Rational Numbers

Construction of the Rational Numbers

The operations on the set of integers are not sufficient to construct the system of decimal numbers. Therefore, the set of integers is extended to form the set of rational numbers.

Notation

Let:
$A = \{ (a, b) \mid a, b \in \mathbb{Z}, b \neq 0 \}$

Thus:
$A = \mathbb{Z} \times \mathbb{Z}^*$, where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$

Theorem

There exists an equivalence relation R on the set A, defined as:
$(a, b) R (c, d) \Leftrightarrow ad = cb$ for all $(a,b), (c,d) \in A$
(i.e. $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$)

Proof

1. Reflexivity:
For any $(a, b) \in A$, we have $ab = ab$, so $(a, b) R (a, b)$.

2. Symmetry:
If $(a, b) R (c, d)$, then $ad = bc \Rightarrow cb = da$, hence $(c, d) R (a, b)$.

3. Transitivity:
Let $(a, b), (c, d), (e, f) \in A$ such that $(a, b) R (c, d)$ and $(c, d) R (e, f)$.
Then:
$ad = bc$ and $cf = ed \Rightarrow adf = bcf = bed \Rightarrow af = be$ (since $d \neq 0$)
$\Rightarrow (a, b) R (e, f)$

Therefore, R is an equivalence relation on A.

Remark

We will write (a, b) ~ (c, d) instead of (a, b) R (c, d), and read this as: "(a, b) is equivalent to (c, d)".

Example

Let:
(5, 6), (20, 9), (3, 4), (15, 20) ∈ A

(3, 4) ~ (15, 20) because (3)(20) = (4)(15) ⇒ 60 = 60 ⇒ (3, 4) ~ (15, 20)

(5, 6) ≁ (20, 9) because (5)(9) = 45, (6)(20) = 120 ⇒ 45 ≠ 120 ⇒ (5, 6) ≁ (20, 9)

Definition

The equivalence class containing the pair (a, b) is said to be a rational number, denoted by: [a, b]

i.e.,
[a, b] = { (c, d) | (c, d) ~ (a, b) }

The Set of Rational Numbers

The set of all equivalence classes is called the set of rational numbers, denoted by:
$\mathbb{Q} = A / R = \mathbb{Z} \times \mathbb{Z}^* / R = \{ [(a, b)] \mid (a, b) \in A \}$

Remark

If [(a, b)] = [(c, d)] ⇒ (a, b) ~ (c, d) ⇒ ad = bc ⇒ a/b = c/d
Thus, each rational number [(a, b)] can be represented by the fraction a/b.

Arithmetic Operations on Rational Numbers

Let [(a, b)], [(c, d)] ∈ $\mathbb{Q}$

1. Addition:
[(a, b)] + [(c, d)] = [(ad + bc, bd)]

2. Multiplication:
[(a, b)] . [(c, d)] = [(ac, bd)]

3. Additive Inverse:
-[(a, b)] = [(-a, b)]

4. Multiplicative Inverse:
If a ≠ 0, then $[(a, b)]^{-1} = [(b, a)]$

The set ℚ, together with the operations defined above, forms a field.

Example

[0,1]={(c,d)|(c,d)~(0,1)}

={(c,d)|c·1=d·0}

={(c,d)|c=0}

⇒[0,1]={(0,1),(0,2),(0,3),…}

={(0,−1),(0,−2),(0,−3),…}

Ordered on Q^+

Let Q^+={x ∈ Q| ab > 0, (a, b) ∈ x}

Theorem

Let x ∈ Q and (a,b)~(c,d). Then if ab > 0, then cd > 0.

Proof

If (a,b)~(c,d)(a, b) , then ad=cb

(cb)(ad) = (cb)(cb)

⇒(ab)(cd) = (cb)(cb) >0

But ab > 0, then cd > 0

Definition

Let x, y ∈ ℚ, then x is less than y (x < y) if and only if y - x ∈ $ℚ^+$.

Theorem

The system $(\mathbb{Q}, \leq)$ is a totally ordered set.

Proof

1. Reflexive:

For all $x \in \mathbb{Q}$, we have $x \leq x$.
Thus, "$\leq$" is reflexive.

2. Anti-symmetric:

If $x \leq y$ and $y \leq x$, then:
$x \leq y$ implies $x < y$ or $x = y$.
$y \leq x$ implies $y < x$ or $y = x$.

Now, if $x < y$ and $y < x$:
then $(y - x) \in \mathbb{Q}^+$ and $(x - y) \in \mathbb{Q}^+$, a contradiction.
Thus, $x = y$.
Therefore, "$\leq$" is anti-symmetric.

3. Transitive:

Let $x \leq y$ and $y \leq k$, then:
(i) If $x < y$ and $y < k$, then $(y - x) \in \mathbb{Q}^+$ and $(k - y) \in \mathbb{Q}^+$,
thus $(k - x) = (k - y) + (y - x) \in \mathbb{Q}^+$, implying $x < k \Rightarrow x \leq k$.
(ii) If $x < y$ and $y = k$, then $x < k \Rightarrow x \leq k$.
(iii) If $x = y$ and $y < k$, then $x < k \Rightarrow x \leq k$.
(iv) If $x = y$ and $y = k$, then $x = k \Rightarrow x \leq k$.
Thus, "$\leq$" is transitive.

<u>Conclusion</u>:
Let $x, y \in \mathbb{Q}$. Then, **exactly one of** the following holds:
1. $x \leq y$
2. $y \leq x$

Thus, **every pair is comparable**.
Therefore, "$\leq$" is a total order on $\mathbb{Q}$.

Arithmetic of the Rational Numbers

لتعريف عمليتي الجمع والضرب في الأعداد النسبية، نحتاج إلى:

Lemma

Let (a,b) ~ (a',b') and (c,d) ~ (c',d'). Then:

1. (ad + cb, bd) ~ (a'd' + c'b', b'd')

2. (ac, bd) ~ (a'c', b'd')

Proof

Since (a, b) ~ (a', b') and (c, d) ~ (c', d'), then:

ab' = a'b and cd' = c'd

Then: (ad + cb) (b'd') = abd'd' + cbb'd' = a'dbd' + c'bbd

= (a'd' + c'b')bd

=> (ad + cb, bd) ~ (a'd' + c'b', b'd')

Also: (ac)(b'd') = (ab')(cd') = (a'b)(c'd) = (a'c')(bd)

So (ac, bd) ~ (a'c', b'd')

Theorem

∃ f, g: Q → Q such that for all x, y ∈Q, if (a, b) ∈ x and (c, d) ∈ y:

1. f(x, y) = (ad + cb, bd)

2. g(x, y) = (ac, bd)

Definition (addition and multiplication in Q)

Let x, y ∈ Q, x = (a,b), y = (c,d), then:

1. The function f defined on Q by f(x,y) = [ad + cb, bd] is said to be "addition in Q".

Notation: f(x,y) = x + y

2. The function g is said to be "multiplication in Q", defined by g(x,y) = [ac, bd]

Notation: g(x,y) = x . y

Example

Let x = [1,2], y = [3,5] ∈ Q

Then:

x + y = [1·5 + 3·2, 2·5] = [11, 10]

x * y = [1·3, 2·5] = [3, 10]

Definition (Subtraction in Q)

Let x, y ∈ Q, x = (a,b), y = (c,d), then:

x - y = x + (-y) = [a,b] + [-c,d] = [ad - cb, bd]

Remarks

1. 0 = [0,1]
2. 1 = [1,1]

Example

Let x = [4,9], y = [6,13]

x - y = [4·13 - 6·9, 9·13] = [52 - 54, 117] = [-2, 117]

Definition (Division and Inverse)

Let x ∈ Q, y ∈ Q - {0}, then:

1. $\frac{x}{y} = x \cdot y^{-1}$
2. $x^{-1} = \frac{1}{x}$

Example

1. Let x = [3,4], y = [1,1]
   - $\frac{1}{x}$= [1,1] · x^{-1} = [1,1] · [4,3] = [4,3]
   - x · x^{-1} = [3,4] · [4,3] = [12,12] = [1,1]
2. x = [2,5]   y = [-3,11] then
   $\frac{x}{y}$ = y. x^{-1} = [-3, 11] . [5, 2]
   = [-15, 22]

Remark: The set V = {x ∈ ℚ | x ≤ 1} has no least element.

Theorem: (ℚ, ≤) is not well-ordered.

Proof:

(ℚ, ≤) is poset but ∃ the set V = {x ∈ ℚ | x ≤ 1} has no least element.

∴ (ℚ, ≤) is not well-ordered.

"**Properties** of ℚ"

1. ℚ is Countable

Proof.

رسم جدول للأعداد النسبية الموجبة في عدد من المتتابعات بدون تكرار

1/1 → 2/1 → 3/1 → 4/1 → ...
↓ ↗ ↗ ↗
1/2 3/2 5/2 7/2 →…
↓ ↗ ↗ ↗
1/3 2/3 4/3 5/3 → …
↓ ↗ ↗ ↗
1/4 3/4 5/4 7/4 →…

The set of all positive rational numbers is:
{1, 1/2, 1/3, 2/3, 3, 1/4, 2/3, 5/2, 4, 1/3, ...}

∴ each rational number in {0, 1, -1, 1/2, -1/2, 2, -2, 1/3, -1/3, ...}

∴ ∃ bijective f: ℚ → ℕ such that:

0 ⟷ 0
1 ⟷ 1
-1 ⟷ 2
1/2 ⟷ 3
-1/2 ⟷ 4
...

∴ ℚ is countable set.

2. Dense ordered **(الترتيب الكثيف)**

Definition: Let "≤" be a partially ordered relation on the set A. Then "≤" is said to be dense if and only if (a, b ∈ A) ∧ (a < b) → (∃ c ∈ A such that a < c < b)

Remark.
The relation "≤" on ℚ is dense. That is, ∀ p, q ∈ ℚ, ∃ r ∈ ℚ such that p < r < q

Theorem: $\nexists\, x \in \mathbb{Q}$ such that $x^2 = 2$

Proof:
Suppose that $x \in \mathbb{Q}$ such that $x^2 = 2$

$\Rightarrow x = a/b$; $a, b \in \mathbb{Z}$ and $b \neq 0$
$\Rightarrow x^2 = 2 \Rightarrow a^2 / b^2 = 2 \Rightarrow a^2 = 2b^2$

$\Rightarrow a^2$ is even $\Rightarrow$ a is even $\Rightarrow a = 2k$; $k \in \mathbb{Z}^+$

$\Rightarrow b^2 = 2k^2 \Rightarrow b^2$ even $\Rightarrow$ b is even

$\Rightarrow b = 2t$; $t \in \mathbb{Z}$

$\Rightarrow x = a/b = 2k / 2t = k / t$

$\Rightarrow \exists\, k < a$ such that $x = k/t \in \mathbb{Q}$!

$\Rightarrow$ contradiction $\Rightarrow \nexists\, x \in \mathbb{Q}$ such that $x^2 = 2$