Definition. Let $x \in \mathbb{Q}$, then

1. x is said to be negative rational number if $-x \in \mathbb{Q}$, (i. e $x > 0$)(i.e. if $x = (a, b) \in \mathbb{Q}$ then $-x = (-a, b) \in \mathbb{Q}$)
2. if $x \neq 0$ and $x = (a, b)$, then $x^{-1} = (b, a)$

Example.

1. $x = (-3, 5)$ is negative rational number because $-(-3, 5) \in \mathbb{Q}^{+}$.
2. $x^{-1} = (-5, 3) = (5, -3)$

Density of Rational Numbers

Remark:

The relation "<" on $\mathbb{Q}$ is dense. That is:

$$\forall\, p, q \in \mathbb{Q}, \exists\, r \in \mathbb{Q} \text{ such that } p < r < q$$

H.W.:

Give an example about density relation on another set.

Important Axioms

1. **Zoran's lemma**:

let S be a partially ordered set. If every totally ordered subset of S has upper bound, then S contains a maximal element.

2. **Axiom of choice**

Let $\{y_\alpha\}_{\alpha\in I}$ be a nonempty family of sets then $\exists\ f: I \to \ \bigcup_{\alpha\in I} y_\alpha$ such that f(x) $\in y_\alpha$.

3. **Division Algorithm**

Theorem.
Let a, b ∈ $\mathbb{Z}$, b > 0, then ∃! q, r ∈ $\mathbb{Z}$ such that a= bq + r for $0 \le r < b$

Corollary.
Let a, b ∈ $\mathbb{Z}$, b ≠ 0, then ∃! q, r ∈ $\mathbb{Z}$ such that a= bq + r for $0 \le r < |b|$

Remark.

q = quotient, r = reminder

i.e

```
        b
     ______
  q |  a
 ___|
       aq
     ______
        r
```

Example.

1. If b = -2 and a= -5, then -5 = (-2) .3 + 1
2. b = 3 and a = 20, then 20 = 3. 6 +2

H.W. Compute the quotients and the reminder r when

1. b = -7 and a = -2
2. b = -7 and a = 61 ,
3. b = 6 and a = 49

Binary Operations (العمليات الثنائية)

Definition:

Let A be a nonempty set. A function $*: A \times A \rightarrow A$ is said to be a **binary operation** on A if:

$$\forall\, a, b \in A \Rightarrow a * b \in A$$

$$\text{i.e } *: (a, b) \rightarrow a*b$$

Definition

Let A be a nonempty set, and $B \subseteq A$. If * is a binary operation on A, then B is said to be a **closed subset** with respect to * if and only if

$$\forall\, a, b \in B \Rightarrow a * b \in B$$

Remark:

If * is a binary operation on A, then A is closed under *.

Examples:

1. Let $A = \mathbb{N}$ and * = "+"

   $$(a, b) \in \mathbb{N} \times \mathbb{N} \Rightarrow a + b \in \mathbb{N}$$

   So "+" is a binary operation on $\mathbb{N}$, and $\mathbb{N}$ is closed under "+"
2. The function "+" is also a binary operation on, $\mathbb{Q}$, and $\mathbb{Z}$

3. Let $A = \{-2, -1, 0, 1, 2\}$, and * = "+"

   Then "+" is **not** a binary operation on A.

   Because: If $a = 2$ and $b = 1 \Rightarrow a + b = 3 \notin A$, so A is not closed under "+"

Homework

1. Is the operation "−" a binary operation on $\mathbb{N}$?
2. Let * be defined by the following table on the set $A = \{1, 2, 3\}$:

*	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Is "*" a binary operation on A?

Field (الحقل)

Definition. Let A be the nonempty set and '+' , '·' be two binary operations on A. Then the mathematical system $(A, +, \cdot)$ is said to be a field if: $\forall$ a, b, c $\in$ A;

1. $a + b = b + a$
2. $\exists$ 0 the identity element such that $\forall a \in A$, $a + 0 = 0 + a = a$
3. $\forall a \in A$, $\exists (-a) \in A$ such that $a + (-a) = (-a) + a = 0$
4. $(a + b) + c = a + (b + c)$
5. $a \cdot b = b \cdot a$
6. $\forall a (\neq 0) \in A$, $\exists a^{-1} \in A$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$
7. $\exists$ 1 the identity element such that $\forall a \in A$, $a \cdot 1 = 1 \cdot a = a$
8. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. $a \cdot (b + c) = a \cdot b + a \cdot c$

   $(b + c) \cdot a = b \cdot a + c \cdot a$

Theorem

The mathematical system $(\mathbb{Q}, +, .)$ is a field (and it's said to be the field of rational numbers).

Theorem

If the algebraic system $(A, +, ., <)$ is an ordered field, then the relation " $<$ " is dense.

Proof. Let $a, b \in A$ such that $a < b$, then

$\rightarrow a + b < b + b$

$\rightarrow a + a < b + a = a + b$

$\rightarrow a + a < a + b < b + b$

$\rightarrow 2a < a + b < 2b$

$2 = 1 + 1 > 0 \rightarrow \frac{1}{2} > 0$

$\rightarrow a < \frac{a+b}{2} < b$

$\rightarrow$ $\forall$ a, b $\in$ A such that a < b, $\exists \frac{a+b}{2} \in$ A such that $a < \frac{a+b}{2} < b$

∴ " < " is dense.

Remark. An order "<" in $\mathbb{Z}$ is not dense.

Proof.

Let x, y ∈ $\mathbb{Z}$ such that x < y and y = x+1. We want to show ∄ k ∈ $\mathbb{Z}$ such that x < k < y

If x < y ⇒ x < x + 1, → ∄ k ∈ $\mathbb{Z}$ such that x < k < x + 1 ⇒ No such k exists

Suppose ∃ p ∈ $\mathbb{Z}$ such that k = x + p. Then x < x + p < x + 1 ⇒ p < 1

But p ∈ $\mathbb{Z}$ ⇒ p ≤ 0 c! x < x + p < x + 1, since x + p ≤ x.

Therefore, ∄ k ∈ $\mathbb{Z}$ such that x < k < y

∴The order is not dense in $\mathbb{Z}$.

Archimedean field

Definition. An ordered field (A, +, ·, <) is called Archimedean if for all x, y ∈ A with x > 0, ∃ n ∈ $\mathbb{N}$ such that nx ≥ y

Theorem. ($\mathbb{Q}$, +, ·, <) is Archimedean.

Proof. Let x, y ∈ $\mathbb{Q}$ such that 0 < x < y, then ∃ a, b, c, d ∈ $\mathbb{Z}$ with b ≠ 0, d ≠ 0, such that x = a/b , y = c/d ⇒ $0 < \frac{a}{b} < \frac{c}{d}$

Multiply both sides by bd: ⇒ 0 < ad < cb ⇒ $\frac{ad}{bd} < \frac{cb}{db}$

Put:

p = ad, q = bc, r = bd

Then:

$x = \frac{a}{b} = \frac{ad}{bd} = \frac{p}{r}$,

$y = \frac{c}{d} = \frac{cb}{db} = \frac{q}{r}$

$\Rightarrow 0 < \frac{p}{r} < \frac{q}{r}$

Now multiply both sides by r: ⇒ $kx = k\cdot\frac{p}{r}$

⇒ ∃ k ∈ $\mathbb{N}$ such that $kx = k\cdot\frac{p}{r} \geq y = \frac{q}{r}$

⇒ kx ≥ y

⇒ ($\mathbb{Q}$, +, ·, <) is Archimedean.