Finite and Infinite Sets

Definition:

1. A set A is said to be **finite** if $\exists$ bijective f: $A \to B$ with $B \subseteq \mathbb{N}$ where $B = \{1, 2, , \ldots, m\}$ and $m \in \mathbb{N}$.
2. A set A is said to be **infinite** if $\nexists$ bijective f: $A \to B$ with $B \subseteq \mathbb{N}$.

Remark:

Let A be a set. The cardinal number of A is the number of elements in the set A. (#(A) or |A|)

Remarks:

1. The set of natural numbers is an infinite set.
2. $B = \{6, 8, 10, 12, ...\}$ is an infinite set.
3. The empty set φ is a finite set (because there exists a bijection g: $\varphi \to \{0\}$).
4. The set $A = \{a, b, c, d\}$ is a finite set.

Definition:

1. A set S is said to be countable if there does not exist a 1-1 function f: $S \to \mathbb{N}$.
2. A set S is said to be uncountable if there does not exist a 1-1 function f: $S \to \mathbb{N}$.
3. $\#(\mathbb{N}) = \aleph_0$ (aleph-null). If #A or $|A| \leq \aleph_0$, then A is countable.
4. A is countable infinite if $|A| = \aleph_0$.
5. Every subset of a countable set is either finite or countable.

Theorem: For any set A, the following statements are equivalent:

1. A is Countable (A مجموعة معدودة).
2. $\exists$ a bijective function α: $A \to \mathbb{N}$.
3. A is either finite or countably infinite

Theorem:

For any set A, the following statements are equivalent:

1. A is Countably Infinite (A منتهية غير معدودة).
2. There exists a bijective function f: $A \to \mathbb{N}$.

3. The elements of A can be arranged in an infinite sequence $a_0, a_1, a_2, \ldots$, where $a_i \neq a_j$ for $i \neq j$.

عناصر المجموعة A يمكن ترتيبها ضمن تسلسل غير منتهٍ من عناصر مختلفة $(a_i \neq a_j)$

Corollary. For any set A, the following statements are equivalent:

1. A is Countably Infinite (A منتهية غير معدودة).
2. The elements of A can be arranged in an infinite sequence $a_0, a_1, a_2, \ldots$, where $a_i \neq a_j$ for $i \neq j$.

Proof. $\Rightarrow$) Suppose A is countable $\Rightarrow$ $\exists$ a bijective $f: \mathbb{N} \to A$.

$\Rightarrow$ $A = f(\mathbb{N}) = \{f(1), f(2), \ldots\}$

Put $f(n) = a_n$ for all n in $\mathbb{N}$. $\Rightarrow$ $A = \{a_1, a_2, \ldots\}$

$\Leftarrow$) suppose that $A = \{a_1, a_2, \ldots\}$ such that $a_i \neq a_j$ for $i \neq j$.

Define $f: \mathbb{N} \to A$ by $f(n) = a_n$ for all n in $\mathbb{N}$. Then

1. f is one-one: let $x, y \in \mathbb{N}$ such that $f(x) = f(y) \Rightarrow a_x = a_y. \Rightarrow x = y$

   if $x \neq y \Rightarrow a_x \neq a_y \Rightarrow f(x) \neq f(y) \Rightarrow$ f is one to one.
2. f is onto: by definition of f, $\forall a_n \in A, \exists n \in \mathbb{N}$ such that $f(n) = a_n$

$\therefore$ f is bijective

$\therefore$ A is countable

Remarks:

1. If A and B are countable sets, then $A \times B$ is countable.

Proof. A and B are countable sets, then:

i. either A and B are finite sets, then

$A = \{a_1, a_2, \ldots, a_m\}$ and $\Rightarrow$ $B = \{b_1, b_2, \ldots, b_n\}$

$\therefore A \times B$ is countable.

ii. Or A and B are infinite sets, then

iii. $A= \{a_1, a_2, \dots\}$ and ⇒ $B= \{b_1, b_2, \dots\}$

⇒ $A \times B = \{(a_i, b_j) | a_i \in A, b_j \in B \text{ and } i, j \in \mathbb{N}\}$ is infinite.

⇒ $A \times B$ can be arranged by

	b_1	b_2	b_3 …..
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3) ….
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3) ……
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3) ….
…..	………………………………………………..		

So, the first element is (a_1, b_1)

⇒ we can list all elements of $A \times B$ in an infinite sequence.

∴$A \times B$ is countable

2. If A and B are countable sets, then $A \cup B$ is countable.
3. The set of all finite subsets of $\mathbb{N}$ is countable.

Cantor's Theorem:

Let A be a set and P(A) be the power set of A. Then there does not exist a surjective function $f: A \to P(A)$.

Theorem:

The set $P(\mathbb{N})$ is uncountable.

Theorem The Cartesian product of $\mathbb{N} \times \mathbb{N}$ is countable.

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Proof. To prove that $\mathbb{N} \times \mathbb{N}$ is countable:

The elements of $\mathbb{N} \times \mathbb{N}$ can be arranged as a matrix:

(1,1) → (1,2) → (1,3) → ...
↓ ↗ ↗ ↗
(2,1) → (2,2) → (2,3) → ...
↓ ↗ ↗ ↗
(3,1) → (3,2) → (3,3) → ...
……..

Hence:

$$\mathbb{N} \times \mathbb{N} = \{ (1,1), (2,1), (1,2), (3,1), (2,2), (1,3), \dots \}$$

We can write it as:

$$\mathbb{N} \times \mathbb{N} = \{ a_0, a_1, a_2, a_3, a_4, a_5, \dots \}$$

Where:

$a_1 = (1,1)$
$a_2 = (2, 1)$
$a_3 = (1, 2)$
$a_4 = (3, 1)$
$a_5 = (2, 2)$
....

The elements of $\mathbb{N} \times \mathbb{N}$ can be numbered,
which means that we can arrange the natural numbers in a sequence that matches all pairs in $\mathbb{N} \times \mathbb{N}$ without repetition.

$$\therefore \mathbb{N} \times \mathbb{N} \text{ is countable.}$$