CHAPTER SEVEN

Number Systems الانظمة العددية

Chapter Four Contents:

1. Natural Number System نظام الاعداد الطبيعية

2. Integer Number System نظام الاعداد الصحيحة

يتناول الفصل الحالي الانظمة العددية المتعارف عليها التي نستخدمها في حياتنا اليومية. ان اكبر واشمل نظام عددي هو نظام الاعداد الحقيقية والذي يتضمن الانظمة التالية:

1) نظام الاعداد الطبيعية N

2) نظام الاعداد الصحيحة Z

3) نظام الاعداد النسبية Q

هناك نظام عددي اخر وهو نظام الاعداد المركبة اوالمعقدة C

تاريخيا اول مجموعة تم تكوينها هي مجموعة الاعداد الطبيعية N ثم مجموعة الاعداد الصحيحة Z ومن الاعداد الصحيحة تم الحصول على الاعداد النسبية Q ومن الاعداد النسبية يمكن الحصول على الاعداد الحقيقية R واخيرا نكون الاعداد المعقدة C من R.

ولتكوين اي مجموعة من هذه المجموعات يجب تعريف عمليتي الجمع والضرب على تلك المجموعة ومن ثم اعطاء خواص الجمع والضرب كمبرهنات ثم نعين ترتيبا على المجموعة ونعين خواصه.

سنبدأ اولا باستعراض كيفية تكوين نظام الاعداد الطبيعية N

The Natural Number: نظام الاعداد الطبيعية

تم تكوين نظام الاعداد الطبيعية بالاعتماد على مجموعة من البديهيات (عبارات رياضية لاتحتاج الى برهان) والتي تسمى **بديهيات بيانو** (Peano's Axioms) والتي تعود لمكتشفها العالم الايطالي **(Giuseppe Peano)**

Peano's Axioms: بديهيات بيانو

1. $1 \in N$

2. $\exists \alpha: N \to N\backslash\{1\}$ bijective map s.t. $\alpha(n) = n^* = n + 1 \quad \forall n \in N$

n^* is called the **successor التابع** of n

3. Induction Axiom بديهية الاستقراء

$\forall M \subseteq N$; If $[(1 \in M \wedge k \in M)$ then $k^* \in M\,]$ then $M = N$

Sum on *N* الجمع على مجموعة الاعداد الطبيعية

ان عملية الجمع على الاعداد الطبيعية تم تعريفها كالتالي

1. $n + 0 = n \quad \forall n \in N$

2. $\forall n \in N$ define $n + 1 = n^*$

3. $\forall n, m \in N$ define $m + n^* = (m + n)^*$

Example 4.1: Use the definition above to find $2 + 1$ and $2 + 3$

$2 + 1 = 2^* \quad [n + 1 = n^*]$

$= 3$

$2 + 3 = 2 + 2^* \quad [n + 1 = n^*]$

$= (2+2)^* \quad [m + n^* = (m+n)^* \;]$

$= (2+1^*)^* = (2+1)^{**} \quad = 2^{***} \quad = 3^{**} \; = 4^* = 5$

<u>Example</u> 4.2: Find $5 + 4$ (**H.W.**)

<u>Theorem</u> 4.3:

1. $n + 1 = 1 + n = n^* \quad \forall n \in N$

2. $(m+n) + p = m + (n+p) \quad \forall m, n, p \in N$

3. $m + n = n + m \quad \forall n, m \in N$ (**H. W.**)

4. $m + n = p + n$ iff $m = p$

<u>ملاحظة</u>: سنعتمد على تحقيق الشرط الثالث من بديهيات بيانو لبرهان المبرهنة اعلاه

<u>Proof 1</u>: Let $M \subseteq N$ s.t. $M = \{m \in N : m + 1 = 1 + m = m^*\}$ T.P. $M = N$ (Axiom3)

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $m = 1 \in M$

i.e., **T. P.** $1 + 1 = 1 + 1 = 2$

$m + 1 = 1 + 1 = 1^* = 2$ ….(1)

$1 + m = 1 + 1 = 1^* = 2$ ….(2)

From (1) and (2), $1 + 1 = 1 + 1 = 2 \Longrightarrow 1 \in M$

2. Suppose $k \in M$ $(k+1 = 1+k = k^*)$….(3)

3. T. P. $k^* \in M$ ($\mathrm{T.P.}\ k^* + 1 = 1 + k^* = k^{**}$)

$k^* + 1 = k^{**}$ ….(4)

$1 + k^* = (1+k)^* = (k+1)^*$ (from 3, $1 + k = k + 1$)

$= k^{**}$…(5)

From (4) and (5) $\Longrightarrow$ $k^{**} \in M$

Hence, $M = N$ (Axiom3)

$\Longrightarrow m + 1 = 1 + m = m^* \quad \forall m \in M$

<u>Proof</u> 2: Let $M \subseteq N$ s.t. $M = \{p \in N: (n+m)+p = n+(m+$
$p \ \ \forall m,n \in N\}$ T.P. $M=N$

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $1 \in M$ (T.P. $(n+m)+1 = n+(m+1)$)

$(n+m)+1 = (n+m)^* = n + m^* = n + (m+1)$

2. Suppose $k \in M$ ($(n+m)+k = n+(m+k)$)

3. T. P. $k^* \in M$ $\big(\mathrm{T.P.}\ (n+m)+k^* = n+(m+k^*)\big)$

$$(n+m)+k^* = \big((n+m)+k\big)^* = \big(n+(m+k)\big)^* = n+(m+k)^*$$
$$= n + (m+k^*)$$

$\therefore\ k^* \in M$

$\therefore M = N$

<u>Proof</u> 4: $\Longrightarrow$) Suppose $m + n = p + n$ **T. P.** $m = p$

Let $M \subseteq N$ s.t. $M = \{n \in N: m + n = p + n \Longrightarrow \mathrm{m = p} \quad \forall m, p \in N\}$
T.P. $M = N$

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $n = 1 \in M$

i.e., $m + 1 = p + 1$ **T. P.** $\mathrm{m = p}$

$m + 1 = p + 1$

$m^* = p^*$

$\alpha(m) = \alpha(p)$ (by axiom2)

$m = p$ (α is 1-1)$\Longrightarrow 1 \in M$

2. Suppose $k \in M$ ($m + k = p + k \Longrightarrow \mathrm{m = p}$)

3. T. P. $k^* \in M$ (if $\mathrm{m + k^* = p + k^*}$ **T. P.** $\mathrm{m = p}$)

$m + k^* = p + k^*$

$(m + k)^* = (p + k)^*$

$\Longrightarrow \alpha(\mathrm{m + k}) = \alpha(\mathrm{p + k})$

$\Longrightarrow m + k = p + k$ (α is 1-1)

$\Longrightarrow m = p$ (from2)

$\Longrightarrow \mathrm{k^* \in M}$

Hence, $M = N$ (Axiom3)

$\Longleftarrow$) Let $m = p$ **T.P.** $m + n = p + n$

Let $M \subseteq N$ s.t. $M = \{n \in N: \mathrm{m = p} \Longrightarrow m + n = p + n \ \forall m, p \in N\}$ T.P. $M = N$

1. T. P. $n = 1 \in M$

Let $m = p$ T.P. $m + 1 = p + 1$

$m = p \Longrightarrow \ \alpha(m) = \alpha(p)$ (α is well defined)

$\Longrightarrow m^{*} = p^{*}$ (by axiom 2)

$\Longrightarrow \mathrm{m} + 1 = p + 1$

$\therefore 1 \in M$

2. Suppose $k \in M$ (if $\mathrm{m} = \mathrm{p} \Longrightarrow m + k = p + k$)

3. T. P. $k^{*} \in M$ (if $\mathrm{m} = \mathrm{p}$ $\mathbf{T.P.}$ $\mathrm{m} + \mathrm{k}^{*} = \mathrm{p} + \mathrm{k}^{*}$)

$m = p \ \Longrightarrow m + k = p + k$ (from2)

$\Longrightarrow \alpha(\mathrm{m} + \mathrm{k}) = \alpha(\mathrm{p} + \mathrm{k})$ (α is well defined)

$\Longrightarrow (m + k)^{*} = (p + k)^{*}$ (by axiom2)

$\Longrightarrow \mathrm{m} + k^{*} = p + k^{*}$

$\Longrightarrow k^{*} \in M$

$\therefore M = N$

Multiplication on *N* عملية الضرب على مجموعة الاعداد الطبيعية

ان عملية الضرب على الاعداد الطبيعية تم تعريفها كالتالي

1. $n.0 = 0 \quad \forall\, n \in N$

2. $n.1 = n \quad \forall\, n \in N$

3. $m.n^{*} = m.(n + 1) = m.n + m \quad \forall\, n, m \in N$

<u>**Example**</u> **4.4:** Use the definition above to find 2.3 and 5.4

2.3=2.2$^{*} = 2.2 + 2$ $\qquad (m.n^{*} = m.n + m)$

$= 2.1^* + 2$

$= (2.1 + 2) + 2 \quad (m.n^* = m.n + m)$

$= (2 + 2) + 2$

$= (2 + 1^*) + 2 \quad (n^* = n + 1)$

$= (2 + 1)^* + 2$

$= 2^{**} + 2 \quad (n^* = n + 1)$

$= 3^* + 2$

$= 4 + 2 = 4 + 1^* = (4 + 1)^* = 4^{**} = 5^* = 6$

Find 5.4 (**H. W.**)

Theorem 4.5:

1. $n.1 = 1.n = n \quad \forall n \in N$
2. $m^*.n = m.n + n \quad \forall n, m \in N$
3. $p.(m + n) = p.m + p.n \quad \forall n, m, p \in N$
4. $(m.n).p = m.(n.p) \quad \forall n, m, p \in N$ (**H. W.**)
5. $m.n = n.m \quad \forall n, m \in N$
6. $(m + n).p = m.p + n.p \quad \forall n, m, p \in N$ (**H. W.**)
7. If $m.p = n.p$ then $m = n$ (**H. W.**)

Proof 1: Let $M \subseteq N$ s.t. $M = \{n \in N: n.1 = 1.n = n\}$ T.P. $M = N$ (Axiom3)

شرط بيانو الثالث يتضمن التالي

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $n = 1 \in M$

i.e., **T. P.** $1.1 = 1.1 = 1$

$1.1 = 1 \Longrightarrow 1 \in M$

2. Suppose $k \in M$ $(k.1 = 1.k = k)$….(1)

3. T. P. $k^* \in M$ $(\text{T.P.}\, \text{k}^*.1 = 1.\text{k}^* = \text{k}^*)$

$k^*.1 = k^*$ (n..1=n) ….(2)

$1.k^* = 1.k + 1$

$= k.1 + 1$ $(1.k = k.1)$

$= k + 1 = k^*$…(3)

From (2) and (3) $\Longrightarrow$ $\text{k}^* \in \text{M}$

Hence, $M = N$ (Axiom3)

$\Longrightarrow n.1 = 1.n = n \quad \forall n \in M$

<u>Proof</u> 2: Let $M \subseteq N$ s.t. $M = \{n \in N\colon\ m^*.n = m.n + n \quad \forall m \in N\}$ T.P. $M = N$ (Axiom3)

شرط بيانو الثالث يتضمن التالي

1. Prove $1 \in M$

2. Assume $k \in M$

3. Prove $k^* \in M$

1. T. P. $n = 1 \in M$

i.e., **T. P.** $m^*.1 = m.1 + 1$

$m^*.1 = m^*$ $[n.1 = n]$ …(1)

$m.1 + 1 = m + 1 \quad [m.1 = m]$

$= m^*$.....(2)

From (1) and (2) $\Longrightarrow 1 \in M$

2. Suppose $k \in M$ $(m^*.k = m.k + k)$....(3)

3. T. P. $k^* \in M$ $(\mathrm{T.P.\,m^*.k^* = m.k^* + k^*})$

$\mathrm{m^*.k^*} = m^*.k + m^*$

$= (m.k + k) + m^*$ (from 3)

$= (m.k + k) + (m + 1)$

$= m.k + (k + m) + 1$ (+ is associative)

$= m.k + (m + k) + 1$ (+ is commutative)

$= (m.k + m) + (k + 1)$ (+ is associative)

$= \mathrm{m.k^* + k^*}$

$\Longrightarrow \mathrm{k^* \in M}$

Hence, $M = N$ (Axiom3)

$m^*.n = m.n + n \quad \forall n, m \in N$

<u>Proof</u> 3: Let $M \subseteq N$ s.t. $M = \{n \in N: p.(m + n) = p.m + p.n \ \forall m, p \in N\}$

T.P. $M = N$ (Axiom3)

1) $1 \in M$? T.P. $p.(m + 1) = p.m + p.1 \ \forall m, p \in N$

$p.(m + 1) = pm + p \quad [m.(n + 1) = m.n + m]$

$= p.m + p.1 \quad (n.1 = n)$

2) Suppose $k \in M$ $[p.(m + k) = p.m + p.k \ \forall m, p \in N]$

3) T.P. $k^* \in M$ $\quad (T.P. \quad p.(m+k^*) = p.m + p.k^*)$

$p.(m+k^*) = p.(m+k)^*$ $\qquad [m+n^* = (m+n)^*]$

$= p.(m+k) + p$ $\qquad [m.n^* = m.n + m]$

$= (p.m + p.k) + p$ $\quad$ [from 2]

$= p.m + (p.k + p)$ $\qquad$ [+ is associative]

$= p.m + p.k^*$ $\qquad [m.n^* = m.n + m]$

$\Longrightarrow \quad k^* \in M$

Hence, $M = N$

$\Longrightarrow \quad p.(m+n) = p.m + p.n \ \ \forall m, n, p \in N$

<u>Proof</u> 5: Let $M \subseteq N$ s.t. $M = \{n \in N: \ m.n = n.m \quad \forall m \in N\}$

T.P. $M = N$ (Axiom3)

1) $1 \in M$? T.P. $m.1 = 1.m \ \ \forall m \in N$

$m.1 = 1.m = m$ $\quad$ [from 1 $(n.1 = 1.n = n \quad \forall n \in N)$]

2) Suppose $k \in M$ $[k.m = m.k \ \ \forall m \in N]$

3) T.P. $k^* \in M$ $\quad (T.P. \quad m.k^* = k^*.m)$

$m.k^* = m.k + m$

$= k.m + m$ $\qquad [k.m = m.k \ \ \forall m \in N]$

$= k^*.m$ $\qquad$ [from 2]

$\Longrightarrow \quad k^* \in M$

Hence, $M = N$

$\Longrightarrow \quad m.n = n.m \quad \forall m, n \in N$

<u>Definition</u> 4.6: Let $x \in N$, then

1. $x^0 = 1$

2. $x^1 = x$

3. $x^{n+1} = x^n . x^1 \quad \forall n \in N$

<u>Theorem 4.7</u>: Let $x, y, n, m \in N$. Then:

1. $x^m . x^n = x^{m+n}$

2. $(x^m)^n = x^{nm}$ **(H. W.)**

3. $(x . y)^n = x^n . y^n$ **(H. W.)**

<u>Proof</u> 1: Let $M \subseteq N$ s.t. $M = \{n \in N\colon\ x^m . x^n = x^{m+n}\ \ \forall m \in N\}$

T.P. $M = N$ (Axiom3)

1) $1 \in M$? T.P. $x^m . x^1 = x^{m+1}\ \ \forall m \in N$

$x^m . x^1 = x^m . x \quad [x^1 = x]$

$\qquad = x^{m+1} \quad [x^{n+1} = x^n . x^1]$

2) Suppose $k \in M$ $[x^m . x^k = x^{m+k}\ \ \forall m \in N]$

3) T.P. $k^* \in M \quad \left(T.P.\quad x^m . x^{k^*} = x^{m+k^*}\ \ \forall m \in N\right)$

$x^m . x^{k^*} = x^m . x^{k+1}$

$\qquad = x^m . x^k . x^1$

$\qquad = x^{m+k} . x^1$ (from2)

$\qquad = x^{m+k+1}$

$\qquad = x^{m+k^*}$

$\Longrightarrow \quad k^* \in M$

Hence, $M = N$

$\Longrightarrow \quad x^m . x^n = x^{m+n} \quad \forall m, n \in N$

Ordered on N الترتيب على الاعداد الطبيعية

The relation $<$ is defined as follows

$m < n \leftrightarrow \exists! r \in N \; s.t. m + r = n$

$m \leq n \leftrightarrow m = n \vee m < n \quad \forall m, n \in N$

Theorem 4.8:

1. $\forall m, n \in N,$ either $m = n$ or $m < n$ or $n < m$ (بدون برهان) قانون الانقسام الثلاثي

2. If $m \leq n$ and $n \leq m$ then $n = m$ (see Example 3.75 (anti symm.) from Mathematical logic and set theory)

3. If $m < n$ and $n < p$ then $m < p$ (see Example 3.75 (transitive) from Mathematical logic and set theory)

4. $m < n \leftrightarrow m + p < n + p \quad \forall p \in N$ **(see Example 3.45)**

5. $m < n \leftrightarrow m.p < n.p \quad \forall p \in N$ **(see Example 3.45)**

6. If $p \leq m \leq p^*$ then $m = p$ or $m = p^*$

Proof 6: Assume $m \neq p$ and $m \neq p^*$ برهان غير مباشر

$\Rightarrow p < m$ and $m < p^*$

$p < m \Rightarrow \exists k \in N \; s.t. p + k = m$….(1)

$m < p^* \Rightarrow \exists r \in N \; s.t. m + r = p^*$….(2)

Substitute (1) in (2) $\Rightarrow p^* = p + k + r$

$\Rightarrow p + 1 = p + k + r$

$\Rightarrow 1 = k + r$ تناقض لايوجد عددين طبيعيين حاصل جمعهما واحد

$\Rightarrow m = p$ or $m = p^*$

Least Element العنصر الاصغر

Let $\emptyset \neq M \subseteq N$ and $a \in N$. Then a is called **least element** in M if $a \leq x \quad \forall x \in M$.

Example 4.9: Let $M = \{3,4,5\} \subseteq N$

Then 3 is the least element of M because $3 \leq x \quad \forall x \in M$

Well-Ordered Set المجموعة المرتبة جيدا

Let $\emptyset \neq M \subseteq N$ and $a \in N$. Then M is called **well-ordered** if each non empty subset of M contains least element

المجموعة M تكون مرتبة ترتيبا جيدا اذا كانت كل مجموعة جزئية وغير خالية منها تمتلك عنصر اصغر

Integer Numbers System نظام الاعداد الصحيحة

ظهرت الحاجة الى تكوين نظام عددي اوسع من نظام الاعداد الطبيعية عندما وجد بان نظام الاعداد الطبيعية غير قادر على ايجاد حلول لمعادلات من النوع التالي

$5 + x = 2$ or $10 + y = 1 \quad \forall x, y \in N$

لذلك تم ايجاد نظام عددي أوسع يحوي الحل ل هكذا نوع من المعادلات ألا وهو نظام الاعداد الصحيحة والتي يرمز لها بالرمز Z

Construction of Integer Numbers انشاء مجموعة الاعداد الصحيحة

Definition 4.10: Let N be the set of natural numbers then

$$N \times N = \{(n, m) : n, m \in N\}$$

Define a relation $\sim$ on $N \times N$ as

$(n_1, m_1) \sim (n_2, m_2)$ iff $n_1 + m_2 = m_1 + n_2 \;\; \forall\, (n_1, m_1), (n_2, m_2) \in N \times N$

Example 4.11: $(6,3) \sim (10,7)$ since $6 + 7 = 3 + 10$

$(5,2) \sim (8,4)$ since $5 + 4 \neq 2 + 8$

Theorem 4.12: The relation $\sim$ on $N \times N$ is an equivalence relation

Proof:

Reflexive? T.P. $(n, m) \sim (n, m) \;\; \forall n, m \in N$

T.P. $n + m = m + n$ (def. of $\sim$)

Since $n + m = m + n$ (+ is commutative on N)

$\therefore\ (n, m) \sim (n, m)$

Symmetric? Let $(n, m) \sim (r, s)$ T. P. $(r, s) \sim (n, m)$

Since $(n, m) \sim (r, s) \Rightarrow \mathrm{n} + s = m + r$ (def of $\sim$)

$\Rightarrow \mathrm{s} + \mathrm{n} = \mathrm{r} + \mathrm{m}$ (+ is commutative on N)

$\Rightarrow r + m = s + n$

$\Rightarrow (r, s) \sim (n, m)$

Transitive? Let $(n, m) \sim (r, s)$ and $(r, s) \sim (p, q)$ T. P. Let $(n, m) \sim (p, q)$

$(n, m) \sim (r, s) \Rightarrow \mathrm{n} + s = m + r$ (def. of $\sim$) ….(1)

$(r, s) \sim (p, q) \Rightarrow r + q = s + p$ (def. of $\sim$) ….(2)

By summing up (1) and (2)

$\Rightarrow n + s + r + q = m + r + s + p$

$\Rightarrow n + (s + r) + q = m + (s + r) + p$ (+ comm. and assoc. on N)

$\Longrightarrow n + q + (s + r) = m + p + (s + r)$ (+ comm. on N)

$\Longrightarrow n + q = m + p$ (by theorem 4.5(4), قانون الحذف)

$(n, m) \sim (p, q)$

$\therefore$ ~ is an equivalence relation

<u>Definition 4.13</u>: Let ~ be a relation on $N \times N$. Since ~ is an equivalence relation, then we can define an equivalence classis on $N \times N$

$H = \{[(n, m)]:\ n, m \in N\} = \{[n, m]:\ n, m \in N\}$

Where $[n, m] = \{(r, s): (n, m) \sim (r, s)\}$

<u>Definition 4.14</u>: The set of all different equivalence classes on $N \times N$ form the set of integer numbers

Mathematically,

$Z^{+} = \{[n, m]:\ n > m \text{ and } n, m \in N\}$

$0 = \{[n, m]: n = m \text{ and } n, m \in N\}$

$Z^{-} = \{[n, m]: n < m \text{ and } n, m \in N\} = \{-[m, n]: m > n\}$

<u>Example 4.15</u>: $[\mathbf{3}, \mathbf{1}] = \mathbf{2};\ \ [\mathbf{2}, \mathbf{3}] = -[\mathbf{3}, \mathbf{2}] = -\mathbf{1};\ \ [\mathbf{3}, \mathbf{3}] = \mathbf{0}$

<u>Example 4.16</u>: Find [(2,3)], [(10,7)], [0,0]

$[(\mathbf{2}, \mathbf{3})] = [2,3] = \{(r, s) \in N \times N: (r, s) \sim (2,3)\}$

$= \{(r, s): r + 3 = s + 2\}$

$= \{(r, s):\ r + 1 + 2 = s + 2\}$

$= \{(r,s): r+1 = s,\ \ r,s \in N\}$

$= \{(1,2),(2,3),(3,4),\ldots.\} = -1$ (from def.7)

من خواص صفوف التكافؤ ان العناصر الموجودة في صف تكافؤ معين لها صفوف تكافؤ متساوية.
اي ان

$[2,3] = [(1,2)] = [3,4] = [(4,5)] = \cdots$

$[\mathbf{10},\mathbf{7}] = \{(r,s) \in N \times N: (r,s) \sim (10,7)\}$

$= \{(r,s): r+7 = s+10\}$

$= \{(r,s):\ r+7 = s+3+7\}$

$= \{(r,s): r = s+3,\ \ r,s \in N\}$

$= \{(4,1),(5,2),(6,3),\ldots.\} = 3$ (from def.7)

$\therefore [10,7] = [4,1] = [5,2] = [6,3] = \cdots$

$[\mathbf{0},\mathbf{0}] = \{(r,s) \in N \times N: (r,s) \sim (0,0)\}$

$= \{(r,s): r+0 = s+0\}$

$= \{(r,s):\ r = s\}$

$= \{(1,1),(2,2),(3,3),\ldots.\} = 0$ (from def.7)

$\therefore [0,0] = [1,1] = [2,2] = [3,3] = \cdots$

كل عدد صحيح يمكن كتابته على شكل صف تكافؤ

Addition and Multiplication on *Z*

Let $x, y \in Z$ such that $x = [n,m]$ and $y = [r,s]$. Define

$x +_Z y = [n,m] +_Z [r,s] = [n+r, m+s]$

$x ._Z\ y = [n,m] ._Z [r,s] = [nr+ms, ns+mr]$

<u>Example 4.17</u>: Find $(3) +_Z (-5)$, $(-1) +_Z (-3)$, $(-2) ._Z (4)$

$(3)+_Z(-5) = [5,2]+_Z[1,6] = [5+1, 2+6]$ (def. of $+_Z$)

$= [6,8] = -2$

$(-1)+_Z(-3) = [5,6]+_z[1,4]$

$= [5+1, 6+4]$ (def. of $+_Z$)

$= [6,10] = -4$

$(-2)._Z\,(4) = [2,4]._Z\,[8,4]$

$= [(2.8)+(4.4), (2.4)+(4.8)]$ (def. of $._Z$)

$= [16+16, 8+32]$

$= [32, 40] = -8$

<u>Example 4.18</u>: (H. W.) Find $(-3)+_Z(4)$, $(-5)._Z\,(-1)$

<u>Theorem 4.19</u>: Properties of the sum on *Z*

1. $x+_Z y = y+_Z x \quad \forall x, y \in Z$
2. $(x+_Z y)+_Z p = x+_Z(y+_Z p) \ \ \forall x, y, p \in Z$
3. $\exists 0 \in Z$ such that $x+_Z 0 = 0+_Z x = x \quad \forall x \in Z$
4. $x+_Z p = y+_Z p \Leftrightarrow x = y \ \ \forall x, y, p \in Z$
5. $\forall x \in Z, \exists y \in Z\ s.t.\ x+_Z y = 0$

<u>Proof 1</u>: T.P. $x+_Z y = y+_Z x \quad \forall x, y \in Z$

Let $x = [n, m]$ and $y = [r, s]$ such that $n, m, r, s \in N$

$x+_Z y = [n, m]+_Z[r, s]$

$= [n+r, m+s]$ (def. of $+_Z$)

$= [r+n, s+m]$ (+ is comm. on N)

$= [r,s] +_Z [n,m]$ (def. of $+_Z$)

$= y +_Z x$

Proof 3: T.P. $\exists 0 \in Z$ such that $x +_Z 0 = 0 +_Z x = x \quad \forall x \in Z$

Let $x = [n,m]$ and $0 = [r,s]$ such that $n,m,r,s \in N$

$x +_Z 0 = x \Longrightarrow [n,m] +_Z [r,s] = [n,m]$

$\Longrightarrow [n+r, m+s] = [n,m]$ (def. of $+_Z$)

$\Longrightarrow (n+r, m+s) \sim (n,m)$ ([a] = [b] $\Longrightarrow a \sim b$)

$\Longrightarrow n+r+m = m+s+n$ (def. of ~)

$\Longrightarrow (n+m)+r = (n+m)+s$ (+ comm. and assoc)

$\Longrightarrow$ r $= s$ (By theorem 4.5(4))

$\therefore 0 = [r,s] = [r,r] = \{[1,1], [2,2], [3,3], \dots\}$

Similarly, $0 +_Z x = x$

$\therefore \exists 0 = [r,r]\ s.t.\ x +_Z 0 = 0 +_Z x = x \quad \forall x \in Z$

Proof 4:

Let $x = [n,m]$, $y = [r,s]$ and $p = [h,k]$ such that $n,m,r,s,h,k \in N$

$x +_Z p = y +_Z p$

$\Leftrightarrow [n,m] +_Z [h,k] = [r,s] +_Z [h,k]$

$\Leftrightarrow [n+h, m+k] = [r+h, s+k]$ (def. of $+_Z$)

$\Leftrightarrow (n+h, m+k) \sim (r+h, s+k)$ ([a] = [b] $\Longrightarrow a \sim b$)

$\Leftrightarrow n+h+s+k = m+k+r+h$ (def. of ~)

$\Leftrightarrow n+s+(h+k) = m+r+(h+k)$ (+ comm. and assoc)

$\Leftrightarrow n + s = m + r$ (By theorem 4.5(4))

$\Leftrightarrow [n, m] \sim [r, s]$ (def. of ~)

$\Leftrightarrow [n, m] = [r, s]$ $(a \sim b \Longrightarrow [a] = [b])$

$\Leftrightarrow x = y$

<u>Proof 5</u>: T. P. $\forall x \in Z, \exists y \in Z\ s.t. x +_Z y = 0$

Let $x = [n, m]$ and $y = [r, s]$

Suppose $x +_Z y = 0$

$[n, m] +_Z [r, s] = [h, h]$

$[n + r, m + s] = [h, h]$ (def. of $+_Z$)

$(n + r, m + s) \sim (h, h)$ $([a] = [b] \Longrightarrow a \sim b)$

$n + r + h = m + s + h$ (def. of ~)

$n + r = m + s$

$[n, m] \sim [s, r]$ (def. of ~)

$[n, m] = [s, r]$ $(a \sim b \Longrightarrow [a] = [b])$

$x = -y = [s, r]$

$\Longrightarrow \forall x \in Z, \exists y \in Z\ s.t. x +_Z y = [s, r] +_Z [r, s] = [s + r, r + s] = 0$

<u>Theorem 4.20</u>: Properties of the multiplication on *Z*

1. $x ._Z\ y = y ._Z\ x$ $\forall x, y \in Z$ (**H. W.**)

2. $(x ._Z\ y) ._Z\ w = x ._Z (y ._Z\ w)$ $\forall x, y, w \in Z$ (**H. W.**)

3. $\exists 1 \in Z$ such that $x ._Z\ 1 = 1 ._Z\ x = x$ $\forall x \in Z$

4. $x \cdot_Z w = y \cdot_Z w \Leftrightarrow x = y \quad \forall x, y, w \in Z$ (**H. W.**)

5. $x \cdot_Z 0 = 0 \cdot_Z x = 0 \quad \forall x \in Z$ (**H. W.**)

6. $x \cdot_Z (y +_Z w) = x \cdot_Z y +_Z x \cdot_Z w \quad \forall x, y, w \in Z$

<u>Proof 3</u>:

Let $x = [n, m]$ and $1 = [2,1] = [3,2] = \cdots$

T. P. $x \cdot_Z 1 = 1 \cdot_Z x = x \quad \forall x \in Z$

$x \cdot_Z 1 = [n, m] \cdot_Z [2,1]$

$= [2n + m, n + 2m]$ (def. of $\cdot_Z$)

$= [n + n + m, n + m + m]$

$= [n + m + n, n + m + m]$

$= [n + m, n + m] +_Z [n, m]$

$0 +_Z [n, m] = x$

Similarly, $1 \cdot_Z x = x$ (**H. W.**)

<u>Proof 6</u>: T. P. $x \cdot_Z (y +_Z w) = x \cdot_Z y +_Z x \cdot_Z w \quad \forall x, y, w \in Z$

Let $x = [n, m], y = [r, s]$ and $w = [p, q]$

$x \cdot_Z (y +_Z w) = [n, m] \cdot_Z ([r, s] +_Z [p, q])$

$= [n, m] \cdot_Z [r + p, s + q]$ (def. of $+_Z$)

$= [n.(r + p) + m.(s + q), n.(s + q) + m.(r + p)]$ (def. of $\cdot_Z$)

$= [nr + np + ms + mq, ns + nq + mr + mp]$

$= [(np + mq) + (nr + ms), (nq + mp) + (ns + mr)]$ (+ is comm. &asso.)

$= [np + mq, nq + mp] +_Z [nr + ms, ns + mr]$ (def. of $+_Z$)

$= ([n, m] \cdot_Z [p, q]) +_Z ([n, m] \cdot_Z [r, s])$ (def. of $\cdot_Z$)

$= (x \cdot_Z w) +_Z (x \cdot_Z y)$

Ordering on Z الترتيب على الاعداد الصحيحة

Let $x = [n, m]$ be an integer number then

1. x is called **positive integer number** if $m < n$.

2. x is called **negative integer number** if $-x$ is positive or if $n < m$.

Example 4.21: $x = [10, 3]$ is positive because $3 < 10$

$y = [1,5]$ is negative because $-y$ is positive (or because $1 < 5$)

Theorem 4.22: Let x and y are positive integer numbers then $x +_Z y$ is positive integer number

Proof: Let x be a positive integer number $\Rightarrow$ $x = [n, m]$ $s.t. m < n$ ….(1)

Let y be a positive integer number $\Rightarrow$ $y = [r, s]$ $s.t. s < r$ ….(2)

From (1) &(2) $\Rightarrow m + s < n + r$ (by thm 4.8(4)) …(3)

$x +_Z y = [n, m] +_Z [r, s] = [n + r, m + s]$ (def. of $+_Z$)

Since $m + s < n + r \Rightarrow x +_Z y$ is positive integer number

Definition 4.23: Let x and y are two integer numbers such that $x = [n, m]$ and $y = [r, s]$. $y - x$ is positive integer number iff $x < y$

In other words,

$y - x = [r, s] - [n, m]$ is positive

$= [r, s] +_Z [m, n]$ $(-x = [m, n])$

$= [r+m, s+n]$ (def. of $+_Z$)

Since, $y - x$ is positive $\Longrightarrow$ $s + n < r + m$

<u>Example 4.24</u>: let $x = [2,3] = [n, m]$ and $y = [5,1] = [r, s]$. Is $x < y$?

$s + n = 3 < 8 = r + m \Longrightarrow y - x$ is positive

$\Longrightarrow x < y$

<u>Example 4.25</u>: let $x = [10, 5] = [n, m]$ and $y = [6,1] = [r, s]$

$s + n = 11 > 1$=6-5 $\Longrightarrow$ x – y is positive

$\Longrightarrow$ y< x

<u>Definition 4.26</u>: Let x be an integer number. Then

1. x is a positive integer number $\Leftrightarrow x > 0$
2. x is a negative integer number $\Leftrightarrow x < 0$
3. $x \le y$ means $x < y$ or $x = y$
4. $x \ge y$ means $x > y$ or $x = y$

<u>Theorem 4.27</u>: For each $x, y, p \in Z$

1. If $x > 0$ and $y > 0$ then $x \cdot_Z y > 0$ (**H. W.**)
2. If $x < 0$ and $y < 0$ then $x \cdot_Z y > 0$ (**H. W.**)
3. If $x < 0$ and $y > 0$ then $x \cdot_Z y < 0$
4. $x < y \Leftrightarrow x +_Z p < y +_Z p$

5. $x < y \Leftrightarrow x \cdot_Z p < y \cdot_Z p \quad \forall p \in Z^+$ **(H. W.)**

Proof 3: Let $x = [n,m] < 0$, $y = [r,s] > 0$ T. P. $x \cdot_Z y < 0$

Since $x < 0 \Rightarrow n < m \Rightarrow m = n + k_1;\ \ k_1 \in N \ldots.(1)$

Since $y > 0 \Rightarrow r > s \Rightarrow r = s + k_2;\ \ k_2 \in N$ …..(2)

$x \cdot_Z$ y=$[n,m] \cdot_Z [r,s] = [nr + ms, ns + mr]$ (def. of $\cdot_Z$)

$= [n(s + k_2) + ms, ns + m(s + k_2)]$ (using (2))

$= [ns + nk_2 + ms, ns + ms + mk_2]$

$= [ns + nk_2 + (n + k_1)s, ns + (n + k_1)s + (n + k_1)k_2]$ (using (1))

$= [ns + (n + k_1)s + nk_2,\ ns + (n + k_1)s + (n + k_1)k_2]$

$= \ ns + (n + k_1)s + nk_2 < ns + (n + k_1)s + nk_2 + k_1k_2$

$\Rightarrow \ \ x \cdot_Z y < 0$

Proof 4: $\Rightarrow$) Let $x = [n,m] < y = [r,s]$ T. P. $x +_Z p < y +_Z p$

Since $x < y \Rightarrow$ y − x is positive

$\Rightarrow [r,s] +_Z [m,n]$ is positive

$\Rightarrow [r + m, s + n]$ is positive

$\Rightarrow s + n < r + m$

$\Rightarrow \mathrm{r + m = s + n + k;\ \ k \in N} \ldots..(\mathbf{1})$

Let $p = [u,v];\ \ u,v \in N$ T. P. $x +_Z p < y +_Z p$

T. P. $(y +_Z p) - (x +_Z p)$ is positive

T. P. $[r + u, s + v] - [n + u, m + v]$ is positive

T. P. $[r+u, s+v]+_z[m+v, n+u]$ is positive

T. P. $[r+u+m+v, s+v+n+u]$ is positive

T. P. $\boldsymbol{r+u+m+v > s+v+n+u}$

From (1), $\mathrm{r+m=s+n+k}$; $\mathrm{k} \in \mathrm{N}$

$\Rightarrow \mathrm{r+m+(u+v)=s+n+k+(u+v)}$

$\Rightarrow (\mathrm{r+u+m+v}) = (s+n+u+v)+k$; $k \in N$

$\Rightarrow \boldsymbol{r+u+m+v > s+v+n+u}$

$\Rightarrow x+_Z p < y+_Z p$

$\Leftarrow$) Suppose $x+_Z p < y+_Z p$ T. P. $x < y$

Since $x+_Z p < y+_Z p \Rightarrow r+u+m+v > s+v+n+u$

$\Rightarrow (\mathrm{r+u+m+v}) = (s+n+u+v)+k$; $k \in N$

$\Rightarrow \mathrm{r+m+(u+v)=s+n+k+(u+v)}$

$\Rightarrow \mathrm{r+m=s+n+k}$

$\Rightarrow \mathrm{r+m} > s+n$

$\Rightarrow [r+m, s+n]$ is positive

$\Rightarrow [r,s]+_Z [m,n]$ is positive

$\Rightarrow$ y − x is positive $\Rightarrow \mathrm{x} < y$

<u>Embedding</u>: الغمر

Let $(M, *, \#, R_1)$ and $(L, *', \#', R_2)$ are two number systems with the T.O.R relation R_1, R_2. We say that

$(M,*,\#,R_1)$ is **embedded** **مغمورة**in $(L,*',\#',R_2)$ if and only if there exist $\alpha: M \to L$ such that α is 1-1 and the following conditions hold: For all $n,m \in M$

1. $\alpha(n * m) = \alpha(n) *' \alpha(m)$

2. $\alpha(n\#m) = \alpha(n)\#'\alpha(m)$

3. $n\ R_1\ m \iff \alpha(n)R_2\,\alpha(m)$

Theorem 4.28: $(N \cup \{0\},+,.,\leq)$ is embedded in $(Z,+,.,\leq)$

Proof: Define $\alpha: N \cup \{0\} \to Z$ s.t. $\alpha(n) = [n,0] \quad \forall n \in N$

1. T. P. α is 1-1

Let $\alpha(n) = \alpha(m)$ T. P. $n = m$

$[n,0] = [m,0] \Longrightarrow (\mathrm{n},0)\sim(\mathrm{m},0)$

$$\Longrightarrow n + 0 = m + 0 \Longrightarrow \mathrm{n} = \mathrm{m} \Longrightarrow \alpha \text{ is 1-1}$$

2. T. P. $\alpha(n+m) = \alpha(n) + \alpha(m)$

$$\alpha(n+m) = [n+m,0] = [n,0]+_Z [m,0] \quad (\text{def. of } +_Z)$$

$$= \alpha(n) + \alpha(m) \qquad (\text{def. of } \alpha)$$

3. T. P. $\alpha(n.m) = \alpha(n).\alpha(m)$

$\alpha(n.m) = [n.m,0] = [n,0]._Z\ [m,0]$ (def. of $._Z$)

$$= \alpha(n).\alpha(m) \qquad (\text{def. of } \alpha)$$

4. T. P. $n < m \iff \alpha(n) < \alpha(m)$

$n < m \iff n + 0 < m + 0$

$\iff [n,0] < [m,0]$

$\iff \alpha(n) < \alpha(m)$

$\therefore$ $(N \cup \{0\},+,.,\leq)$ is embedded in $(Z,+,.,\leq)$