In this semester, we shall study the following four chapters:

Chapter one: Definitions and Preliminaries.

Chapter two: Modules homomorphism

Chapter three: Sequences

Chapter four: Noetherian and Artinian modules

Chapter one (Definitions and Preliminaries.)

Definition. ***(Modules)***

Let R be a ring. A (left) R-module is an additive abelian group M together with a function f : R x M → M defined by: f(r,a)=ra such that for all r,s ∈R and a,b ∈ M :

1. r(a + b) = ra + r b.
   }(distributive laws)
2. (r + s)a = ra + sa.
3. r(sa) = (rs)a. (associative law)

If R has an identity element 1_R and

4. $1_R a = a$ for all a ∈ M,

then M is said to be a ***unitary*** left R-module.

Remarks.

1. A (unitary) right R-module is defined similarly by a function f:MxR→M denoted by (a,r) → ar and satisfying the obvious analogues of (1)-(4).
2. If R is commutative, then every left R-module M can be given the structure of a right R-module by defining ar = ra for r ∈ R, a ∈ M.

3. Every module M over a commutative ring R is assumed to be both a left and a right module with ar = ra for all r ∈R, a∈ M.
4. We shall refer to left R-module by R- module. Also, in this course, all R-modules are unitary.

Remarks.

1. If 0_M is the additive identity element of M and 0_R is the additive identity element of a ring R (where M is an R-module), then for all r ∈ R, a ∈ M : r $0_M = 0_M$ and $0_{R.}$ a = 0_M.
2. (-r)a = -(ra) = r(-a) and n(ra) = r(na) for all r ∈ R, a ∈ M and n ∈ ℤ(ring of integers).

Examples.

1. Every commutative ring is an R-module.
   Proof. Define f: R x R → R by $f(r_1, r_2) = r_1r_2$ for all r_1, r_2 ∈R.then
   a. $(r_1+r_2)r = r_1r + r_2r$
   b. $r(r_1+ r_2) = rr_1+ rr_2$
   c. $(r_1r_2)r = r_1(r_2r)$
2. Every additive abelian group G is a unitary ℤ -module.
   Proof. Define α: ℤ x G→ G by: α(n, m) = nm for all n ∈ ℤ and m ∈G.
   i.e $\alpha(n, m) = \underbrace{m + m + \cdots + m}_{n-times} = nm$
   since G is group and m ∈ G, then there is –m ∈G such that
   $(-nm) = -\underbrace{m - m - \cdots - m}_{n-times}$
   Now,
   i. $(n_1+n_2)m = n_1m + n_2m$
   ii. $n(m_1+ m_2) = \underbrace{(m_1 + m_2) + (m_1 + m_2) + \cdots + (m_1 + m_2)}_{n-times}$
   $= nm_2 + nm_2$
   iii. $(n_1 n_2)m = n_1(n_2m)$
   also, since ℤ has identity element, then
   iv. 1. m = m

3. Every ideal in a ring R is an R- module
4. Every vector space V over a field F is F-module.
5. If Q is the set of rational numbers, then Q is $\mathbb{Z}$-module.
   Proof. Define β: $\mathbb{Z}$ x Q → Q by:
   $\beta(m, \frac{n}{t}) = m\frac{n}{t} = \frac{mn}{t}$ for all $m \in \mathbb{Z}$ and $\frac{n}{t} \in Q$.
6. If $\mathbb{Z}_n$ is the group of integers modulo n, then $\mathbb{Z}_n$ is $\mathbb{Z}$-module.
   Proof. define $\alpha: \mathbb{Z} \times \mathbb{Z}_n \to \mathbb{Z}_n$ by: $\alpha(n, \bar{a}) = n\bar{a}$ for all $n \in \mathbb{Z}$, $\bar{a} \in \mathbb{Z}_n$.
7. Let A be an abelian group and
   S= $end_R(A) = Hom_R(A, A)$= {f: A→ A; f is a group homomorphism}
   Define " + " on S by: for all f, g ∈ S and a ∈ A,
   (f+g)(a) = f(a) + g(a)
   Then
   1. (S, +) is an abelian group:
      i. S is closed under "+"
      ii. 0(a) = 0 (zero function 0 : A→ A)
      iii. (-f(a)) = -(f(a)) (additive inverse)

      (f+ (-f)(a) = f(a) + -(f(a)) = 0

      iv. "+" is an associative operation

      iv. "+" is an abelian:

      (f+g)(a) = f(a) + g(a) = g(a) + f(a) = (g+f)(a)

      (S, +) is an abelian group

   2. Define " . " on S by: for all f, g ∈ S and a ∈ A,
      f.g ≡ fog and (fog)(a) = (f(g(a))
   (S, +, .) is a ring with identity I: A→ A (where foI = Iof = f)
   3. Now, one can consider A as a unitary S-module:
   with α : S x A → A, α(f, a) = f(a) f ∈ S and a ∈ A

8. If R is a ring, every abelian group can be consider as an R-module with trivial module structure by defining ra =0 for all r ∈ R and a ∈ A.
9. ***The R-module $M_{n.}(R)$***. let

$$M_{n.}(R) = \text{the set of nxn matrices over } R$$

$$=\{(a_{ij})_{nxn} \mid a \in R\}$$

$M_n(R)$ is an additive abelian group under matrix addition. If $(a_{ij}) \in M_{n.}(R)$ and a∈ R, then the operation $a.(a_{ij}) = (a.a_{ij})$ makes $M_{n.}(R)$ into an R-module. $M_{n.}(R)$is also a left R-module under the operation $a.(a_{ij})= (a.a_{ij})$.

10. **The Module R[X]**. If R[X] is the set of all polynomials in X with their coefficients in R,

i.e $R[X] = \{(a_0 ,a_1 ,\dots ,a_n)| a_i\in R, i = 1,2,\dots,n,\}$

then (R[X], +) is an additive abelian group under polynomial addition on R[X] is an R-module via the function R x R[X]→ R[X] defined by : $a.(a_0 + x.a_1 +\dots + x^n.a_n) = (a.a_0) +(a.a_1).x + \dots + (a.a_n).x^n$

Definition. Let R be a ring, A an R-module and B a nonempty subset of A. B is a ***submodule*** of A provided that B is an additive subgroup of A and rb ∈ B for all r ∈ R and b ∈ B.

Remark. Let R be a ring, A an R-module and B a nonempty subset of A. B is a submodule iff:

1. for all a, b ∈ B, a+b ∈ B
2. for all r ∈ R and a ∈ B, ra ∈B.

Another characterization for a submodule concept

Remark. A nonempty subset B of an R-module A a submodule iff: ax + by ∈ B, for all a, b ∈ R and x, y ∈ B.

Examples.

1. let M an R-module and x ∈M, the set

$R_x = \{rx | r \in R\}$ is a submodule of M such that

a. $r_1x - r_2x = r_1x + (-r_2)x \in R_x$.

b. $r_1(r_2x) = (r_1r_2)x$

2. let R be a commutative ring with identity and S be a set. Consider the set

$$X = R^s = \{f : S \to R; f \text{ is a function}\}.$$

The two operation "+" and "." on X denoted by

$(f+g)(s) = f(s) + g(s)$ and $(f.g)(s) = f(s) . g(s)$ for $s \in S$ and $f, g \in X$

Then (X, +) is an abelian group (H.W).

The function $\alpha : R \times X \to X$ denoted by $\alpha(r, f) = rf$ since $(rf)(s) = r(f(s))$ for all $s \in S$, $r \in R$ and $f \in X$, then X is an R-module(H. W)

And $Y = \{f : \in X : f(s) = 0$ for all but at most a finite number of $s \in S\}$, the Y is a submodule of an R-module X. (H.W)

3. ***Finite Sums of Submodules.*** If $M_1, M_2, \ldots, M_n$ are submodules of an R-module M, then $M_1 + M_2 + \ldots + M_n = \{x_1 + x_2 + \ldots + x_n | x_i \in M_i$ for $i=1,2,\ldots,n\}$ is a submodule of M for each integer $n \geq 1$.
4. If one take n=2 in (3) then

$$N+K = \{x+y | x \in N, y \in K\}$$

is a submodule of M for each submodule N and K of M

Proof. let $w_1, w_2 \in N+K$. Then

i. $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$ for $x_1, x_2 \in N$ and $y_1, y_2 \in K$. Now, $w_1 + w_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \in N+K$.

ii. let $w = x + y \in N+K$, $r \in R$. so, $rw = r(x+y) = rx + ry \in N+K$.

5. let N_α; $\alpha \in I$(I is the index set), be a family of submodules of an R-module M, then $\bigcap_{\alpha \in I} N_\alpha$ is also a submodule of M.

Proof. H.W.