Arithmetic of the Rational Numbers

لتعريف عمليتي الجمع والضرب في الأعداد النسبية، نحتاج إلى:

Lemma

Let $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$. Then:

1. $(ad + cb, bd) \sim (a'd' + c'b', b'd')$
2. $(ac, bd) \sim (a'c', b'd')$

Proof

Since $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then:

$ab' = a'b$ and $cd' = c'd$

Then: $(ad + cb)(b'd') = abd'd' + cbb'd' = a'dbd' + c'bbd$

$= (a'd' + c'b')bd$

$\Rightarrow (ad + cb, bd) \sim (a'd' + c'b', b'd')$

Also: $(ac)(b'd') = (ab')(cd') = (a'b)(c'd) = (a'c')(bd)$

So $(ac, bd) \sim (a'c', b'd')$

Theorem

$\exists$ f, g: Q $\to$ Q such that for all x, y $\in$Q, if $(a, b) \in x$ and $(c, d) \in y$:

1. $f(x, y) = (ad + cb, bd)$
2. $g(x, y) = (ac, bd)$

Definition (addition and multiplication in Q)

Let x, y $\in$ Q, x = (a,b), y = (c,d), then:

1. The function f defined on Q by $f(x,y) = [ad + cb, bd]$ is said to be "addition in Q".

   Notation: $f(x,y) = x + y$

2. The function g is said to be "multiplication in Q", defined by $g(x,y) = [ac, bd]$

   Notation: $g(x,y) = x . y$

Example

Let $x = [1,2]$, $y = [3,5] \in Q$

Then:

$x + y = [1\cdot5 + 3\cdot2, 2\cdot5] = [11, 10]$

$x * y = [1\cdot3, 2\cdot5] = [3, 10]$

Definition (Subtraction in Q)

Let x, y ∈ Q, x = (a,b), y = (c,d), then:

x - y = x + (-y) = [a,b] + [-c,d] = [ad - cb, bd]

Remarks

1. 0 = [0,1]
2. 1 = [1,1]

Example

Let x = [4,9], y = [6,13]

x - y = [4·13 - 6·9, 9·13] = [52 - 54, 117] = [-2, 117]

Definition (Division and Inverse)

Let x ∈ Q, y ∈ Q - {0}, then:

1. $\frac{x}{y} = x \cdot y^{-1}$
2. $x^{-1} = \frac{1}{x}$

Example

1. Let x = [3,4], y = [1,1]
   - $\frac{1}{x}$= [1,1] · x^{-1} = [1,1] · [4,3] = [4,3]
   - x · x^{-1} = [3,4] · [4,3] = [12,12] = [1,1]
2. x = [2,5]   y = [-3,11] then
   $\frac{x}{y}$ = y. x^{-1} = [-3, 11] . [5, 2]
   = [-15, 22]

Remark: The set V = {x ∈ ℚ | x ≤ 1} has no least element.

Theorem: (ℚ, ≤) is not well-ordered.

Proof:

(ℚ, ≤) is poset but ∃ the set V = {x ∈ ℚ | x ≤ 1} has no least element.

∴ (ℚ, ≤) is not well-ordered.

"**Properties** of ℚ"

1. ℚ is Countable

Proof.

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1/1 → 2/1 → 3/1 → 4/1 → ...
↓ ↗ ↗ ↗
1/2 3/2 5/2 7/2 →...
↓ ↗ ↗ ↗
1/3 2/3 4/3 5/3 → ...
↓ ↗ ↗ ↗
1/4 3/4 5/4 7/4 →...

The set of all positive rational numbers is:
{1, 1/2, 1/3, 2/3, 3, 1/4, 2/3, 5/2, 4, 1/3, ...}

∴ each rational number in {0, 1, -1, 1/2, -1/2, 2, -2, 1/3, -1/3, ...}

∴ ∃ bijective f: ℚ → ℕ such that:

0 ⟷ 0
1 ⟷ 1
-1 ⟷ 2
1/2 ⟷ 3
-1/2 ⟷ 4
...

∴ ℚ is countable set.

2. Dense ordered **(الترتيب الكثيف)**

Definition: Let "≤" be a partially ordered relation on the set A. Then "≤" is said to be dense if and only if $(a, b \in A) \wedge (a < b) \rightarrow (\exists\ c \in A \text{ such that } a < c < b)$

Remark.

The relation "≤" on ℚ is dense. That is, $\forall\ p, q \in \mathbb{Q}, \exists\ r \in \mathbb{Q}$ such that $p < r < q$

Theorem: $\nexists\, x \in \mathbb{Q}$ such that $x^2 = 2$

Proof:
Suppose that $x \in \mathbb{Q}$ such that $x^2 = 2$

$\Rightarrow x = a/b$; $a, b \in \mathbb{Z}$ and $b \neq 0$
$\Rightarrow x^2 = 2 \Rightarrow a^2 / b^2 = 2 \Rightarrow a^2 = 2b^2$

$\Rightarrow a^2$ is even $\Rightarrow$ a is even $\Rightarrow a = 2k$; $k \in \mathbb{Z}^+$

$\Rightarrow b^2 = 2k^2 \Rightarrow b^2$ even $\Rightarrow$ b is even

$\Rightarrow b = 2t$; $t \in \mathbb{Z}$

$\Rightarrow x = a/b = 2k / 2t = k / t$

$\Rightarrow \exists\, k < a$ such that $x = k/t \in \mathbb{Q}$!

$\Rightarrow$ contradiction $\Rightarrow \nexists\, x \in \mathbb{Q}$ such that $x^2 = 2$