3. if f is injective, surjective and homomorphism, then is said to be isomorphism (and written $M \approx N$) .

Examples.

1. $2\,\mathbb{Z}_{\mathbb{Z}} \approx 3\,\mathbb{Z}_{\mathbb{Z}}$.

Proof. Define $g: 2\,\mathbb{Z} \to 3\mathbb{Z}$ as $g(2n) = 3n$ for all $n \in \mathbb{Z}$.

i. g is well-define.

ii. g is homomorphism : for $2n, 2n_1, 2n_2 \in 2\mathbb{Z}$, $r \in \mathbb{Z}$

$g(2n_1 + 2n_2) = g(2(n_1 + n_2)) = 3\,(n_1 + n_2) = 3\,n_1 + 3n_2 = g(2n_1) + g(2n_2)$

$g(r(2n)) = g(2rn) = 3rn = r(3n) = rg(2n)$

iii. g is one – to – one. If $g(2n_1) = g(2n_2)$, then

$\to\; 3n_1 = 3n_2 \;\to\; n_1 = n_2 \;\to\; 2\,n_1 = 2n_2$.

iv. g is onto. for all $y = 3n \in 3\,\mathbb{Z}$, there is $x = 2n \in 2\,\mathbb{Z}$ such that $g(2n) = 3n$.

Hence $2\,\mathbb{Z} \approx 3\,\mathbb{Z}$(i.e g is an isomorphism).

2. Let R be a ring and $\{ M_i \mid i \in I \}$ a family of submodules of an R-module M such that:

i. M is the sum of the family $\{ M_i \mid i \in I \}$

ii. for each $k \in I$, $M_k \cap \sum_{i\in I, i\neq k} M_i = 0$

Then $M \approx \bigoplus_{i\in I} M_i$

(Hint : define $\beta: \bigoplus_{i\in I} M_i \to M$ by $\beta(f) = \sum_{i\in I} f(i)$)

3. Let $\{ M_i \mid i \in I \}$be family of R-modules.

i. For each $k \in I$, the canonical projection $\rho_k: \prod_{i\in I} M_i \to M_k$ defined by $\rho_k(f) = f(k)$ is an R- module epimorphism .

ii. For each $k \in I$, the canonical injection $J_k: M_k \to \prod_{i\in I} M_i$

defined by for $x \in M_k$, $(J_k(x))i = \begin{cases} x & if\ i = k \\ 0 & otherwise (i \neq k) \end{cases}$

is an R-module monomorphism.

iii. $\rho_k \circ J_k = I_{M_k}$.

Proof. $\rho_k \, o J_k : M_k \to M_k$ with $(\rho_k \, o J_k)(x) = \rho_k \, (J_k(x)) = J_k(x)(k) = x$

iv. $J_k \, o \, \rho_k \neq I_{M_k}$.

4. Let K be a submodule of a module M. the function $\pi: M \to \frac{M}{K}$ defined by $\pi(x) = x+K$ for all $x \in M$, is an R-homomorphism and onto. This homomorphism is called the natural epimorphism.

Exercises. Prove :

1. If R is a ring, the map R[x] → R[x] given by f → f(x)(for example, $(x^2 + 1) \to x(x^2 + 1)$) is an R-module homomorphism, **but not** a ring homomorphism (prove that).
2. Hom(R, M) ≈ M
3. for each $n \in \mathbb{Z}$, $\frac{\mathbb{Z}}{n\mathbb{Z}} \approx \mathbb{Z}_n$.

Theorem. Let f : M → N be a homomorpism, then

1. ***kernel of f*** (kerf = {x ∈M| f(x) = 0}) is a submodule of M.
2. ***Image of f*** (Imf={n ∈N| n = f(m) for some m ∈ M}) is a submodule of N.
3. f is a monomorpism iff kerf = 0.
4. f : M→N is an R-module isomorphism if and only if there is A homomorphism g : N →M such that $gf = I_M$ and $fg = I_N$.

Proof. H.W.

Proposition. Let R be an integral domain and M be an R-module, then:

1. If f : $M \to \acute{M}$ be a module homomorphism, then $f(\tau(M)) \leq \tau(\acute{M})$.
2. If $M = M_1 \oplus M_2$, then $\tau(M) = \tau(M_1) \oplus \tau(M_2)$.

Definition. An R-module, M is called ***simple*** if M ≠ {0} and the only submodules of M are M and {0}

Proposition. Every simple module M is cyclic (i.e M = Rm for every nonzero m ∈ M).

Proof. Let M be a simple R-module and m ∈ M. Both Rm and

B = { c ∈ M| Rc = 0 } are submodules of M. Since M is simple, then each of them is either 0 or M. But RM ≠ 0 implies B ≠ M. Consequently B = 0, whence Ra = M for all nonzero m ∈ M. Therefore M is cyclic

Remark. The converse is not true in general: that is a cyclic module need not be simple for example, the cyclic Z-module Z_6.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}_3$ is simple.
2. The $\mathbb{Z}$-module $\mathbb{Z}_p$ is simple for each prime integer's p.
3. The $\mathbb{Z}$-module $\mathbb{Z}_4$ is not simple, since the submodule $\{\bar{0}, \bar{2}\} \neq 0$ and $\{\bar{0}, \bar{2}\} \neq \mathbb{Z}_4$.
4. The $\mathbb{Z}$-module $\mathbb{Z}$ is not simple.(why?)
5. Every division ring D is a simple ring and a simple D-module

Lemma. (**Schur's lemma**)

1. Every R-homomorphism from a simple R-module is either zero or monomorphism.
2. Every R-homomorphism into a simple R-module is either zero or epimorphism.
3. Every R-homomorphism from a simple R-module into simple R-module is either zero or isomorphism.

Proof 1. Let M be a simple module and f: M→ N be an R-module homomorphism. Then kerf is a submodule of M . But M is simple.

So either kerf = {0} , implies f is one-to-one

or kerf = M, implies f is zero homomorphism.

Proof 2. Let N be a simple module and f: M→ N be an R-module homomorphism. Then Imf is a submodule of N . But N is simple.

So either Imf = {0} , implies f zero homomorphism or $\quad$ Imf = N, implies f is onto.

Proof 3. as a consequence to (1) and (2), the proof of (3) holds.

Examples. 1. An R-module homomorphism f: $\mathbb{Z}_4 \rightarrow \mathbb{Z}_5$ is zero.

2. An R-module homomorphism f: $\mathbb{Z}_3 \rightarrow \mathbb{Z}_5$ is zero.

Exercise. Let $M \neq \{0\}$ be an R-module. Prove that:

If N_1, N_2 are submodules of M, with N_1 simple and $N_1 \cap N_2 \neq 0$, then $N_1 \leq N_2$

Remark. Let A, B be two simple R-module, then Hom(A, B) is either zero or for all f ∈ Hom(A, B) is an isomorphism, where Hom(A, B) = {f:A→B| f is homomorphism}

Isomorphism theorems

First isomorphism theorem. Suppose f: M→ N is an R-module homomorphism. Then $\frac{M}{kerf} \approx f(M)$.

Proof. Define h : $\frac{M}{kerf} \rightarrow f(M)$ by: h(m + kerf) = f(m) for all m ∈ M.

1. h is well define: Let m_1+ kerf , m_2 + kerf $\in \frac{M}{kerf}$ such that

$$m_1 + kerf = m_2 + kerf \text{ implies } m_1 - m_2 \in kerf$$

and so

$$f(m_1 - m_2) = f(m_1) - f(m_2) = 0 \rightarrow f(m_1) = f(m_2)$$

Hence

$$h(m_1 + kerf) = h(m_2 + kerf)$$

$$\therefore \text{h is well define}$$

2. h is a homomorphism since f is homomorphism.

3. h is a monomorphism: for that suppose that

$$h(m_1+ kerf) = h(m_2+ kerf).$$

from definition of h, $f(m_1) = f(m_2)$ implies $f(m_1) - f(m_2) = f(m_1- m_2)=0$

so $m_1- m_2 \in kerf \rightarrow m_1+ kerf = m_2+ kerf$

4. h is an epimporphism: let y∈ f(m) ∈ f(M), ∃ m + kerf ∈ $\frac{M}{kerf}$ such that h(m+ kerf) = f(m) = y

$$\therefore \text{ h is an epimorphism}$$

So h is an isomorphism and by this, $\frac{M}{kerf} \approx f(M)$

Remark. If f is an epimorphism, then $\frac{M}{kerf} \approx N$

<u>Second isomorphism theorem.</u> Let N and K be submodules of an R-module M, then $\frac{K+N}{N} \approx \frac{K}{N\cap K}$

Proof. Define α: $K \rightarrow \frac{K+N}{N}$ by α(x) = x + N for each x ∈ K.

1. α is well-define (prove)
2. α is homomorphism (prove)
3. α is epimorphism (prove)
4. kerα = { x ∈ K| α(x) = 0}
   ={ x ∈ K | x + N = N}
   ={ x ∈ K | x ∈ N}
   = N∩K

   Then by the first isomorphism theorem, $\frac{K}{N\cap K} \approx \frac{K+N}{N}$

<u>Third isomorphism theorem</u>. Let N, K be submodules of M, and K≤ N, then $\frac{\frac{M}{K}}{\frac{N}{K}} \approx \frac{M}{N}$.

Proof. Define g: $\frac{M}{K} \rightarrow \frac{M}{N}$ by :g(m + K) = m + N for all m ∈ M.

1. g is well-define:
   suppose $m_1 + k = m_2 + K$ iff $m_1 - m_2 \in K \leq N$ iff $m_1 + N = m_2 + N$

   $\therefore$ g is well defined
2. g is a homomorphism (prove)
3. g is an epimorphism (prove)
4. kerg = {m+K| g(m+ k) = N}
   ={m+K| m+ N = N}
   = {m+K| m∈ N}
   = $\frac{N}{K}$ (where $K \leq N$ and $m \in N$)

   $\therefore$ kerg $= \frac{N}{K}$

Then by the first isomorphism theorem, $\frac{\frac{M}{K}}{\frac{N}{K}} \approx \frac{M}{N}$.

Exercise. Let M be a cyclic R-module, say M=Rx. Prove that M≈R/ ann(x), where ann(x) = {r ∈ R| rx = 0}.

[Hint: Define the mapping f: R→M by f(r) = rx]

Chapter three (Sequence)

Short exact sequence

Definition. A sequence $M_1 \xrightarrow{f} M \xrightarrow{g} M_2$ of R-modules and R-module homomorphismsis said to be ***exact*** *at* M Im f = ker g while a sequence of the form

$$\partial: \quad \dots \to M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \cdots$$

n∈ ℤ, is said to be an *exact sequence* if it is exact at M_n for each n ∈ ℤ. A sequence such as

$$0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$$