Chapter four (Noetherian and Artinian modules)

Ascending and Descending chain condition

Definition. An R-module M is said to be satisfy the ascending chain condition (resp. descending chain condition) if for every ascending (resp. descending) chain of submodules

$$M_1 \leq M_2 \leq M_3 \leq \ldots \leq M_n \leq \ldots$$

$$(\text{resp.} \quad M_1 \geq M_2 \geq M_3 \geq \ldots \geq M_n \geq \ldots)$$

there exists $m \in \mathbb{Z}_+$ such that $M_n = M_m$ whenever $n \geq m$.

Definition. A module which satisfies the ascending chain condition is said to be ***Noetherian***.

Definition. A module which satisfies the descending chain condition is said to be ***Artinian***.

Remark. A ring R is said to be *Noetherian* (*Artinian)* if it is *Noetherian* (*Artinian*) as an R-module. i.e., if it satisfies a.c.c. (d.c.c.) on ideals.

Example. Every simple module is both Noetherian and Artinian.

Theorem 1. Let M be an R-module. Then the following statements are equivalent:

1. M satisfies the ascending (descending) chain condition.
2. For any nonempty family $\{M_\alpha\}_{\alpha \in I}$ of submodules of M, there exist a maximal (minimal) element M_0 satisfies the maximal condition (resp. minimal condition)
   (i.e $\exists M_0 \in \{M_\alpha\}_{\alpha \in I}$ such that whenever $M_0 \leq M_\beta$, then $M_0 = M_\beta$)
   (resp. i.e $\exists M_0 \in \{M_\alpha\}_{\alpha \in I}$ such that whenever $M_\beta \leq M_0$, then $M_0 = M_\beta$)

Proof. (1→2) consider the set

$$\mathcal{F} = \{M_i \mid M_i \leq M\}$$

$\mathcal{F} \neq \varphi$

Suppose $\mathcal{F}$ has no maximal element.

Let $M_1 \in \mathcal{F}$ implies M_1 is not maximal element.

$\exists$ $M_2 \in \mathcal{F}$ such that $M_1 \leq M_2$. Since M_2 is not max. element, then there is $M_3 \in \mathcal{F}$ such that $M_2 \leq M_3$.

Continuing in this way, we get

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

A chain of submodules of M. if this sequence is an infinite, then it does not satisfy the ACC. C!

$$\therefore\ \mathcal{F} \text{ has maximal element}$$

(2 → 1) suppose M satisfies the maximal condition for submodules, and let

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

be ascending chain of submodules of M.

Let $\mathcal{H} = \{M_\alpha\}_{\alpha \in I}$ be a family of the submodules of M. Then $\mathcal{H} \neq \varphi$ and has maximal element M_m. implies whenever $n \geq m$, $M_m = M_n$.

$$\therefore\ \mathcal{H} \text{ satisfies the ascending chain condition.}$$

Theorem 2. Let M be an R-module. Then the following statements are equivalent:

1. M is Noetherian.
2. Every submodule of M is finitely generated.

Proof. (1 → 2) suppose M is Noetherian module and K be submodule of M. Let $\mathcal{F} = \{A \mid A \text{ is finitely generated submodule of } K\}$

$\mathcal{F} \neq \varphi$ (the zero submodule of A is in $\mathcal{F}$)

Since M is Noetherian module , so $\mathcal{F}$ has maximal element say K_0.

Hence K_0 is finitely generated submodule of K

$$\text{i.e } K_0 = Rk_1 + Rk_2 + \ldots + Rk_n$$

Suppose $K_0 \neq K \rightarrow \exists\, a \in K$ and $a \notin K_0$ and so

$$K_0 + Ra = K_0 = Rk_1 + Rk_2 + \ldots + Rk_n + Ra$$

$\because K_0 + Ra$ is a finitely generated submodule of K, then $K_0 + Ra \in \mathcal{F}$ is a contradiction with the maximalist of K_0. Hence $K_0 = K$

$$\therefore \text{K is a finitely generated}$$

($2 \rightarrow 1$) suppose that every submodule of M is finitely generated.

Let $K_1 \leq K_2 \leq K_3 \leq \ldots$ be an ascending chain of submodules of M.

Put $K = \bigcup_{i=1}^{\infty} K_i$ $\rightarrow$ K is submodule of M.

$\rightarrow$ K is a finitely generated submodule of M

$\rightarrow K = Rk_1 + Rk_2 + \ldots + Rk_n$

$\rightarrow$ each K_j is in K_i's

$\rightarrow \exists$ m such that $k_1, k_2, \ldots, k_r \in K_m \qquad \forall\, n \geq m$

$\therefore$ M is Noetherian module.

Examples.

1. The $\mathbb{Z}$- module $\mathbb{Z}$ is Noetherian module (every submodule of the $\mathbb{Z}$- module $\mathbb{Z}$ (= $n\mathbb{Z}$ cyclic) is finitely generated) which is not Artinian ($2\mathbb{Z} > 4\mathbb{Z} > 8\mathbb{Z} > \ldots > 2^n\mathbb{Z} > \ldots$. is a chain of ideals of Z that does not terminate)
2. The ring of integers $\mathbb{Z}$ is Noetherian (every principal ideal ring is Noetherian).

3. Q is not Noetherian module (since the $\mathbb{Z}$- module Q is not finitely generated).
4. A division ring D is Artinian and Noetherian since the only right or left ideals of D are 0 and D.
5. Every finite module is an Artinian module.

Remark. Every nonzero Artinian module contains a simple submodule.

Proof. let $0 \neq M$ be an Artinian module.

If M is a simple module, then we are done.

If not, $\exists$ $0 \neq M_1$ submodule of M. If M_1 is a simple, then we are done.

If not, $\exists$ $0 \neq M_2$ submodule of M_1. If M_2 is a simple, then we are done.

If not, $\exists$ $0 \neq M_3$ submodule of M_2. If M_3 is a simple, then we are done.

So there is a descending chain

$$M \geq M_1 \geq M_2 \geq M_3 \geq \dots$$

of submodules of M. Since M is an Artinian module, then the family $\{M_i\}_{i \in I}$ of the chain has minimal element and this element is the simple submodule.

Proposition. Let $0 \to N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \to 0$be a short exact sequence of R-modules and module homomorphism. Then M is Noetherian (resp. Artinian) iff both N (Artinian) and $\frac{M}{N}$ are Noetherian (Artinian) (resp. Artinian).

Proof.$\rightarrow$)Suppose that M is a Noetherian module and N submodule of M . So every submodule of N is a submodule of M. so N is Noetherian. Let

$$\frac{M_1}{N} \leq \frac{M_2}{N} \leq \frac{M_3}{N} \leq \dots$$

be an ascending chain of submodules of $\frac{M}{N}$, where

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

is an ascending chain of submodules of M which contain N. But M Noetherian, ∃m such that $M_n = M_m$ for all $n \geq m$.

$$\therefore \frac{M}{N} \text{ is Noetherian module.}$$

←) Suppose that N and $\frac{M}{N}$ are Noetherian modules. Let

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

be an ascending chain of submodules of M. Then

$$M_1 \cap N \leq M_2 \cap N \leq M_3 \cap N \leq \dots$$

is an ascending chain of submodules of N, so there is an integer $m_1 \geq 1$ such that $M_n \cap N = M_{m_1} \cap N$ for all $n \geq m_1$. Also,

$$\frac{M_1+N}{N} \leq \frac{M_2+N}{N} \leq \frac{M_3+N}{N} \leq \dots$$

is an ascending chain of submodules of $\frac{M}{N}$ and there is an integer $m_2 \geq 1$ such that $\frac{M_n+N}{N} = \frac{M_{m_2}+N}{N}$ for all $n \geq m_2$. Let $m = \max.\{m_1, m_2\}$. Then for all $n \geq m$,

$$M_n \cap N = M_m \cap N \text{ and } \frac{M_n+N}{N} = \frac{M_m+N}{N}$$

If $n \geq m$ and $x \in M_n$, then $x + N \in \frac{M_n+N}{N} = \frac{M_m+N}{N}$, so there is a $y \in M_m$ such that $x + N = y + N$ implies that $x - y \in N$ and since $M_m \leq M_n$ we have $x - y \in M_n \cap N = M_m \cap N$ when $n \geq m$ If $x - y = z \in M_m \cap N$, then $x = y + z \in M_m$, so $M_n \leq M_m$. Hence, $M_n = M_m$ whenever $n \geq m$, so M is Noetherian.

Remark. In general, if the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact, then B is Noetherian (Artinian) if and only if each of A and C is Noetherian (Artinian).

Example. Let M_1 and M_2 be R-modules. Then $M_1 \oplus M_2$ is Noetherian (Artinian) iff each of M_1 and M_2 is Noetherian (Artinian). (i.e every finite direct sum of Noetherian (Artinian)is Noetherian (Artinian)

(The proof is done using the short exact sequence

$$0 \to M_1 \overset{J_1}{\to} M_1 \oplus M_2 \overset{\rho_2}{\to} M_2 \to 0)$$

Theorem. Let α : $M \to \acute{M}$ be an epimorphism. If M is Noetherian (Artinian), then so is $\acute{M}$.

Proof. Since kerα is a submodule of M, then the sequence

$$0 \to ker\alpha \overset{i}{\to} M \overset{\pi}{\to} \frac{M}{ker\alpha} \to 0$$

is a short exact sequence. By hypothesis, M is Noetherian, implies that $\frac{M}{ker\alpha}$is Noetherian. But $\frac{M}{ker\alpha} \approx \acute{M}$(first isomorphism theorem) and $\frac{M}{ker\alpha}$ is Noetherian, so $\acute{M}$ is a Noetherian.

Theorem. The following are equivalent for a ring R.

1. R is right Noetherian.
2. Every finitely generated R-module is Noetherian.

Proof.(1→ 2) let M be a finite generated over a Noetherian ring R.

∃ $x_1, x_2, \ldots, x_n \in$ M such that M = $Rx_1 + Rx_2 + \ldots + Rx_n$. since R is Noetherian, then so is the finite direct sum of copies of R. Define

α : $R^{(n)} \to$ M by : $\alpha(r_1, r_2, \ldots, r_n) = r_nx_1 + r_nx_2 + \ldots + r_nx_n$.

It's clear that α is a well-define, homomorphism and onto. So, Imα = M is Noetherian.

(2 → 1)Since R = <1>, so R is finitely generated and hence R is Noetherian.

References

1. P.E. Bland, "Rings and Their modules", New York, 2011.
2. T.W. Hungerford, "Algebra", New York, 2000.
3. D.M. Burton, "Abstract and linear algebra", London, 1972.
4. M.F. Atiyah, "Introduction to Commutative Algebra", University of Oxford, 1969.