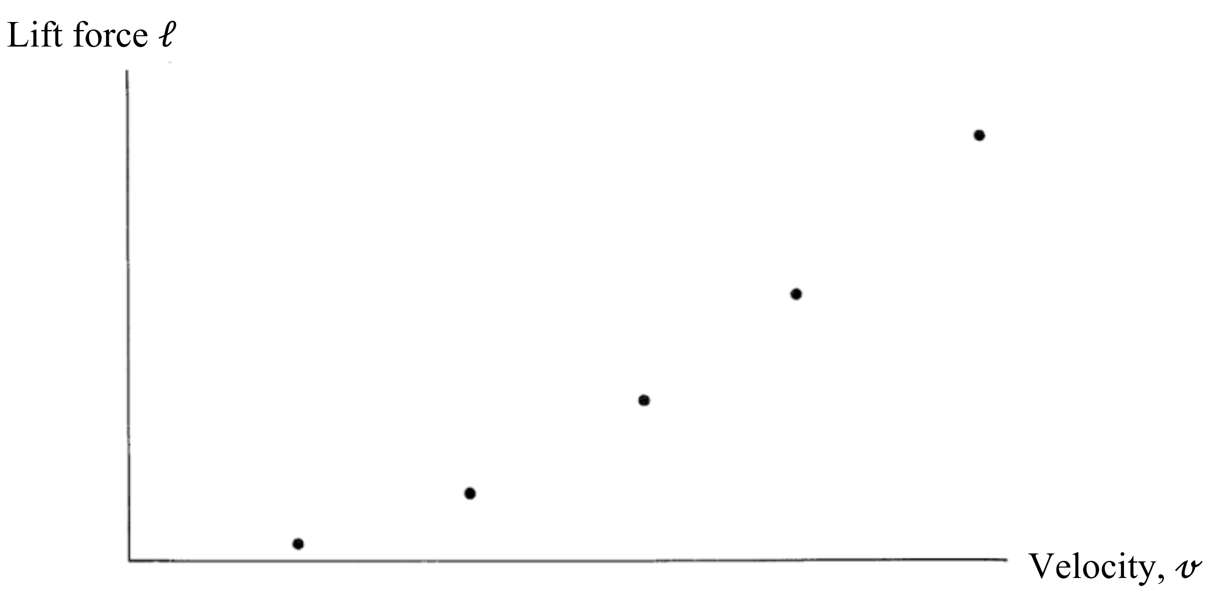


Figure (1.1): Physical model of the aircraft

But there is an entirely different approach to this problem. One based on a mathematical model. This m. m. consists of a single equation which relates the lift force to other factors. It is

$$\ell = C_\ell \frac{\rho}{2} s v^2 \qquad \dots (1)$$

ℓ = lift force

C_ℓ = lift coefficient depends on the shape of the plane

ρ = density of the air

v = velocity

and s =total surface area of the tops of the wings.

We can estimate s from the blueprints of the plane we propose to build. ρ is a measurement we can make in the atmosphere. (It may differ a little from one airport to another.) C_ℓ is known. Then the product $C_\ell \left(\frac{\rho}{2}\right) s$ in equation (1) becomes a known constant. If we call this constant a, then equation (1) becomes an equation linking only two variables, ℓ an v:

$$\ell = a\, v^2 \qquad \dots (2)$$

Using this equation, we can generate the graph shown in figure (2) with a moment's worth of calculation and plotting.

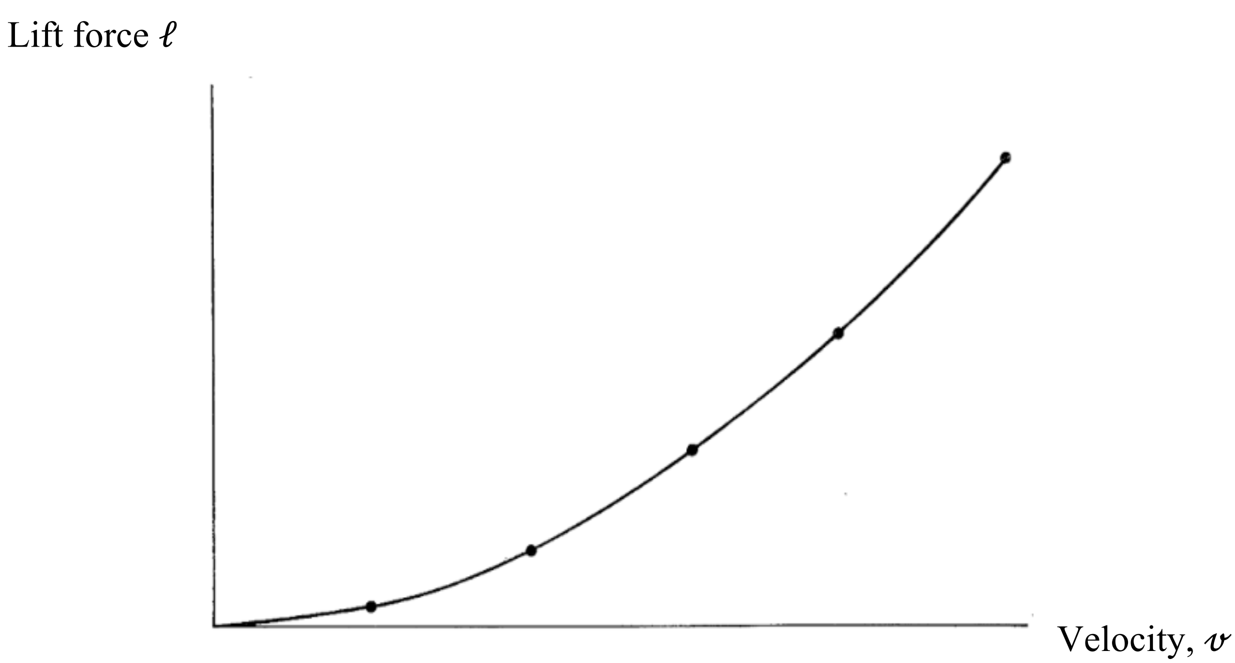


Figure (1.2): Mathematical model of the aircraft

Which approach is better, experiments on the physical model or predictions from the mathematical model? Building a physical model is the time consuming: it might take days to make a good model plane. It is also expensive. In both these respects, the mathematical model is superior. However, it has another advantage. It tells us things that our physical experiments do not.

THE MODELING PROCESS

It is useful to view mathematical modeling as a process as illustrated in Figure 1.2. The modeling process is represented by a loop, where the starting point is step 1, located in the box in the upper left-hand corner of Figure 1.2.

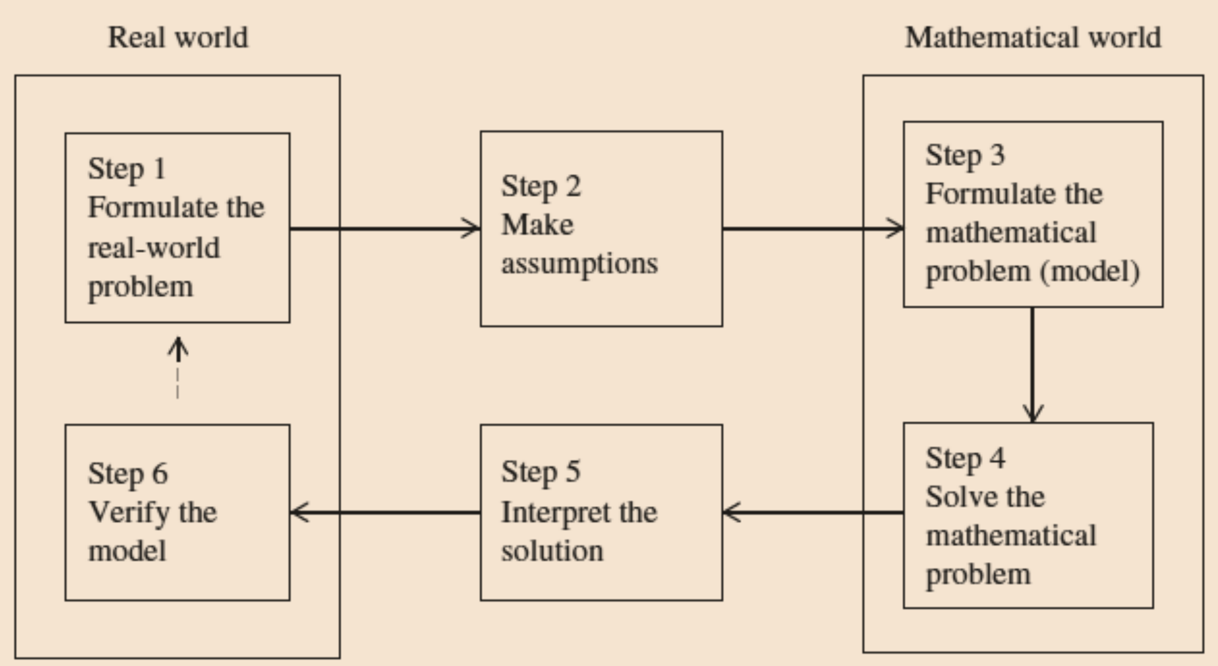


FIGURE 1.2. The mathematical modeling process.

Steps in Building a Mathematical Model

Abstract Three steps in mathematical modelling are discussed: formulation, mathematical manipulation, and evaluation.

1) Formulation

A) Stating the question. The question we start with is often too vague or too "big". If it's vague, make it precise. If it's too big, subdivide it into manageable parts.

B) Identifying relevant factors. Decide which quantities and relationship are important for your question and which can be neglected.

C) Mathematical description. Each important quantity should be represented by a suitable mathematical entity, e.g., a variable, a function, a geometric figure, etc. Each relationship should be represented by an equation, inequality, or other suitable mathematical assumption.

2) Mathematical manipulation

The mathematical formulation rarely gives us answers directly. We usually have to do some mathematics. This may involve a calculation, solving an equation, proving a theorem, etc.

3) Evaluation

In deciding whether our model is a good one, there are many things we could take into account. The most important question concerns whether or not the model gives correct answers. If the answers are not accurate enough or it the model has other shortcomings, then we should try to identify the sources of the shortcomings. Mistakes may have been made in the mathematical manipulation. But in many cases what is needed is a new formulation. After a new formulation, we will need to do new mathematical manipulation and new evaluation. Thus mathematical modelling can be a repeated cycle of the three modelling steps, as shown in the flowchart of a figure (3).

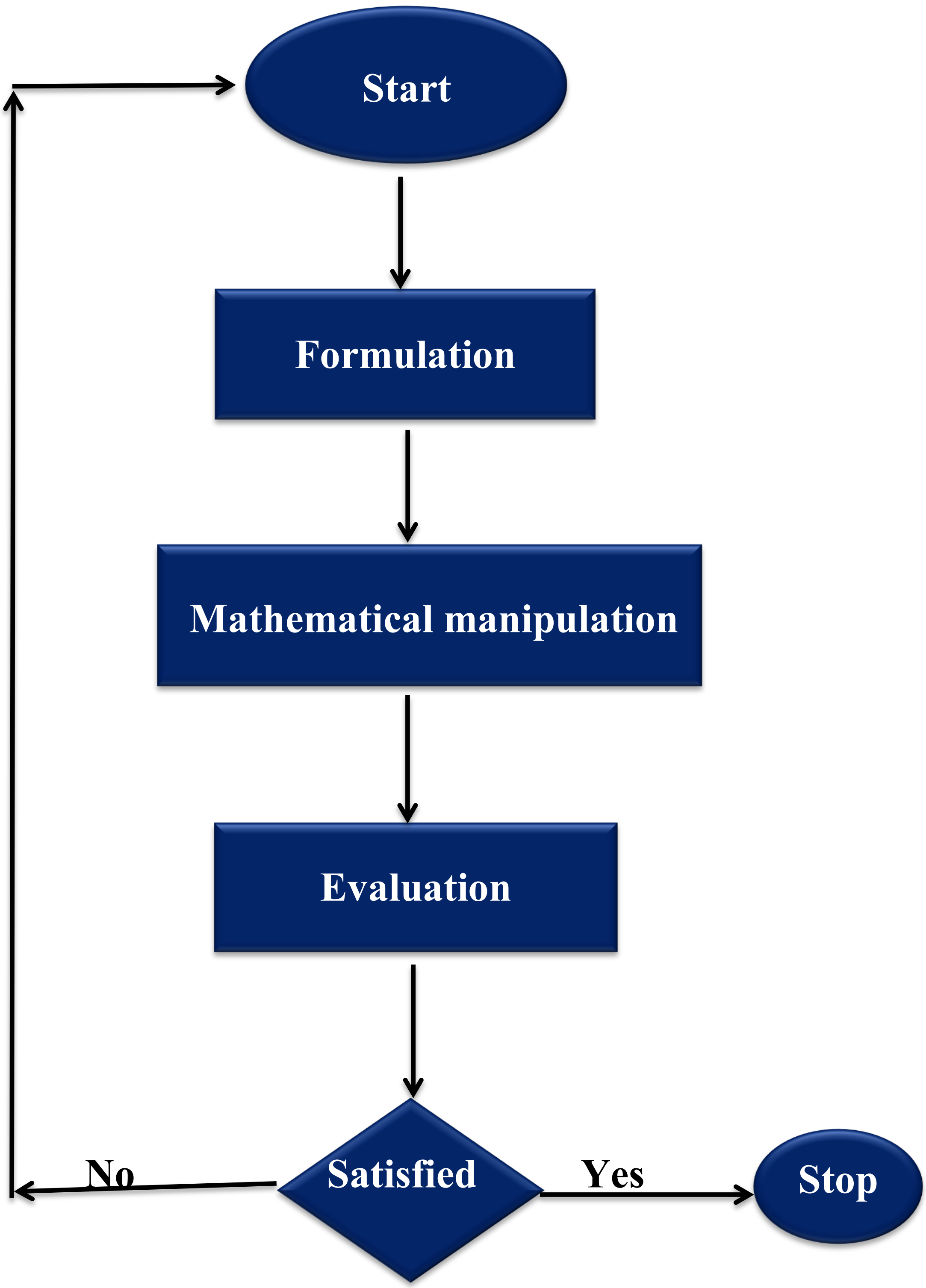


Figure (1.4): Steps of Mathematical Modelling

Illustration 1 Galileo's Gravitation Models

1) Formulation:

I) Stating the question.

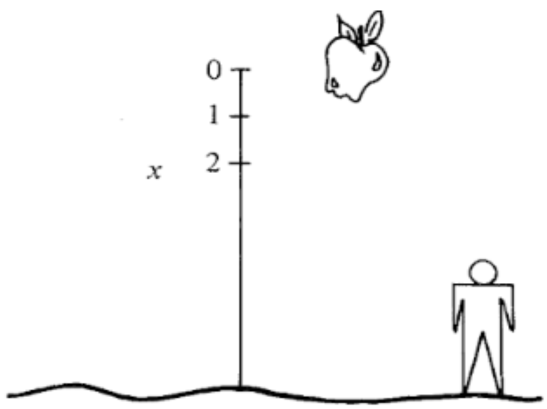
Two particular questions Galileo asked were:

a) What formula describes how a body gains velocity as it falls?
b) What formula describes how far a body falls in a given amount of time?

II) Identifying relevant factors.

Assumption 1

If a body falls from rest, then its velocity at any point is proportional to the distance already fallen.



III) Mathematical description.

We will set up a distance scale to measure an object's fall and use the variable x to measure distance along this scale. At a certain instant in time, say $t = 0$, we begin observing the object's fall. For convenience let $x = 0$ at this initial point of observation. Let $x(t)$ denote the distance the object has fallen after t seconds.

Between the time instantst_0 and$t_0 + h$ the distance travelled is $x(t_0 + h) - x(t_0)$. Since the time elapsed is h seconds, the average velocity is

$$\frac{x(t_0 + h) - x(t_0)}{h}$$

If we let $h \to 0$, this quotient approaches $\frac{dx}{dt}$ evaluated at t_0, a quantity we call the instantaneous velocity. We sometimes denote this $v(t_0)$ instead of $(dx/dt)|_{t_0}$.

The same ideas apply to acceleration. Between time instants t_0 and $t_0 + h$ the change in velocity is $v(t_0 + h) - v(t_0)$. Since the time elapsed is h seconds, the average change in velocity is

$$\frac{v(t_0 + h) - v(t_0)}{h}$$

Let $h \to 0$, we obtain $(dv/dt)|_{t_0}$ which is the same as $\frac{d\left(\frac{dx}{dt}\right)}{dt} = \frac{d^2x}{dt^2}$ evaluated at t_0. This is called the instantaneous acceleration at the time t_0.

The mathematical description of assumption 1 is

$$\frac{dx}{dt} = ax \qquad \dots (1)$$

where a is a constant yet t_0 be determined.

2) Mathematical manipulation

$$\frac{dx}{dt} = ax$$

$$\int \frac{dx}{x} = \int a\, dt$$

$$\ln x = at + c$$

$$e^{\ln x} = e^{at+c}$$

$$x = e^c e^{at}$$

$$x = ke^{at}$$

$$\text{by } x(0) = 0$$

$$0 = ke^0 \rightarrow k = 0$$

$$x = 0 \qquad \text{for all t}$$

This says that the object will never move, no matter how long we wait.

Evaluation

Since the conclusion is completely absurd and there are no mistakes in the mathematical manipulation, we need a reformulation

1) Formulation (again)

Assumption 2

If a body falls from rest, then its velocity at any point is proportional to the time it has been falling. In particular, for each second of fall, the object gains an extra 32 feet/second in velocity.

The mathematical description of this assumption is

$$\frac{dx}{dt} = 32t$$

2) Mathematical manipulation

$$\int dx = \int 32t\, dt$$

$$x = 16t^2 + c$$

$$\text{by } x(0) = 0 \to c = 0$$

$$x = 16t^2$$

3) Evaluation

This law of falling bodies agrees well with observations in many circumstances.

Example:

a) An object falls, starting at rest, for 2 seconds. How far does it fall and what is its velocity after 2 seconds?

b) How long does it take an object to fall 144 feet?

Solutions:

a) $$\frac{dx}{dt} = 32.t$$

$$\frac{dx}{dt} = 32.2 = 64 \quad \text{feet/second}$$

$$x = 16.2^2 = 64 \quad \text{feet}$$

b) $$144 = 16t^2$$

$$t^2 = 9 \quad \text{seconds}$$

$$t = 3 \quad \text{seconds}$$

Illustration 2 The Manufacturing Progress Curve

If you have a complicated job to do and you have to do it many times, you'll probably get better at it. Partly this is a matter of learning: practice does make perfect. But ingenuity will also play a role: you will invent shortcuts; devise new tools to assist you, etc. The same is true for a team of workers and managers assembling complex products, like airplanes or automobiles. T.P. wright studied this in aircraft assembly planes in 1936 and proposed the following model. Since then, it has been used in many branches of manufacturing.

1) Formulation

I) **Stating the question**. How does the time required to produce an airplane of a given type depend on the number of planes of that type already produced?

II) **Describing relevant factors**: we consider only the time for assembly and the number already assembled. All other factors are ignored. For example, we don't consider whether workers are given cash incentives for efficient work, even though this may be relevant.

The assumption when the number of planes is doubled, the time for production decreases to about 80 percent of its former value.

III)Mathematical description.

Let $T(x)$ be the time required for the x th plane write's assumption means that, if the first plane took 100,000 worker-hours then the second would take 80,000. the fourth would need 64,000. In general, if the time for the first plane is T_1, then the following table describes

Plane no. x	1	2	4	8	…
Hours T	T_1	$0.8\, T_1$	$(0.8)^2\, T_1$	$(0.8)^3\, T_1$	…

The formula we are looking for appears to be

$$T = (0.8)^n \, T_1 \qquad \dots(*)$$

where n is the number of doubling, starting with the number 1, required to obtain x. The formula doesn't involve x directly. Also, it doesn't allow us to compute T at any number of planes not a power of 2 like 3,5,…

2) Mathematical manipulation

The number of doublings to reach x, starting with 1 is, by definition, $\log_2 x$. Thus, the equation $(*)$ can be rewritten

$$n = \log_2 x \; iff \; x = 2^n$$

$$T(x) = 0.8^{\log_2 x} \, T_1$$

Taking base-2 logarithms, we get

$$\log_2 T(x) = (\log_2(x))(\log_2(0.8)) + \log T_1$$

$$= \log_2 x^{\log_2 0.8} + \log T_1$$

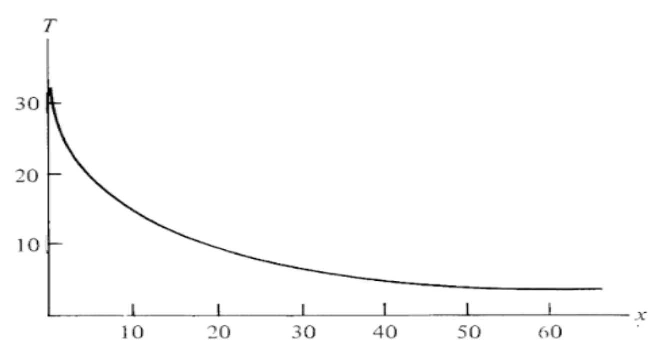
$$\log_2 T(x) = \log_2 T_1 x^{\alpha}$$

where $\alpha = \log_2 0.8 = \log_{10} 0.8 \, / \, \log_{10} 2 = -0.322$

$$\therefore T(x) = T_1 x^{\alpha} \qquad \dots(**)$$

$$T(x) = T_1 x^{-0.322}$$

this is called the manufacturing progress curve for airplane, and 80% is called the progress rate. As figure



Example:

A new assembly process for airplanes, which uses more automated machine assembly and less human labor, is introduced. Consequently, the opportunities for learning decline and progress rate shifts from 80 to 90 percent. Assume the first plane made with the new process took 100,000 hours.

a) What is the equation for $T(x)$?
b) How long does it take to produce the hundredth plane?

Solution:

a) From eq. (**) we obtain

$$T(x) = 100{,}000\, x^{\alpha}$$

where $\alpha = \log_2 0.9 = \frac{\log_{10} 0.9}{\log_{10} 2} = -0.152$

b) $T(100) = 100{,}000\, (100)^{-0.152}$

$= 49{,}659$ hours.

Evaluation:

$$\therefore \lim_{x \to \infty} x^{-0.322} = 0$$

Formulation (again)

$$T(x) = T_m + T_e(x)$$

$$T(x) = T_m + T_0 x^{\alpha}$$

الحد الادنى من ساعات عمل لتجميع الطائرة تحت ظروف عمل مثالية

أعتماداً على الموديل القديم

Macro and Micro Population Models

Model 1- Exponential Growth

Macro and micro refer to the measurement of size but in different directions. One refers to large measurements, and one refers to small measurements.

We illustrate this in the next sections by presenting models for population projection, each occupying a different position on the macro-micro spectrum. The first can be thought of as an answer to the question of how many people there will be in future. The second, the Leslie-matrix model, can be regarded as an answer to the question of how many people of various ages there will be in the future. The last, the sex and family-planning model, can be considered an answer to the question of the how many people there will be in your family in the future.

We will use the variable t to measure time in years, with $t = 0$ denoting the present. Let $P(t)$ denote the size of population. Let $B(t)$ be the number of births in the 1-year interval between time t and $t + 1$ and let $D(t)$ denote the number of deaths between t and $t + 1$. The main assumption of our model is that certain rates stay the same.

Definition

1. $\frac{B(t)}{P(t)}$ is called the birth rate for the time (t) to $(t + 1)$.
2. $\frac{D(t)}{P(t)}$ is called the death rate for the time interval (t) to $(t + 1)$.

Assumptions

1. The birth rate is the same for all intervals. Likewise, the death rate is the same for all intervals. This means that there is a constant b, called the birth rate, and a constant d, called the death rate so that, for all $t \geq 0$

$$b = \frac{B(t)}{P(t)} \quad \text{and} \quad d = \frac{D(t)}{P(t)} \qquad \dots(1)$$

2. There is no migration into or out of the population; i.e., the only source of population change is birth and death.

   As a result of assumptions 1 and 2 we deduce that, for $t \geq 0$

$$P(t+1) = P(t) + B(t) - D(t)$$
$$= P(t) + bP(t) - dP(t)$$
$$= (1 + b - d)P(t) \quad \dots(2)$$

If $t = 0$ in equation (2) gives

$$P(1) = (1 + b - d)P(0) \qquad \dots(3)$$

Setting $t = 1$ in equation (2) and substituting equation (3) gives

$$P(2) = (1 + b - d)P(1) \;\; P(1)$$
$$= (1 + b - d)(1 + b - d)P(0)$$
$$= (1 + b - d)^2 P(0)$$

Continuing this way yields

$$P(t) = (1 + b - d)^t P(0) \qquad \dots(4)$$

for $t = 0,1,2, \dots$

The constant $1 + b - d$ is often abbreviated r and called the growth rate, or in more high-flown language, the Malthusian parameter, in honor of Robert Malthus who first brought this model to popular attention. In term of r eq. (4) becomes

$$P(t) = P(0)r^t \qquad t = 0,1,2, \dots \qquad \dots(5)$$

$P(t)$ is an example of an exponential function. Any function of the form Cr^t, where C and r are constant, is an exponential function.

Example (1)

Suppose the current population is 250,000,000 and the rates are $b = 0.02$ and $d = 0.01$. What will the population be in 10 year?

Solution:

From equation (4)

$$P(10) = (1.01)^{10}(250{,}000{,}000)$$

$$= (1.104622125)(250{,}000{,}000)$$

$$= 276{,}155{,}531.25$$

Naturally, this result is absurd, since one can't have 0.25 of a person. This is a good illustration that the fundamental assumption of the model is not exactly true, but only approximately.

Example (2)

How many years will it take for the population of example 1 to double its initial size?

Solution:

We seek a value of t for which $\frac{P(t)}{P(0)} = 2$.

This requires that

$$\frac{(1.01)^t P(0)}{P(0)} = 2$$

$$\Longrightarrow (1.01)^t = 2$$

$$\Longrightarrow t\ \log(1.01) = \log 2$$

$$\Longrightarrow t = \frac{\log 2}{\log(1.01)} \approx 69.66\ \text{years}$$

Example (3)

How many births will occur between $t = 10$ and $t = 11$?

Solution:

From equation (1),

$$b = \frac{B(t)}{P(t)} \Longrightarrow B(t) = bP(t) \qquad \ldots (*)$$

Substitute eq. (4) into (*), we get

$$B(t) = b(1 + b - d)^t P(0)$$

$$\therefore B(10) = 0.02(1.01)^{10}(250{,}000{,}000)$$

$$= 5{,}523{,}110.6$$

Exercises

1) Suppose the current population is 1,500,000, $b = 0.03$, and $d = 0.01$. What is the population in 5 years?
2) What can conclude about $P(t)$ if the birth rate equals the death rate?
3) Show that, for any fixed number of years, say k, the percent by which the population increases in k years is a function of k, b and d alone, it does not depend on the population size at the start of $k -$ year.

4) The crude birth rate b^* differs from our birth rate b in that births are computed per 1000 people in the population at the middle of the year in which those births occurred. Thus $b^* = 1000\frac{B(t)}{P\left(t+\frac{1}{2}\right)}$. find the formula relating b^* to b and d.

Model 2- The Leslie Matrix

How many people aged 65 to 70 will there be in 10 years? This is often more useful to know than how many people there will be altogether. If we want to know how much social security will have to be paid out in 10 years or how many schools, nursing homes will be needed, we'll need a model which recognizes and projects age groups.

Our model will have is that men are completely ignored. In our model we imagine the female population divided into age categories $[0,\Delta), [\Delta,2\Delta), \dots, [(n-1)\Delta, n\Delta)$. Here Δ, which is the width of each age interval, can be any convenient number, and n is a number sufficiently large that only a negligible number of women survive beyond $n\Delta$ years. In practice $\Delta = 5$ and $n = 20$ are often used.

We will use the variable t to measure time (in years) with $t = 0$ being the present. Our model will not be able to tell us the populations of the age groups for all times in the future, but only for a series of instants in the future spaced Δ years a part: $t = \Delta$, $t = 2\Delta$, etc.

Le t $F_i(t)$ denote the number of females at time t in the i th age group, i.e., with ages in the interval $[i\Delta, (i+1)\Delta)$. We define the column vector $\vec{F}(t)$ by

$$\vec{F}(t) = \begin{bmatrix} F_0(t) \\ F_1(t) \\ \vdots \\ F_{n-1}(t) \end{bmatrix}$$

and call this the age distribution vector for time t. $F(0)$ is the current age distribution and known to us from census data. Our task is to predict $F(\Delta)$, $F(2\Delta)$, …

A graphical representation of $F(t)$ is often made in the form of a "population pyramid" (figure (1)).

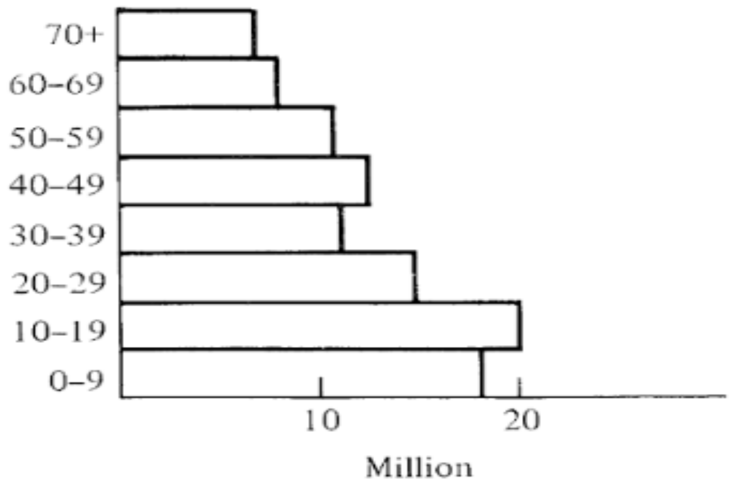


Figure (1)
Population pyramid

Obviously, to do this trick of prediction, it is necessary to have some information about birth and death rates for the various age groups. Therefore let d_i be the death rate for i th age group; specifically, d_i is the fraction of the i th age group which, on account of death, will not be present in $(i+1)st$ age group Δ years later. The fraction surviving is therefore $1-d_i$, which we denote P_i. This survival rate (P_i) is assumed to be in effect for all future times.

This means

$$F_{i+1}(t+\Delta) = (1-d_i)F_i(t)$$

$$= P_i\, F_i(t) \qquad \dots (1)$$

for $t = 0, \Delta, 2\Delta, \dots$.

Let m_i denoted the Δ–year maternity rate for the i th group. This means that for any t value, the average women in the i th age group at time t will, by having babies in between times t and $t+\Delta$, contribute m_i children to the lowest age group at time $(t+1)$. The m_i are assumed constant in time.

Thus, the number of newborns (age 0) at time $t + \Delta$ is

$$F_0(t + \Delta) = \sum_{i=0}^{n-1} m_i\, F_i(t) \qquad \dots (2)$$

Equations (1) and (2) can be written as a single matrix equation

$$\begin{bmatrix} F_0(t+\Delta t) \\ F_1(t+\Delta t) \\ \vdots \\ F_{n-1}(t+\Delta t) \end{bmatrix} = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & & m_{n-1} \\ p_0 & 0 & 0 & \cdots & & 0 \\ 0 & p_1 & 0 & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & p_{n-2} & 0 \end{bmatrix} \begin{bmatrix} F_0(t) \\ F_1(t) \\ \vdots \\ F_{n-1}(t) \end{bmatrix} \qquad \dots (3)$$

which is valid for $t = 0, \Delta, 2\Delta, \dots$ up to the latest time one wishes to project to.

The $n \times n$ matrix is called the Leslie Matrix denoted by M. Equation (3) can then be abbreviated

$$\vec{F}(t + \Delta) = M\vec{F}(t) \qquad t = 0,1,2, \dots \qquad \dots (4)$$

Particular instances of equation (4) for $t = 0$ and $t = \Delta$

$$\vec{F}(\Delta) = M\vec{F}(0)$$

$$\vec{F}(2\Delta) = M\vec{F}(\Delta)$$

$$= MM\vec{F}(0)$$

$$= M^2\vec{F}(0)$$

Likewise,

$$\vec{F}(3\Delta) = MM^2\vec{F}(0)$$

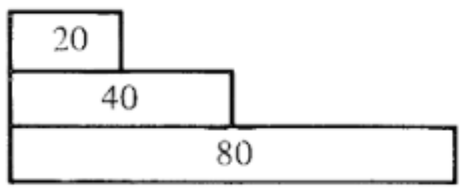
$$= M^3\vec{F}(0)$$

In general,

$$\vec{F}(k\Delta) = M^k \vec{F}(0) \quad k = 0,1,\ldots$$

Example (1)

Imagine a population divided into three age groups. Initially $(t = 0)$ the population of females is divided into three age groups, as in the pyramid below.



Suppose that, as one-time unit passes, everyone in the oldest group dies and one-quarter of those in each of the other age groups dies. Suppose also that the age-specific maternity rates are $m_0 = 0, m_1 = 1$ and $m_2 = 2$. Find the age distribution vectors $F(\Delta)$ and $F(2\Delta)$ and represent them as population pyramids.

Solution:

The information given about mortality implies $p_0 = \frac{3}{4}$ and $p_1 = \frac{3}{4}$. Therefore, the Leslie matrix is

$$M = \begin{bmatrix} 0 & 1 & 2 \\ \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \end{bmatrix}$$

and the initial vector is $\begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix}$

to find $F(\Delta)$, we use equation (5) with $k = 1$

$$F(\Delta) = \begin{bmatrix} F_0(\Delta) \\ F_1(\Delta) \\ F_2(\Delta) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix} = \begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix}$$

and

$$F(2\Delta) = \begin{bmatrix} F_0(2\Delta) \\ F_1(2\Delta) \\ F_2(2\Delta) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 120 \\ 60 \\ 45 \end{bmatrix}$$

or

$$F(2\Delta) = \begin{bmatrix} F_0(2\Delta) \\ F_1(2\Delta) \\ F_2(2\Delta) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ \frac{3}{4} & 0 & 0 \\ 0 & \frac{3}{4} & 0 \end{bmatrix}^2 \begin{bmatrix} 80 \\ 40 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{4} & \frac{3}{2} & 0 \\ 0 & \frac{3}{4} & \frac{3}{2} \\ \frac{9}{16} & 0 & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 60 \\ 30 \end{bmatrix} = \begin{bmatrix} 120 \\ 60 \\ 45 \end{bmatrix}$$

the population pyramids for $t = 0, \Delta$ and 2Δ