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System of Numbers

1. **Natural Numbers**
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References

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The Natural Numbers الأعداد الطبيعية

Definition:

Let $1=\varphi$, where φ is empty set.

$2=\{1\}$

$3=\{1, 2\}$,

$4=\{1, 2, 3\}$,

$5=\{1, 2, 3, 4\}$

Definitions.

1. Successor of element: $n^+= n+1$
2. Successor of set**:** The successor of S is denoted by S^+ and defined by $S^+=S\cup\{S\}$
3. Successor set: Let S be any set. We say that S is successor set if
    1. $1\in S$
    2. If $x\in S$, then $x^+\in S$. (i.e $S = \{1, 2, \ldots\}$)

Remarks.

1. $S\subset S^+$
2. $S\in S^+$

Examples.

a. If $S=\{a, b\}$, then $S^+=S\cup\{S\}=\{a,b,\{a,b\}\}$

b. $1=\varphi$

c. $2=1^+=\{1\}=\varphi\cup\{1\}$

d. $3=2^+=\{1,2\}=\{1\}\cup\{2\}=2\cup\{2\}$

e. $4=3^+=\{1,2,3\}=\{1,2\}\cup\{3\}=3\cup\{3\}$

f. $5=4^+=\{1,2,3,4\}=\{1,2,3\}\cup\{4\}=4\cup\{4\}$

Remark. If $n=\{1,2,\ldots,n-1\}$, then $n^+=\{1,2,\ldots,n\}$,

السؤال الذي يطرح هنا: هل يوجد شي ممكن تسميته مجموعة الأعداد الطبيعية؟ ان الطريقة التي ذكرناها سابقا لا نستطيع بواسطتها تكوين مجموعة كل الأعداد الطبيعية حيث يمكن فقط تكوين الأعداد $1,2,\ldots,n$. لذا لا نستطيع أن نتكلم عن مجموعة كل الأعداد الطبيعية

Remark.

Any successor set contain the numbers $1, 2, \ldots, n$

Proof. Since if S is successor set, then $\varphi \in S \Rightarrow 1 \in S \Rightarrow 1^{+} \in S \Rightarrow 2 \in S \Rightarrow 1, 2, \ldots, n \in S$

Axiom of Infinity (بديهية المالانهاية)

There exists successor set (توجد مجموعة تابعية)

Theorem.

1. The family of successor sets is nonempty
2. The intersection of any nonempty family of successor sets is also successor set.

Proof:

1. Direct from Axiom of Infinity.
2. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be nonempty family of successor sets

$\Rightarrow \varphi \in A_\lambda$ for all $\lambda \in \Lambda \Rightarrow \varphi \in \bigcap_{\lambda\in\Lambda} A_\lambda$

Let $x \in \bigcap_{\alpha\in\Lambda} A_\alpha \Rightarrow x \in A_\alpha$ for all $\lambda \in \Lambda$

$\Rightarrow x^{+} \in A_\lambda$ for all $\lambda \in \Lambda \Rightarrow x^{+} \in \bigcap_{\lambda\in\Lambda} A_\lambda$ so that $\bigcap_{\lambda\in\Lambda} A_\lambda$ is successor set.

Definition.

The intersection of all successor sets is called the set of natural numbers and denoted by $\mathbb{N}$. Each element of $\mathbb{N}$ is called the natural number. The set of natural numbers is smallest successor set.

Lemma (*): The set S is said to be transitive set is it satisfy $x \in n^{+} \rightarrow x \subseteq n^{+}$

Examples.

1. The set $A = \{a,b\}$ is not transitive , since $a \in A$, but $a \not\subset A$
2. The natural number 3 is transitive set , since $3 = \{1, 2\}$, and

$1 \in 3 \Rightarrow 1 = \varphi \subseteq 3$

$2 \in 3 \Rightarrow 2 = \{1\} \subseteq 3$

$\{\varphi, \{\varphi\}\} \in 3 \Rightarrow \{\varphi, \{\varphi\}\} \subseteq 3$, since $\varphi \in 3$ and $\{\varphi\} \in 3$

Remark. Every natural number satisfy the property $x \in n^+ \to x \subseteq n^+$

Peano's Axioms

The Peano's axioms for natural numbers are:

1. $1 \in \mathbb{N}$
2. If $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$
3. If $n \in \mathbb{N}$, then $n^+ \neq 1$
4. If X is a successor subset of $\mathbb{N}$, then $X = \mathbb{N}$
5. If $n, m \in \mathbb{N}$ such that $n^+ = m^+$, then $n = m$.

Proof 1. Since $\mathbb{N}$ is successor set , then $1 \in \mathbb{N}$

Proof 2. Let $n \in \mathbb{N}$, since $\mathbb{N}$ is successor set , then $n^+ \in \mathbb{N}$

Proof 3. Let $n \in \mathbb{N}$, $n^+ = n \cup \{n\} \Rightarrow n \in n^+ \Rightarrow n^+ \neq \varphi \Rightarrow n^+ \neq 1$

Proof 4. $\because \mathbb{N}$ is intersection of successor set and X is successor set,

$\therefore \mathbb{N} \subseteq X$, but $X \subseteq \mathbb{N}$, then $X = \mathbb{N}$.

Proof 5. Let $n, m \in \mathbb{N}$ such that $n^+ = m^+$

Since $n \in n^+$ and $n^+ = m^+$, then $n \in m^+$, but $m^+ = m \cup \{m\}$, then either $n \in m$ or $n = m$

If n=m, we are done. Or $n \in m$, by lemma (*), we have $n \subseteq m$

by the same argument we have $m \subseteq n$, $\to n = m$.

Remark. Axiom (4) is called The Principle of Mathematical Induction**.**

Arithmetic of the Natural Numbers حساب الأعداد الطبيعية

Addition on $\mathbb{N}$ الجمع على الأعداد الطبيعية

Theorem.

Let $m \in \mathbb{N}$, , $\exists!$ $+: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$+(m, n^+) = \left(+(m,n)\right)^+, \forall n \in \mathbb{N}.$$

Definition.

We write $+(m,n) = m + n$ name "addition"

Theorem.

1. $m + n^+ = \left((m+n)\right)^+, \forall n, m \in \mathbb{N}.$

Example:

$1 + 2 = 1 + 1^+ = (1+1)^+ = 2^+ = 3$

<u>Cancellation law for addition</u>

Let $n, m, k \in \mathbb{N}$ and $m + k = n + k$, then $n = m$.

Properties of addition on $\mathbb{N}$

Theorem: for all $n, m, k \in \mathbb{N}$**:**

1. $n^+ = 1 + n$
2. $(m + n) + k = m + (n + k)$ (Associative property)
3. $m + n = n + m.$ (Commutative property)

Proof.

1. Let $X = \{n \in \mathbb{N} : n^+ = 1 + n\} \Rightarrow X \subseteq \mathbb{N}$

Since $1^+ = 2 = 1 + 1$, then $1 \in X$

Let $n \in X$. To prove $n^+ \in X$

Since $n \in X \Rightarrow n^+ = 1 + n$

$\Rightarrow (n^+)^+ = (1+n)^+ = 1 + n^+ \Rightarrow n^+ \in X$. By the axiom of induction $X = \mathbb{N}$.

2. Let $X_{mn} = \{k \in \mathbb{N} : (m+n) + k = m + (n+k)\} \Rightarrow X_{mn} \subseteq \mathbb{N}$

Since $(m+n)+1 = (m+n)^+$, $m + (n+1) = m + n^+ = (m+n)^+$, then $1 \in X_{mn}$

Let $k \in X_{mm}$. To prove $k^+ \in X_{mm}$

Since $k \in X_{mn} \Rightarrow (m+n)+k = m+(n+k)$

$$(m+n)+k^+ = ((m+n)+k)^+ = (m+(n+k))^+ = m+(n+k)^+ = m+(n+k^+)$$

$\Rightarrow k^+ \in X_{mn}$. By the axiom of induction $X = \mathbb{N}$.

3. let $X_m = \{n \in \mathbb{N} : m+n = n+m\} \Rightarrow X_m \subseteq \mathbb{N}$

Since $n+1 = n^+$, $1+n = n^+$, then $1 \in X_m$

Let $n \in X_m$. To prove $n^+ \in X_m$

Since $n \in X_m \Rightarrow m+n = n+m$

$m+n^+ = (m+n)^+ = 1+(n+m) = (1+n)+m = n^+ + m \Rightarrow n^+ \in X_m$. By the axiom of induction $X_m = \mathbb{N}$

Multiplication on ℕ

Definition. Let m ∈ ℕ, **. :** ℕ × ℕ → ℕ such that

.(n, m**) = n . m** for all m , n ∈ ℕ.

Then m.n is said to be multiply natural numbers.

Remark. let n, m ∈ ℕ**,** then $m.n^+ = m.n + m$

Properties of multiplication on $\mathbb{N}$

Theorem.

1. $1 \cdot n = n$, for all $n \in \mathbb{N}$
2. $m \cdot (n + k) = m \cdot n + m \cdot k$ for all $n, m, k \in \mathbb{N}$ (Left distributive over addition)
3. $(n + k) \cdot m = n \cdot m + k \cdot m$ for all $n, m, k \in \mathbb{N}$ (right distributive over addition)
4. $(m \cdot n) \cdot k = m \cdot (n \cdot k)$ for all $n, m, k \in \mathbb{N}$ (Associative Properties)
5. $m \cdot n = n \cdot m$ for all $n , m \in \mathbb{N}$ (Commutative Properties)

Proof

1. Let $X = \{n \in \mathbb{N} : 1 \cdot n = n\} \Rightarrow X \subseteq \mathbb{N}$

Since $1 \cdot 1 = 1$, then $1 \in X$

Let $n \in X$. To prove $n^+ \in X$

Since $n \in X \Rightarrow 1 \cdot n = n$

$1 \cdot n^+ = 1 \cdot n + 1 = n + 1 = 1 + n = n^+ \Rightarrow n^+ \in X$. By the axiom of induction $X = \mathbb{N}$.

2. Let $X_{mn} = \{k \in \mathbb{N} : (m \cdot n) \cdot k = m \cdot (n \cdot k)\} \Rightarrow X_{mn} \subseteq \mathbb{N}$

Since $(m . n).1 = m.n$ and $m \cdot (n \cdot 1) = m \cdot n \Rightarrow (m . n).1 = m \cdot (n \cdot 1)$ then $1 \in X_{mn}$

Let $k \in X_{mm}$. To prove $k^+ \in X_{mn}$

Since $k \in X_{mn} \Rightarrow (m \cdot n) \cdot k = m \cdot (n \cdot k)$

$$(m \cdot n) \cdot k^+ = (m \cdot n) \cdot k + m \cdot n = m \cdot (n \cdot k) + m \cdot n = m \cdot (n \cdot k + n) = m \cdot (n \cdot k^+)$$

$\Rightarrow k^+ \in X_{mn}$. By the axiom of induction $X_{mn} = \mathbb{N}$.

3. Let $X_m = \{n \in \mathbb{N} : m \cdot n = n \cdot m\} \Rightarrow X_m \subseteq \mathbb{N}$

Since $m \cdot 1 = m$, $1 \cdot m = m$, then $1 \in X_m$

Let $n \in X_m$. To prove $n^+ \in X_m$

Since $n \in X_m \Rightarrow \quad m \cdot n = n \cdot m$

$m \cdot n^+ = m \cdot n + m = n \cdot m + m \cdot 1 = (n+1) \cdot m = (1+n) \cdot m = n^+ \cdot m$

$\Rightarrow \quad n^+ \in X_m$. By the axiom of induction $X_m = \mathbb{N}$.

Definition (3.23):

Let $n, m \in \mathbb{N}$. Define m^n as follows

$$m^{n^+} = \underbrace{m^+ \times m^+ \times \ldots \times m^+}_{n-time} \text{ for all } n, m \in \mathbb{N}$$

Theorem (3.24):

1. $m^{n+k} = m^n \times m^k$ for all $n, m, k \in \mathbb{N}$
2. $(m \times n)^k = m^n \times n^k$ for all $n, m, k \in \mathbb{N}$
3. $(m^n)^k = m^{n \times k}$ for all $n, m, k \in \mathbb{N}$

Finite and Infinite Sets

Definition:

1. A set A is said to be **finite** if ∃ bijective f: A → B with B ⊆ ℕ where B = {1, 2, , …, m} and m ∈ ℕ.
2. A set A is said to be **infinite** if ∄ bijective f: A → B with B ⊆ ℕ.

Remark:

Let A be a set. The cardinal number of A is the number of elements in the set A. (#(A) or |A|)

Remarks:

1. The set of natural numbers is an infinite set.
2. B = {6, 8, 10, 12, ...} is an infinite set.
3. The empty set φ is a finite set (because there exists a bijection g: φ → {0}).
4. The set A = {a, b, c, d} is a finite set.

Definition:

1. A set S is said to be countable if there does not exist a 1-1 function f: S → ℕ.
2. A set S is said to be uncountable if there does not exist a 1-1 function f: S → ℕ.
3. #(ℕ) = $\aleph_0$ (aleph-null). If #A or |A| ≤ $\aleph_0$, then A is countable.
4. A is countable infinite if |A| = $\aleph_0$.
5. Every subset of a countable set is either finite or countable.

Theorem: For any set A, the following statements are equivalent:

1. A is Countable (A مجموعة معدودة).
2. ∃ a bijective function α: A → ℕ.
3. A is either finite or countably infinite

Theorem:

For any set A, the following statements are equivalent:

1. A is Countably Infinite (A معدودة غير منتهية).
2. There exists a bijective function f: A → ℕ.

3. The elements of A can be arranged in an infinite sequence $a_0, a_1, a_2, \ldots$, where $a_i \neq a_j$ for $i \neq j$.

عناصر المجموعة A يمكن ترتيبها ضمن تسلسل غير منتهٍ من عناصر مختلفة $(a_i \neq a_j)$

Corollary. For any set A, the following statements are equivalent:

1. A is Countably Infinite (A منتهية غير معدودة).
2. The elements of A can be arranged in an infinite sequence $a_0, a_1, a_2, \ldots$, where $a_i \neq a_j$ for $i \neq j$.

Proof. ⇒) Suppose A is countable ⇒ ∃ a bijective $f: \mathbb{N} \to A$.

⇒ $A = f(\mathbb{N}) = \{f(1), f(2), \ldots\}$

Put $f(n) = a_n$ for all n in $\mathbb{N}$. ⇒ $A = \{a_1, a_2, \ldots\}$

⇐) suppose that $A = \{a_1, a_2, \ldots\}$ such that $a_i \neq a_j$ for $i \neq j$.

Define $f: \mathbb{N} \to A$ by $f(n) = a_n$ for all n in $\mathbb{N}$. Then

1. f is one-one: let $x, y \in \mathbb{N}$ such that $f(x) = f(y) \Rightarrow a_x = a_y$. $\Rightarrow x = y$

   if $x \neq y \Rightarrow a_x \neq a_y \Rightarrow f(x) \neq f(y) \Rightarrow$ f is one to one.
2. f is onto: by definition of f, $\forall a_n \in A, \exists n \in \mathbb{N}$ such that $f(n) = a_n$

∴ f is bijective

∴ A is countable

Remarks:

1. If A and B are countable sets, then $A \times B$ is countable.

Proof. A and B are countable sets, then:

i. either A and B are finite sets, then

$A = \{a_1, a_2, \ldots, a_m\}$ and ⇒ $B = \{b_1, b_2, \ldots, b_n\}$

∴ $A \times B$ is countable.

ii. Or A and B are infinite sets, then

iii. A= {a_1, a_2, …} and ⇒ B= {b_1, b_2, …}

⇒ $A \times B = \{(a_i, b_j) | a_i \in A, bj \in B$ and $i, j \in \mathbb{N}\}$ is infinite.

⇒ A × B can be arranged by

	b_1	b_2	b_3 …..
a_1	(a_1, b_1) ↓	(a_1, b_2)	(a_1, b_3) ….
a_2	(a_2, b_1) ↓	(a_2, b_2)	(a_2, b_3) ……
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3) ….
…..	………………………………………………..		

So, the first element is (a_1, b_1)

⇒ we can list all elements of A × B in an infinite sequence.

∴A × B is countable

2. If A and B are countable sets, then A ∪ B is countable.
3. The set of all finite subsets of ℕ is countable.

Cantor's Theorem:

Let A be a set and P(A) be the power set of A. Then there does not exist a surjective function f: A → P(A).

Theorem:

The set P(ℕ) is uncountable.

Theorem The Cartesian product of $\mathbb{N} \times \mathbb{N}$ is countable.

الضرب الديكارتي للأعداد الطبيعية مجموعة معدودة.

Proof. To prove that $\mathbb{N} \times \mathbb{N}$ is countable:

The elements of $\mathbb{N} \times \mathbb{N}$ can be arranged as a matrix:

(1,1) → (1,2) → (1,3) → ...
↓ ↗ ↗ ↗
(2,1) → (2,2) → (2,3) → ...
↓ ↗ ↗ ↗
(3,1) → (3,2) → (3,3) → ...
……..

Hence:

$$\mathbb{N} \times \mathbb{N} = \{ (1,1), (2,1), (1,2), (3,1), (2,2), (1,3), \ldots \}$$

We can write it as:

$$\mathbb{N} \times \mathbb{N} = \{ a_0, a_1, a_2, a_3, a_4, a_5, \ldots \}$$

Where:

$a_1 = (1,1)$
$a_2 = (2,1)$
$a_3 = (1,2)$
$a_4 = (3,1)$
$a_5 = (2,2)$
....

The elements of $\mathbb{N} \times \mathbb{N}$ can be numbered,
which means that we can arrange the natural numbers in a sequence that matches all pairs in $\mathbb{N} \times \mathbb{N}$ without repetition.

$\therefore \mathbb{N} \times \mathbb{N}$ is countable.