1. g is well-define:
   suppose $m_1 + k = m_2 + K$ iff $m_1 - m_2 \in K \leq N$ iff $m_1 + N = m_2 + N$

   $\therefore$ g is well defined
2. g is a homomorphism (prove)
3. g is an epimorphism (prove)
4. kerg = {m+K| g(m+ k) = N}
   ={m+K| m+ N = N}
   = {m+K| m∈ N}
   = $\frac{N}{K}$ (where $K \leq N$ and $m \in N$)

   $\therefore$ kerg = $\frac{N}{K}$

Then by the first isomorphism theorem, $\frac{\frac{M}{K}}{\frac{N}{K}} \approx \frac{M}{N}$.

Exercise. Let M be a cyclic R-module, say M=Rx. Prove that M≈R/ ann(x), where ann(x) = {r ∈ R| rx = 0}.

[Hint: Define the mapping f: R→M by f(r) = rx]

Chapter three (Sequence)

Short exact sequence

Definition. A sequence $M_1 \overset{f}{\to} M \overset{g}{\to} M_2$ of R-modules and R-module homomorphismsis said to be ***exact** at* M Im f = ker g while a sequence of the form

$$\partial: \quad \dots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \cdots$$

n∈ ℤ, is said to be an *exact sequence* if it is exact at M_n for each n ∈ ℤ. A sequence such as

$$0 \to M_1 \overset{f}{\to} M \overset{g}{\to} M_2 \to 0$$

that is exact at M_1, at M and at M_2 is called a ***short exact sequence***.

Remarks.

1. If an exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is short exact then
   i. f is a monomorphism
   ii. g is an epimorphism
2. A sequence $0 \to M_1 \xrightarrow{f} M$ is exact iff f is monomorphism
3. A sequence $M \xrightarrow{g} M_2 \to 0$ is exact iff g is epimorphism
4. If the composition(between two homomorphisms f and g) gof = 0, then Imf ≤ kerg.

Examples.

1. If N is a submodule of M, then $0 \to N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \to 0$ is a short exact sequence, where i is the canonical injection and π is the natural epimorphism. for example : since kerf is a submodule of M, then $0 \to kerf \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{kerf} \to 0$ is a short exact sequence.
2. Consider the sequence

   $$\mu: \quad 0 \to M_1 \xrightarrow{J_1} M_1 \oplus M_2 \xrightarrow{\rho_2} M_2 \to 0$$

   ImJ_1 = $M_1 \oplus \{0\}$; $J_1(x) = (x, 0)$
   kerρ_2 = $M_1 \oplus \{0\}$; $\rho_2(x, y) = (0, y)$
   for any $x \in M_1$, $y \in M_2$ and $(x,y) \in M_1 \oplus M_2$
   J_1 is a monomorphism and ρ_2 is an epimorphism

   $\therefore \mu$ is short exact sequence
3. The sequence $0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0$ of $\mathbb{Z}$-modules is a short exact sequence

Remark. ***Commutative Diagrams***

The following diagram

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ g_1\downarrow & & \downarrow g_2 \\ C & \xrightarrow{f_2} & D \end{array}$$

is said to be ***commutative*** if $g_2 o f_1 = f_2 o g_1$.Similarly, for a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & h\searrow & \swarrow g \\ & C & \end{array}$$

is commutative if gof = h and we say that g ***completes the diagram commutatively.***

Theorem. **(*The short five lemma*).** Let R be a ring and

$$\begin{array}{ccccccccc} 0 & \rightarrow & A & \xrightarrow{f_1} & B & \xrightarrow{g_1} & C & \rightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \rightarrow & A' & \xrightarrow{f_2} & B' & \xrightarrow{g_2} & C' & \rightarrow & 0 \end{array}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

1. If α and γ are monomorphisms, then β is a monomorphism.
2. If α and γ are epimorphisms, then β is an epimorphism.
3. if α and γ are isomorphisms, then β is an isomorphism.

Proof 1.

To show that β is a monomorphism, must prove ker β = 0.

Let b ∈ ker β → β(b) = 0 → g_2(β(b)) = g_2(0) = 0. Since the diagram is commutative, then:

γog_1(b) = γ(g_1(b)) = 0 → g_1(b) ∈ kerγ = {0}(γ is a monomorphism) →g_1(b) = 0 → b ∈ kerg_1= Imf_1= f_1(A). There is a ∈ A such that

$$f_1(a) = b \rightarrow \beta(f_1(a)) = \beta(b).$$

Since

βof_1 = f_2oα → f_2oα(a) = β(b) → f_2(α(a)) = 0 → α(a) ∈ kerf_1= {0}(f_2 is a monomorphism), so

$$\alpha(a) = 0 \rightarrow a \in \ker\alpha = \{0\} \text{ (}\alpha\text{ is a monomorphism)} \rightarrow a = 0.$$

But f_1(a)=b and a=0 → b=f_1(a)= f_1(0)= 0 → b=0.

$$\ker\beta = \{0\} \rightarrow \beta \text{ is a monomorphism}$$

Proof 2.

Let $\acute{b} \in \acute{B}$ → $g_2(\acute{b}) \in \acute{C}$ → $g_2(\acute{b}) = \acute{c}$. Since γ is an epimorphism, there is c ∈ C such that

$$\gamma(c) = \acute{c} \rightarrow g_2(\acute{b}) = \gamma(c).$$

But g_1 is an epimorphism, then there is b ∈ B such that

$$g_1(b) = c \rightarrow g_2(\acute{b}) = \gamma(c) = \gamma(g_1(b)) = \gamma o g_1(b) = g_2 o\beta(b)$$

so

$$g_2(\acute{b}) = g_2(\beta(b)) \rightarrow g_2(\beta(b) - \acute{b}) = 0 \text{ (}g_2\text{ is homomorphism).}$$

and

$$\beta(b) - \acute{b} \in \ker g_2 = \mathrm{Im} f_2 \rightarrow \beta(b) - \acute{b} \in \mathrm{Im} f_2.$$

There is $\acute{a} \in \acute{A}$ such that $f_2(\acute{a})$ = β(b) - $\acute{b}$. But α is an epimorphism, there is a ∈ A such that α(a) = $\acute{a}$. Since β$o f_1$ = f_2oα (the diagram is commutative).

Then

$$\beta\,(f_1(a)) = f_2(\alpha(a)) = f_2(\acute{a}) = \beta(b) - \acute{b}$$

so

$$\acute{b} = \beta(b) - \beta\,(f_1(a)) = \beta(b - f_1(a)) \ (\beta \text{ is homomorphism})$$

i.e there is b – f_1(a) ∈ B such that β(b – f_1(a)) = $\acute{b}$

Hence β is an epimorphism.

Proof 3. is an immediate consequence of (1) and (2).

<u>Exercise.</u> Consider the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \to & 0 \\ h \downarrow & & & & & & \\ D & & & & & & \end{array}$$

where the row is exact and hof = 0. Prove that, there exact a unique homomorphism k: C → D such that kog = h.

<u>Definition</u>. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. This sequence is said to be ***splits*** if Imf is a direct summand of B.
(i.e there is D ≤ B such that B = Imf ⊕ D).

Example. The sequence $0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0$ of ℤ-modules and ℤ-homomorphism is a short exact sequence which is not split (where Imi = 2 ℤ is not direct summand of ℤ).

Theorem. Let R be a ring and

$$\mathcal{F}: \quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent

1. $\mathcal{F}$ splits.
2. f has a left inverse (i.e ∃ h: B → A homomorphism with hof = I_A).
3. g has a right inverse(i.e ∃ k:C→B a homomorphism with gok = I_C).

Proof. (1 → 2) since $\mathcal{F}$ splits, then Imf is a direct summand of B.

(i.e. ∃ $B_1 \leq B$ such that B = Imf ⊕ B_1).

Define h: B → A by $h(x) = h(a_1+b_1) = a$ for $x = a_1+b_1 \in$ Imf ⊕ B_1.

where $a_1 \in$ Imf (i.e ∃ a ∈ A such that $f(a) = a_1$) and $b_1 \in B_1$.

a. Since f is one-to-one, then h is well-define.
b. h is a homomorphism
c. let w ∈ A, hof(w) = h(f(w)) = h(f(w)+ 0) = w (by definition of h)

∴ h is a left inverse of f.

(2 → 3) suppose f has a left inverse say h(i.e. hof = I_A).

Define k: C → B by: k(y) = b - foh(b) where g(b) = y with $b \in B_1$.

a. k is well define:
   let $y, y_1 \in C$ such that $y = y_1$ with $g(b) = y$ and $g(b_1) = y_1$ for b, $b_1 \in B_1$.
   Now,
   $$g(b) = g(b_1) \rightarrow b_1 - b \in \ker g = \text{Imf}$$
   so, b_1-b ∈ Imf → ∃ a ∈ A such that $f(a) = b_1 - b$.
   Then $h(f(a)) = h(b_1- b)$. But hof = I_A,
   so $a = hof(a) = h(f(a)) = h(b_1- b) = h(b_1) - h(b)$

∴ $a = h(b_1) - h(b) \rightarrow f(a) = f(h(b_1)) - f(h(b)) = b_1 - b$

∴ $b - f(h(b)) = b_1- f(h(b_1)) \rightarrow k(y) = k(y_1) \rightarrow$ k is well define.

b. k is homomorphism (why?)

c. gok = I_C. for that

let y ∈ C, gok (y) = g(k(y)) = g(b-foh(b)) where g(b) = y.

→ gok(y) = g(b) + gofoh(b) . But Im f = kerg. So, gof = 0.

→gok(y) = g(b) + 0 = y

∴ gok = I_C

(3 → 1) suppose that g has a right inverse say k: C → B such that gok = I_C

Let B_1 = {b ∈B| kog(b) = b}

a. $B_1 \neq \varphi$ (0 ∈ B_1 where g(0) = k(g(0)) = k(0) = 0)

b. B_1 is a submodule of B. (prove?)

c. B = Imf ⊕ B_1, for that:

i. Let w = Imf ∩ B_1 → w = f(a) ∈ B_1 for some a ∈ A with kog(w)=w → k(g(f(a))) = k(0) =0. But k(g(f(a))) = k(g(w)) = w.

Thus w = 0 and so Imf ∩ B_1 = 0.

ii. Let b ∈ B → b = b - kog(b) + kog(b).

Since kog(kog(b)) = kog(b), then kog(b)∈ B_1 and g(b-kog(b)) = g(b) - gokog(b) = g(b) – Iog(b) = g(b) - g(b) = 0 (where gok = I_C).

→ b-kog(b) ∈ kerg = Imf

∴ b = b-kog(b) + kog(b) ∈ Imf + B_1

∴ B = Imf ⊕ B_1→ Imf is a direct summand of B which implies $\mathcal{F}$ splits.

Exercise If the short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

splits, then B ≈ Imf ⊕ Img