Definition. An ideal I of the ring R is said to be ***semiprime*** if and only if $I = \sqrt{I}$.

Remark. the definition of semiprime ideal means that :

An ideal I is semiprime if and only if $a^n \in I$, for some $n \in \mathbb{Z}_+$, then $a \in I$.

Theorem. An ideal of a ring R is semiprime if and only if $\frac{R}{I}$ has nonzero nilpotent elements.

Proof. $\Longrightarrow$) Suppose that I is a semiprime ideal and a+I is a nilpotent element in $\frac{R}{I}$.

$\rightarrow \exists\, n \in \mathbb{Z}_+$ such that $(a + I)^n = I$

$\rightarrow a^n + I = I \rightarrow a^n \in I \rightarrow a \in \sqrt{I}$

But I is semiprime ideal $\rightarrow \sqrt{I} = I \rightarrow a \in I \rightarrow a + I$ is zero element in $\frac{R}{I}$

$\therefore \frac{R}{I}$ has no nonzero nilpotent elements.

$\Longleftarrow$)Suppose that $\frac{R}{I}$ has no nonzero nilpotent elements and we want to prove $I = \sqrt{I}$. It's enough to prove $\sqrt{I} \subseteq I$ (it's always $I \subseteq \sqrt{I}$). Let $r \in \sqrt{I} \rightarrow \exists\, n \in \mathbb{Z}_+$ such that $r^n \in I \rightarrow r^n + I = (r + I)^n = I \rightarrow r + I$ is nilpotent element in $\frac{R}{I}$. But $\frac{R}{I}$ has no non zero nilpotent elements.

$\rightarrow r + I = I \rightarrow r \in I \rightarrow I = \sqrt{I} \rightarrow$ I is semiprime ideal.

Theorem. Every prime ideal of a ring R is semiprime.

Proof. Let I be a prime ideal in a ring R. It's enough to prove $\sqrt{I} \subseteq I$ (it's always $I \subseteq \sqrt{I}$) . Let $x \in \sqrt{I} \rightarrow \exists\, n \in \mathbb{Z}_+$ such that $x^n \in I$. Suppose that n is the smallest positive integers such that $x^n \in I$. Since $x^n = x \,.\, x^{n-1} \in I$ and I prime ideal of R → either $x \in I$ or $x^{n-1} \in I$. But $x^{n-1} \notin I \rightarrow x \in I \rightarrow \sqrt{I} \subseteq I \rightarrow$ I is semiprime ideal ($I = \sqrt{I}$).

Definition. The prime radical ideal of a ring R, (denoted by Rad(R)) is the set:

$$Rad_P(R) = \{P \mid P \text{ is a prime ideal of } R\}$$

Remark. If $Rad_P(R) = \{0\}$, we say that the ring R is without prime radical, or has zero prime radical.

Theorem. Let I be an ideal of R. Then the nil radical of I is:

$$\sqrt{I} = \cap\{P \mid I \subseteq P, P \text{ is prime ideal}\}$$

Examples.

1. Nil $\mathbb{Z} = Rad_P(\mathbb{Z}) = \cap\{P \mid I \subseteq P, I \subseteq P \text{ is prime ideal}\} = (0)$
2. $Rad_P(\mathbb{Z}_6) = (0)$
3. Let $R = \mathbb{Z}$ and $n \in \mathbb{Z}$ $n > 1$, then the ideal $(n) \subseteq (P) \leftrightarrow$ p divides n → the nil radical:
   $\sqrt{(n)} = \cap_{p \backslash n}(P)$ if $n = P_1^{K_1}.P_2^{K_2} \ldots P_m^{K_m}$ where $k_1, k_2, \ldots, k_m \in \mathbb{Z}_+$ and $p_1, p_2, \ldots, p_m$ are distance prime integers.
   $\rightarrow \sqrt{(n)} = (P_1) \cap (P_2) \cap \ldots (P_m)$
   $= (P_1.P_2 \ldots P_m)$

H.W. Find $Rad_P(\mathbb{Z}_{18})$?

Definition. Let I be a proper ideal in a commutative ring with identity R, then I is said to be ***primary ideal*** if whenever **a.b ∈ I, and a ∉I, then b^n ∈ I** for some n ∈ $\mathbb{Z}_+$.

Theorem. The ideal (n) is primary in $\mathbb{Z}$ if and only if $n = p^k$ for some k ∈ $\mathbb{Z}_+$ and prime integers p.

Remark*. Every Prime ideal is primary.

Proof. obvious (put n = 1).

Remark. the converse of remark* is not true in general. For example

Example.

a. The ideal (4) is primary not prime ideal in the ring $\mathbb{Z}$.

Proof. **1. (4) is primary ideal:**

Let a, b ∈ R, such that a.b ∈ (4) → a.b = 4r for r ∈ $\mathbb{Z}$.

→ $2^2 = 4 \backslash a.b \rightarrow 2 \backslash a.b$

either 2\ a or 2 \ b

if 2\a → a = 2s → $a^2 = 4s^2$ ∈ (4) for s ∈ $\mathbb{Z}$.

In the same way if 2\b.

∴ (4) is primary ideal.

2. The ideal (4) is not prime:

Since 2.2 = 4 ∈ (4) but 2 ∉ (4).

∴ (4) is not prime ideal.

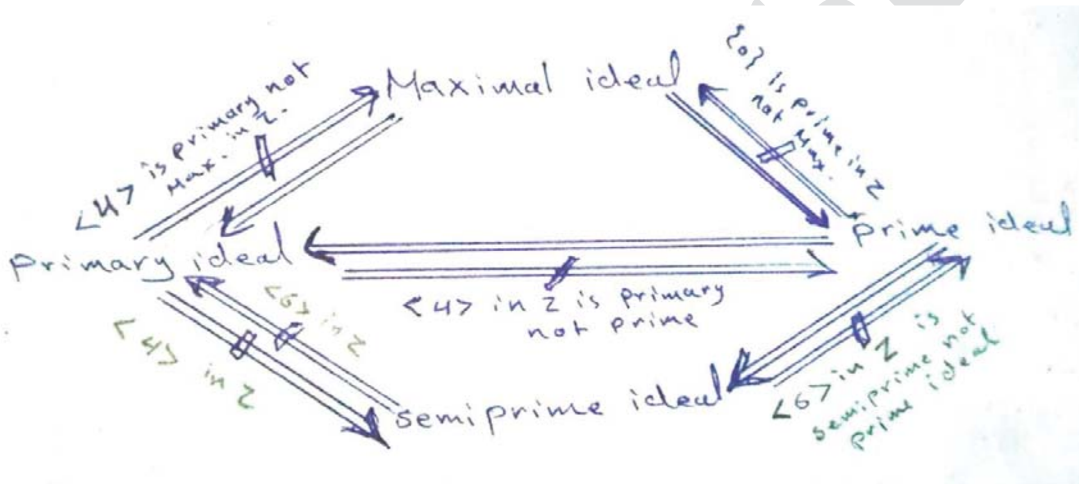
b. The ideal (6) is not primary ideal in $\mathbb{Z}$.

Proof. Since $2.3 = 6 \in (6)$ but neither $2 \in (6)$ nor $3^n \in (6)$ for $n \in \mathbb{Z}_+$

(i.e: $2 \notin (6)$ and $\nexists\ n \in \mathbb{Z}_+$ such that $3^n \in (6)$).

$\therefore$ (6) is not primary ideal.

Remark. The following diagram gives the relation among the maxima, prime, semiprime and primary ideal



Theorem. In a commutative ring R, on the ideal I is primary if and only if every zero divisor element of $\frac{R}{I}$ is nilpotent element.

Proof. ⇒) Let a+I (≠ I) be a zero divisor of $\frac{R}{I}$

→ ∃ b+I ≠ I such that (a + I)(b + I) = I → a.b + I = I

→ a.b ∈ I but I is a primary ideal and b ∉ I → $a^n \in I$ for some $n \in Z_+$

→ $a^n + I = I$ → $a^n + I = (a + I)^n = I$ for some $n \in Z_+$.

∴ a + I is nilpotent element.

⇐) Let a.b ∈ I and a ∉ I for a, b ∈ R → a.b + I = I

→ (a + I)(b + I) = I → a+I ≠ I.

If b + I = I → b ∈ I → **I is primary ideal**.

If b+I ≠ I → b + I is zero divisor → b+ I is nilpotent element.

→$(b + I)^n = I$ for some $n \in Z_+$→ $b^n + I= I$→ $b^n \in I$→ **I is primary ideal**

Examples.

a. The ideal (4) in ℤ is primary which is not semiprime.

Proof.

1. The ideal **(4) in ℤ is primary:** since $\frac{\mathbb{Z}}{(4)} \cong \mathbb{Z}_4$ has only zero divisor $\bar{2}$ and $\bar{2}$ is a nilpotent element in $\mathbb{Z}_4$(by the previous therorm).

2. The ideal **(4) in ℤ is not semiprime**: since $\frac{\mathbb{Z}}{(4)} \cong \mathbb{Z}_4$ has a nilpotent element $\bar{2} \neq 0$(by the previous theorem).

b. The ideal **(6) is semiprime which is not primary** ideal in $\mathbb{Z}$:

1. The ideal **(6) is not primary** Since the zero divisor of $\frac{\mathbb{Z}}{(6)} \cong \mathbb{Z}_6$ are $\{\bar{2}, \bar{3}, \bar{4}\}$which is not nilpotent elements

→ (6) is not primary ideal.

2. The ideal **(6) is semiprime:** since $\frac{\mathbb{Z}}{(6)} \cong \mathbb{Z}_6$ has no non zero nilpotent elements.

Theorem. Let f: R → R' be an epimorphism, then:

1. If M is a maximal ideal of R such that kerf ⊆ M, then f(M) is maximal ideal of R'.
2. If M' is a maximal ideal of R', then $f^{-1}(M)$ is maximal ideal of R.
3. There is an isomorphism between the maximal ideals of R' and the maximal ideals of R which is contain kerf.