6. let N be a submodule of an R-module M and $\frac{M}{N} = \{m+N \mid m \in M\}$.
   clearly that ($\frac{M}{N}$,+) is an abelian group where for each m, m_1, $m_2 \in$ M, r ∈ R:

i. $(m_1+N) + (m_2+N) = (m_1+ m_2) +N$

ii. and $r.(m_2+N) = (r.\ m_2)+ N$.

   then $\frac{M}{N}$ is an R-module, which is called the ***quotient module*** of M by N.

Remark. ***(Modular Law).***

There is one property of modules that is often useful. It is known as the modular law or as the modularity property of modules. If N , L and K are modules, then $N \cap (L+K) = (N \cap L)+(N \cap K)$.

If N , L and K are submodules of an R-module M and $L \leq N$, then $N \cap (L+K) = L + (N \cap K)$.

Definition. Let M be an R-module. If there exists $x_1, x_2, \ldots, x_n \in M$ such that $M= Rx_1+ Rx_2+ \ldots +Rx_n$. M is said to be ***finitely generated*** module. If $M = Rx = <x> = \{rx \mid r \in R\}$ is said to be ***cyclic*** module.

Examples.

1. $\mathbb{Z}_n = <\bar{1}>$ is cyclic $\mathbb{Z}$-module for all $n \in \mathbb{Z}$.
2. $n\mathbb{Z} = < n >$ is cyclic $\mathbb{Z}$-module for all $n \in \mathbb{Z}$.
3. If F is any field, then the ring F[x,y] has the submodule(ideal) $<x,y>$ which is not cyclic.
4. Q is not finitely generated $\mathbb{Z}$-module.

Direct sums and products

Definition. Let R be a ring and $\{M_i \mid i \in I\}$ be an arbitrary (possibly infinite) of a nonempty family of R-modules. $\prod_{i \in I} M_i$ is the ***direct product*** of the abelian groups M_i, and $\bigoplus_{i \in I} M_i$ the ***direct sum*** of the of the abelian groups M_i, where

$$\prod_{i\in I} M_i = \{f: I \to \bigcup_{i\in I} M_i \mid f(i) \in M_i, \text{ for all } i\in I\}$$

Define a binary operation "+" on the direct product (of modules) $\prod_{i\in I} M_i$ as follows: for each f,g $\in \prod_{i\in I} M_i$ (that is, f,g : I $\to \bigcup_{i\in I} M_i$ and f(i),g(i) $\in M_i$ for each i), then f+g : I $\to \bigcup_{i\in I} M_i$ is the function given by i $\to$ f(i)+g(i).

i.e $\quad (f+g)(i) = f(i)+g(i) \quad$ for each i $\in$I.

Since each M_i is a module, f(i)+g(i) $\in M_i$ for every i, whence f+g $\in \prod_{i\in I} M_i$. So ($\prod_{i\in I} M_i$, +) is an abelian group

Now, if r $\in$ R and f $\in \prod_{i\in I} M_i$, then rf : I $\to \bigcup_{i\in I} M_i$ as (rf)(i) = r(f(i)).

1. $\prod_{i\in I} M_i$ is an ***R-module*** with the action of R given by r(f(i)) = (rf(i)) (i.e define α: R x $\prod_{i\in I} M_i \to \prod_{i\in I} M_i$ by α(r,f) = rf)
2. $\bigoplus_{i\in I} M_i$ is a ***submodule*** of $\prod_{i\in I} M_i$. (H.W.)

Remark. $\prod_{i\in I} M_i$ is called the (external) direct product of the family of R-modules $\{M_i \mid i \in I\}$ and $\bigoplus_{i\in I} M_i$ is (external) direct sum. If the index set is finite, say i = { 1 ,2, … , n}, then the direct product and direct sum coincide and will be written $M_1 \oplus M_2 \oplus \ldots \oplus M_n$.

Definition. ((internal) direct sum) Let R be a ring and N, K submodules of an R-module M such that:

1. M = N + K
2. N ∩ K = 0

Then N and K is said to be ***direct summand*** of M and M = N⊕ K ***internal direct sum*** of N and K.

Definition. Let R be an integral domain. An element x of an R-module M (x∈ M) is said to be ***torsion*** element of M if ∃(0≠) r ∈ R with rx = 0.

Example.

1. Let M = $\mathbb{Z}_6$ as $\mathbb{Z}$-module. Then every element in $\mathbb{Z}_6$ is torsion:

$\bar{3} \in \mathbb{Z}_6$, $\exists\ 2 \in \mathbb{Z}$ such that $2.\ \bar{3} = \bar{0}$
$\bar{2} \in \mathbb{Z}_6$, $\exists\ 3 \in \mathbb{Z}$ such that $3.\ \bar{2} = \bar{0}$
$\bar{1} \in \mathbb{Z}_6$, $\exists\ 6 \in \mathbb{Z}$ such that $6.\ \bar{1} = \bar{0}$
$\bar{4} \in \mathbb{Z}_6$, $\exists\ 3 \in \mathbb{Z}$ such that $3.\ \bar{4} = \bar{0}$
$\bar{5} \in \mathbb{Z}_6$, $\exists\ 6 \in \mathbb{Z}$ such that $6.\ \bar{5} = \bar{0}$

2. Every element in $\mathbb{Z}_n$ as $\mathbb{Z}$-module is torsion.
3. The only torsion element in M = Q as $\mathbb{Z}$-module is zero (if $(0\neq)\ x \in$ Q, then $\nexists\ (0\neq)\ r \in \mathbb{Z}$ such that $rx = 0$.

Remark. Let M be an R-module where R is an integral domain, then the set of all torsion elements of M, denoted by $\tau(M)$ is a submodule of M

$(\tau(M) = \{x \in M \mid \exists(0\neq)\ r \in R \text{ such that } rx = 0\})$

Proof. 1. $\tau(M) \neq \varphi$ $(0 \in \tau(M))$

2. if $x, y \in \tau(M)$, then $\exists(0\neq)\ r_1, r_2 \in R$ such that $r_1x = 0$ and $r_2y = 0$. Since R is an integral domain, $r_1 \neq 0$ and $r_2 \neq 0$, so $r_1.\ r_2 \neq 0$. Hence

$r_1.r_2(x+y) = r_1.r_2\ x + r_1.r_2y = r_2.r_1\ x + r_1.r_2y = 0 + 0 = 0$. Thus $x+y \in \tau(M)$

3. let $(0\neq)\ r \in R$ $w \in \tau(M)$, $\exists(0\neq)\ r_1 \in R$ with $r_1w = 0$. Now, $r_1(rw) = 0$ implies $rw \in \tau(M)$.

$\therefore\ \tau(M)$ is a submodule of M.

Remark. In general, If R is not integral domain, then $\tau(M)$ may not submodule of M in general.

Definition. Let M be a module over integral domain R. If $\tau(M) = 0$, Then M is said to be ***torsion free*** module. If $\tau(M) = M$, then M is said to be ***torsion*** module.

Examples. 1. The $\mathbb{Z}$-module Q, is torsion free module.

2. The $\mathbb{Z}$-module $\mathbb{Z}_n$, is torsion module.

Remark. Let M be a module over an integral domain R, then $\frac{M}{\tau(M)}$ is torsion free R-module. (i.e $\tau(\frac{M}{\tau(M)}) = \tau(M)$)

Proof. Let m+ $\tau(M) \in \tau(\frac{M}{\tau(M)})$, $\exists (0 \neq)$ r ∈ R such that $r(m + \tau(M)) = \tau(M)$. $\rightarrow rm + \tau(M) = \tau(M) \rightarrow rm \in \tau(M)$

$\rightarrow \exists (0 \neq)$ s ∈ R such that s(rm) = (sr)m = 0

$\because sr \neq 0 \rightarrow m \in \tau(M) \rightarrow m + \tau(M) = \tau(M) \rightarrow \tau(\frac{M}{\tau(M)}) = \tau(M)$.

Exercises.

1. Every submodule of torsion module over integral domain is torsion module.
2. Every submodule of torsion free module over integral domain is torsion free module.

Definition. Let M be a module over an integral domain R. An element x∈M is said to be ***divisible*** element if for each $(0 \neq)$ r ∈ R ∃y ∈ M such that ry = x.

Examples.

1. 0 is divisible element in every module M.
2. Every element in a $\mathbb{Z}$-module Q is divisible element.
3. 0 is the only divisible element in $2\mathbb{Z}$ as $\mathbb{Z}$-module.

Remark. Let M be a module over an integral domain R. the set of all divisible element of M denoted by $\partial(M) = \{m \in M | \forall (0 \neq) r \in R, \exists y \in M$ such that $m = ry\}$

Definition. Let M be a module over an integral domain R. M is said to be ***divisible*** module if $\partial(M) = M$.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}$ is not divisible.
2. The module Q over the ring $\mathbb{Z}$ is divisible.
3. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is not divisible.

Proposition. Let R be an integral domain and M be an R-module. Then:

1. ∂(M) is a submodule of M.
2. If M is divisible module, then so is $\frac{M}{N}$ for all submodule N of M.
3. M is divisible module iff M = rM for all 0$\neq$ r $\in$ R.
4. If $M = M_1 \oplus M_2$, then ∂(M) = $\partial(M_1) \oplus \partial(M_2)$.

Proof. 1. Let x,y $\in \partial$(M), then

$\forall$ 0$\neq$ r $\in$ R, $\exists$ $x_1 \in$M such that x = rx_1

$\forall$ 0$\neq$ r $\in$ R, $\exists$ $y_1 \in$M such that y = ry_1

i) x + y = r(x_1+y_1) , for all 0 $\neq$ r$\in$ R. implies x+y $\in \partial$(M).

ii) let x $\in \partial$(M) and 0 $\neq$ s $\in$ R, then $\forall$ 0$\neq$ r $\in$ R, $\exists$ y$\in$M such that x = ry. Since R is an integral domain, r $\neq$ 0 and s $\neq$0, then rs $\neq$ 0.

So sx = s(ry) = (sr)y. implies that sx $\in \partial$(M).

$\therefore \partial$(M) is a submodule of M

2. Let x + N $\in \frac{M}{N}$ where x $\in$ M. Since M is divisible and x $\in$ M, then for $\forall$ 0 $\neq$ r $\in$ R, $\exists$ y $\in$ M such that x + N = ry +N = r(y+N).

$\therefore \frac{M}{N}$ is divisible module

3. $\rightarrow$)Suppose that M is divisible module. To prove M = Rm, must prove that: a. M $\leq$ rM b. rM $\leq$ M

for that :

a. Let m $\in$ M. Since M = ∂(M) (M is divisible), so m $\in \partial$(M).

For all $0 \neq r \in R$, $\exists\, n \in M$ such that $m = rn \in rM$. Hence $M \leq rM$.

b. Since M is a module then $rM \leq M$.

$$\therefore M = rM$$

$\leftarrow$) Suppose that $M = rM$ for all $0 \neq r \in R$. if $m \in M = rM$, then $m = rn$ for $n \in M$ and all $0 \neq r \in R$. implies that $m \in \partial(M)$. Thus $M \leq \partial(M)$.

let $x \in \partial(M)$, $\forall\, 0 \neq r \in R$, $\exists\, y \in M$ such that $x = ry$. Thus $\partial(M) \leq M$. Hence $M = \partial(M)$. So M is divisible module.

Remark. Point (2) in the previous proposition means: the quotient of divisible module is divisible.

Exercise. Is every submodule of divisible module divisible?

Definition. Let M be an R-module and x∈M. Then the set

$$\mathbf{ann_R(x)} = \{r \in R \mid rx = 0\}$$

is said to be **annihilator of the element x in R**.

Remarks.

1. Let M be an R-module. Then the set

   $$\mathbf{ann_R(M)} = \{r \in R \mid rM = 0\}$$
   $$= \{ r \in R \mid rm = 0 \text{ for all } m \in M\}$$

   is said to be **annihilator of the module M in R**.

2. Let M be an R-module. If $ann_R(M) = 0$, then M is said to be **faithful** module.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}$ is faithful ($ann_{\mathbb{Z}}(\mathbb{Z}) = 0$)
2. The $\mathbb{Z}$-module Q is faithful ($ann_{\mathbb{Z}}(Q) = 0$)
3. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is not faithful ($ann_{\mathbb{Z}}(\mathbb{Z}_6) = 6\,\mathbb{Z}$)

4. $ann_{\mathbb{Z}_6}(\{\bar{0}, \bar{3}\}) = \{\bar{0}, \bar{2}, \bar{4}\}$
5. $ann_{\mathbb{Z}}(\{\bar{0}, \bar{3}\}) = 2\mathbb{Z}$
6. $ann_{\mathbb{Z}}(\{\bar{0}, \bar{2}, \bar{4}\}) = 3\mathbb{Z}$
7. $ann_{\mathbb{Z}_6}(\{\bar{0}, \bar{2}, \bar{4}\}) = \{\bar{0}, \bar{3}\}$
8. $ann_{\mathbb{Z}}(\mathbb{Z}_n) = n\mathbb{Z}$

Definition. Let N and K be submodules of an R-module M. The set

$$(N: K) = \{r \in R \mid rK \leq N\}$$

is an ideal of R which is called residual.

Remark.

1. If N = 0, then
$$(0: K) = \{r \in R \mid rK = 0\} = ann_R(K)$$
2. If N = 0 and K = M, then
$$(0: M) = \{r \in R \mid rM = 0\} = ann_R(M)$$

<u>Chapter two (Module homomorphisms)</u>

<u>Definition.</u> Let M and N be modules over a ring R . A function f : M → N is an ***R-module homomorphism*** (simply homomorphism) provided that for all x, y ∈ M and r ∈ R :

1. f(x+y) = f(x) + f(y)
2. f(rx) = rf(x).

If R is a field, then an R-module homomorphism is called a ***linear transformation***.

Remarks.

1. if f is injective and homomorphism, then is said to be monomorphism.
2. if f is surjective and homomorphism, then is said to be epimorphism.