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قسم الرياضيات

المرحلة الثانية

المعادلات التفاضلية الاعتيادية

Ordinary Differential Equations

CHAPTER SEVEN

POWER SERIES SOLUTION OF THE LINEAR DIFFERENTIAL EQUATION

اساتذة المادة

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Introduction:

The power series method is a standard basic method for solving linear differential equation with variable coefficients. It gives solutions in the form of power series , this explains the name.

1.Basic definitions:

1. **Power series:** An infinite series of the form

$$\sum_{n=0}^{\infty} \boldsymbol{a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots} \qquad (\mathbf{1})$$

is called a **power series in (x-x_0)**.
where a_0,a_1,a_2,… are constants, called the coefficients of the series.
x_0 is called the center of the series.

$\boldsymbol{for\ example}$: the power series
$\sum_{n=0}^{\infty}(x+1)^n$ is centered at $x_0 = -1$

2. **A power series in x:** It is an infinite series (1) when $x_0 = 0$

$$\sum_{n=0}^{\infty} \boldsymbol{a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots} \qquad (\mathbf{2})$$

3. **Ordinary point:** A point $x = x_0$ is called an ordinary point of the equation:

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

If both the functions P(x) and Q(x) are analytic at x=x_0.

4. **Singular point:** If the point x=x_0 is not an ordinary point of the diff. eq.(3) , then it is called a *Singular point* of eq.(3).
There are two types of singular points:
(i) *regular Singular point*
(**ii**)*irregular Singular point*
A singular point x=x_0 of the diff. eq. (3) is called **regular singular point** of the Diff. eq. (3) if both $\boldsymbol{(x-x_0)P(x)}$ and $\boldsymbol{(x-x_0)^2 Q(x)}$ are analytic at x=x_0.

A singular point, which is not regular is called an **irregular singular point.**

5. Standard equation: equation (3) that in the form

$y'' + P(x)y' + Q(x)y = 0$

Is called a standard equation.

Example1.1: Determine whether x=0 is an ordinary point or a regular singular point of the differential equation $2x^2y'' + 7x(x+1)y' - 3y = 0$.

Sol. Dividing by $2x^2$, the given equation becomes,

$$y'' + \frac{7(x+1)}{2x}y' - \frac{3}{2x^2}y = 0 \qquad (4)$$

Comparing (4) with (3) we have

$P(x) = \frac{7(x+1)}{2x}$ and $Q(x) = -\frac{3}{2x^2}$

Since both $P(x)$ and $Q(x)$ are undefined at x=0, so both $P(x)$ and $Q(x)$ are not analytic at x= 0. Thus x=0 is not ordinary point so x=0 is a singular point

Also, $(x-0)P(x) = \frac{7(x+1)}{2}$ and $(x-0)^2Q(x) = -\frac{3}{2}$

Are analytic at x=0. Then x=0 is a regular singular point.

EXERCISES:

1.Show that x=0 is an ordinary point of $y'' - xy' + 2y = 0$.

2.Show that x=0 is an ordinary point of $(x^2-1)y'' + xy' - y = 0$, but x=1 is a regular singular point.

3. Determine the nature of the point x=0 for the equation $xy'' + y\sin x = 0$

2. Maclaurin Series:

Some functions can be expressed by the power series and are called Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (5)$$

Where $|x| < 1$,its called the geometric series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (6)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (7)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (8)$$

3.The power series method (the power series solution about x=0):

Suppose the second order linear diff. eq. in the standard form is:

$$y'' + P(x)y' + Q(x)y = 0 \qquad (3)$$

and x=0 is an ordinary point. Therefore, to solve the above equation we take the following power series,

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad (9)$$

Differentiating (9) twice w.r.t. (x), we get

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \qquad (10)$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \qquad (11)$$

Substituting equations. (9), (10) and (11) in (3) and collecting the like terms into x (which have the same powers), equating to zero the coefficients of the smallest power of x starting with the constant terms, the terms containing x, the terms containing x^2, etc.

This gives us a relation between the coefficients which helps in determining the nature of the solution.

Remark: A solution of the form $y = \sum_{n=0}^{\infty}(x - x_0)^n$ is said to be a solution about the ordinary point x_0

Example 3.1: Solve $y' - y = 0 \quad about\ x = 0$ (12)

Sol: starting with (9) and (10)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{9}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \tag{10}$$

Substituting in (12), we get

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots \quad) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad) = 0 \tag{13}$$

Now we collect the terms of similar power to *x*:

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + \cdots = 0 \tag{14}$$

Then

$$(a_1 - a_0) = 0 \ \rightarrow a_1 = a_0$$

$$(2a_2 - a_1) = 0 \rightarrow a_2 = \frac{a_1}{2} = \frac{a_0}{2!}$$

$$(3a_3 - a_2) = 0 \rightarrow a_3 = \frac{a_2}{3} = \frac{a_0}{3(2)} = \frac{a_0}{3!}$$

…

Substituting in (9), we get:

$$y = a_0 + a_0 x + \frac{a_0}{2!}x^2 + \frac{a_0}{3!}x^3 + \cdots \tag{15}$$

Or

$$y = a_0\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) = a_0 e^x$$

Example3.2: Solve $y' = 2xy \quad about\ x = 0$ (16)

(use the power series method)

Sol: we arrange the equation as follows

$$y' - 2xy = 0 \tag{17}$$

From (9) and (10)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Substituting in (17) we get

$$(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - 2x(\, a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = 0 \tag{18}$$

$$(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots) - (2\, a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + 2a_5 x^6 + \cdots) = 0$$

Then

$a_1 = 0$ ⁕

$2a_2 = 2a_0 \ \rightarrow a_2 = a_0$

$3a_3 = 2a_1 \rightarrow a_3 = 0$ ⁕

$4a_4 = 2a_2 \rightarrow a_4 = \frac{a_2}{2} \rightarrow a_4 = \frac{a_0}{2!}$

$5a_5 = 2a_3 \rightarrow a_5 = 0$ ⁕

$6a_6 = 2a_4 \rightarrow a_6 = \frac{a_4}{3} \rightarrow a_6 = \frac{a_0}{3!}$

…

Note that the odd coefficients are equal to zero.

Substituting in (18) we get:

$$y = a_0 \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right) \tag{19}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = a_0 e^{x^2} \tag{20}$$

Example 3.3: Solve $y'' + y = 0$ about x=0 (21)

(use the power series method)

Sol: From (9) and (11)

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{9}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \tag{11}$$

Substituting in (21), we get

$$(2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots) + (a_0 + a_1 x + a_2 x^2 + \cdots) = 0 \tag{22}$$

Collecting like powers of x, we find

$$(2a_2 + a_0) + (3 \cdot 2a_3 + a_1)x + (4 \cdot 3a_4 + a_2)x^2 + \cdots = 0 \tag{23}$$

Then

$$2a_2 + a_0 = 0 \quad \rightarrow \quad a_2 = -\frac{a_0}{2} = -\frac{a_0}{2!}$$

$$3 \cdot 2a_3 + a_1 = 0 \ \rightarrow \ a_3 = -\frac{a_1}{3 \cdot 2} \quad \rightarrow a_3 = -\frac{a_1}{3!}$$

$$4 \cdot 3a_4 + a_2 = 0 \ \rightarrow a_4 = -\frac{a_2}{4 \cdot 3} \quad \rightarrow a_4 = \frac{a_0}{4!}$$

…

Substituting in (9) we get

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + a_1(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots) \tag{24}$$

$$= a_0 \cos x + a_1 \sin x$$

4.Some properties of power series:

I. $\sum_{n=p}^{k} F(n) = F(p) + F(\,p+1) + \cdots + F(k) \quad ,k > p$ (25)

Where k and p are integer numbers.

II.$\sum_{n=p}^{\infty} a_n F(n) x^{n+p} = \sum_{n=0}^{\infty} a_{n+p} F(n+p) x^{n+2p}$ (26)

III. $\sum_{n=k}^{\infty} a_n x^{n-k} + \sum_{n=m}^{\infty} a_n x^{n-m} = \sum_{n=0}^{\infty} a_{n+k} x^n + \sum_{n=0}^{\infty} a_{n+m} x^n$

$$= \sum_{n=0}^{\infty} (a_{n+k} + a_{n+m}) x^n \qquad (27)$$

IV. If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \to a_n = b_n, \quad \forall n \geq 0$ (28)

For example: $if \quad \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$ then

$$n a_n = a_{n-1} \to a_n = \frac{a_{n-1}}{n}$$

So , if n=1→$a_1 = a_0$

if n=2→$a_2 = \frac{a_1}{2} \to a_2 = \frac{a_0}{2}$

if n=3→$a_3 = \frac{a_2}{3} \to a_3 = \frac{a_0}{3 \cdot 2} = \frac{a_0}{3!}$

V.**(Identity property of power series)**:

If $\sum_{n=0}^{\infty} a_n x^n = 0$, for every x number in the interval of convergence (i.e. in a neighbourhood of 0), then $a_n = 0$, for all n.

Example 4.1: Solve $y'' + xy = 0$ (29)

Sol: Let $y = \sum_{n=0}^{\infty} a_n x^n$ and $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Substituting in (29) ,we get:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0 \qquad (30)$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \qquad (31)$$

$$2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \qquad (32)$$

From property II, we get,

$$2a_2 + \sum_{k=1}^{\infty} (k+1)(k+2) a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k = 0 \qquad (33)$$

Where for the second term we take *k*=*n*-2→ *n*=*k*+2, so if *n*=3 then *k*=1

And for the third term we take k=n+1→ n=k-1, so if n=0 then k=1

From property III, eq. (33) will be:

$$2a_2 + \sum_{k=1}^{\infty} [(k+1)(k+2)a_{k+2} + a_{k-1}]\, x^k = 0 \qquad (34)$$

From property V, we get:

$$2a_2 = 0 \rightarrow a_2 = 0 \qquad (35)$$

$$(k+1)(k+2)a_{k+2} + a_{k-1} = 0$$

Then

$$a_{k+2} = -\frac{a_{k-1}}{(k+1)(k+2)} \qquad (36)$$

So, if

K=1→$a_3 = -\frac{a_0}{(2)(3)}$

K=2→$a_4 = -\frac{a_1}{(3)(4)}$

K=3→$a_5 = -\frac{a_2}{(4)(5)} \;\; = 0$

K=4→$a_6 = -\frac{a_3}{(5)(6)} \;\; = \frac{a_0}{2{\cdot}3{\cdot}5{\cdot}6}$

K=5→$a_7 = -\frac{a_4}{(6)(7)} = \frac{a_1}{3{\cdot}4{\cdot}6{\cdot}7}$

K=6→$a_8 = -\frac{a_5}{(7)(8)} \;\; = 0$

K=7→$a_9 = -\frac{a_6}{(8)(9)} = \frac{a_0}{2{\cdot}3{\cdot}5{\cdot}6{\cdot}8{\cdot}9}$

…

Substituting in

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \cdots \qquad (37)$$

$$y = a_0 + a_1x - \frac{a_0}{2{\cdot}3}x^3 - \frac{a_1}{3{\cdot}4}x^4 + \frac{a_0}{2{\cdot}3{\cdot}5{\cdot}6}x^6 + \frac{a_1}{3{\cdot}4{\cdot}6{\cdot}7}\;\; x^7 + \frac{a_0}{2{\cdot}3{\cdot}5{\cdot}6{\cdot}8{\cdot}9}x^9 + \cdots \qquad (38)$$

EXERCISES:

Solve the following equations using the power series method:

1.$\frac{dy}{dx} = x^2 - y$

2. $(1 - x^2)y'' - 2xy' + 6y = 0$

3. $y'' + y' - xy = 0$

4.$y'' - y = 0$

5. $y' = x + y$

6. $(x^2 + 1)y'' + xy' - y = 0$

7. $y'' + x^2y' + xy = 0$