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THEOREM:
Let f be continuous on the symmetric interval [-a, a].

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

EXAMPLE: Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

SOL: Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval [- 2, 2]

, so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$
$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$
$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

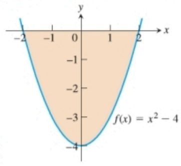
DEFINITION: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b.

$$A = \int_a^b f(x)dx$$

If f(x) is negative then $A = \int_a^b |f(x)|dx$

<u>**EXAMPLE**</u>

Let $f(x) = x^2 - 4$, compute (a) the definite integral over the interval [-2,2], and (b) the area between the graph and the x-axis over [-2,2].



Solution:

(a) $\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3},$

(b) The area between the graph and the x-axis is $\left|-\frac{32}{3}\right| = \frac{32}{3}$

EXAMPLE: Find the area between the graph $\boldsymbol{f(x) = x^3 - 2x^2 - x}$

SOL: f(x)=0 then $\boldsymbol{(x^2 - 1)(x - 2) = 0}$ that is x=1, -1 and x=2

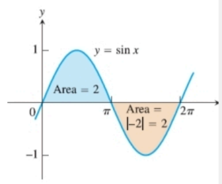
$A = \boldsymbol{A_1 + A_2 = \int_{-1}^{1} |f(x)|dx + \int_{1}^{2} |f(x)|dx}$

$$= \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right] + \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]$$

EXAMPLE: Let the *function* $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.



Solution

(a) The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

(b) To compute the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ we should find the points in which f is intersect x-axis i.e. f(x)=0 this implies to $\sin x=0$ i.e. $x=0$, $x=\pi$ *or* $x=2\pi$

Now subdivide $[0, 2\pi]$ into two pieces: the interval $[0, \pi]$ and the interval $[\pi, 2\pi]$.

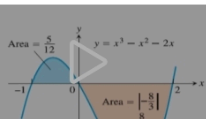
$$\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$
$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

...ea $= |2| + |-2| = 4$.

EXAMPLE:

Find the area of the region between the x-axis and the graph of

$f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2$



Solution

First find the zeros of f. $f(x) = x^3 - x^2 - 2x = 0$

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(b) To compute the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ we should find the points in which f is intersect x-axis i.e. f(x)=0 this implies to sin x=0 i.e. $x=0$, $x=\pi$ or $x=2\pi$
Now subdivide $[0, 2\pi]$ into two pieces: the interval $[0, \pi]$ and the interval $[\pi, 2\pi]$.

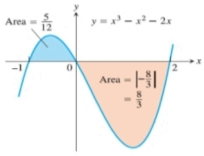
$$\int_0^{\pi} \sin x\, dx = -\cos x\Big]_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$$
$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos\pi] = -[1-(-1)] = -2$$

Area $= |2| + |-2| = 4.$

EXAMPLE:

Find the area of the region between the x-axis and the graph of

$f(x) = x^3 - x^2 - 2x, \qquad -1 \le x \le 2$



Solution

First find the zeros of f. $f(x) = x^3 - x^2 - 2x = 0$
$$x(x^2 - x - 2) = 0$$
$$x(x+1)(x-2) = 0$$

x = 0, -1, and 2 . The zeros subdivide [-1,2] into two subintervals: [-I, 0], on which f≥ 0, and [0, 2], on which $f\le 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$
$$\int_{0}^{2}(x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

Total enclosed area $= \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$

EXAMPLE: Find $\int_{-1}^{2}|x-1|dx$

Since $|x-1| = \begin{cases} x-1 & x\ge 1 \\ -x+1 & x<1 \end{cases}$ then $\int_{-1}^{2}|x-1|dx = \int_{-1}^{1}(-x+1)dx + \int_{1}^{2}(x-1)dx$

.3 Indefinite Integrals and the Substitution Method

Since any two antiderivatives of f differ by a constant, the indefinite integral notation means that for any antiderivative F of f,

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

THEOREM:
The Substitution Rule If u = g(x) is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

tion: Running the Chain Rule Backwards

ifferentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\ldots 1\Big) = u^n \frac{du}{dx}.$$

Therefore $\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx}dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx.$

Sol: let $u = x^3 + x$ then $du = \frac{du}{\ldots}dx = (3x^2+1)\,dx$

THEOREM:
The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Therefore $\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx}dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx.$

Sol: let $u = x^3 + x$. then $du = \frac{du}{dx}dx = (3x^2 + 1)\,dx,$

so that by substitution we have :

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u.$$

EXAMPLE:

Find the integral $\int \sqrt{2x + 1}\,dx.$

SOL: let u=2x+1 and n=1/2, $du = \frac{du}{dx}dx = 2\,dx$

because of the constant factor 2 is missing from the integral. So we write

$$\int \sqrt{2x + 1}\,dx = \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}_{u}\cdot\underbrace{2\,dx}_{du}$$

$$= \frac{1}{2}\int u^{1/2}\,du \qquad \text{Let } u = 2x + 1,\ du = 2\,dx.$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}(2x + 1)^{3/2} + C \qquad \text{Substitute } 2x + 1 \text{ for } u.$$

EXAMPLE: Find $\int \sec^2(5t + 1)\cdot 5\,dt.$

SOL: Let $u = 5t + 1$ and $du = 5\,dx$. Then,

$$\int \sec^2(5t + 1)\cdot 5\,dt = \int \sec^2 u\,du \qquad \text{Let } u = 5t + 1,\ du = 5\,dt.$$

$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \tan(5t + 1) + C \qquad \text{Substitute } 5t + 1 \text{ for } u.$$

EXAMPLE: $\int \cos(7\theta + 3)\,d\theta.$

SOL: Let $u = 7\theta + 3$ so that $du = 7\,d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7. Then,

$$\int \cos(7\theta + 3)\,d\theta = \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\,d\theta \qquad \text{Place factor 1/7 in front of integral.}$$

$$= \frac{1}{7}\int \cos u\,du \qquad \text{Let } u = 7\theta + 3,\ du = 7\,d\theta.$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$

$$= \frac{1}{7}\sin(7\theta + 3) + C \qquad \text{Substitute } 7\theta + 3 \text{ for } u.$$

LE: $\int x^2 \sin(x^3)\,dx = \int \sin(x^3)\cdot x^2\,dx$

$$= \int \sin u\cdot\frac{1}{3}\,du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx.$$

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}(-\cos u) + C \qquad \text{Integrate.}$$

$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Rep ce } u \text{ by } x^3.$$