Find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x),$$

l'Hôpital's Rule now applies to give

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$= \frac{1}{1} = 1.$$

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{L}.$$

Here a may be either finite or infinite.

Therefore, $\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$

EXAMPLE 8 Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

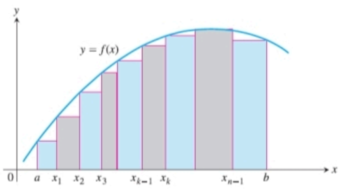
$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1/x}{1}$$

$$= \frac{0}{1} = 0.$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■

.2 INTEGRATION:



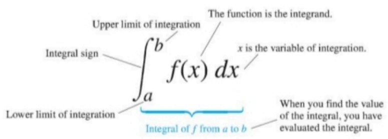
1) The Definite Integral

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right),$$

$\Delta x_k = \Delta x = (b-a)/n$ for all k

$$J = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

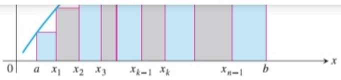
$\Delta x = (b-a)/n$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists

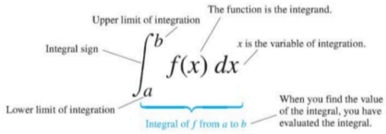
$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right),$$



$\Delta x_k = \Delta x = (b-a)/n$ for all k

$$J = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta$$

$\Delta x = (b-a)/n$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6 $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

EXAMPLE:

Let $\int_{-1}^{1} f(x)\,dx = 5,\quad \int_1^4 f(x)\,dx = -2,$ and $\int_{-1}^{1} h(x)\,dx = 7.$

Then:

1. $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -(-2) = 2$
2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$
   $= 2(5) + 3(7) = 31$
3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_1^4 f(x)\,dx = 5 + (-2) = 3$

2.1 Integration by Substitution

THEOREM Substitution in Definite Integrals: If g' is continuous on the interval *[a,* b] and *f* is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

SOL:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

(a) $\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = \int_1^0 u\cdot(-du)$

Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$,
$-du = \csc^2\theta\,d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) =$
When $\theta = \pi/2$, $u = \cot(\pi/2) =$

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

EXAMPLE:

Let $\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$

Then:

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SOL:

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EXAMPL

(a) $\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = \int_1^0 u\cdot(-du)$

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When $\theta = \pi/2,\ u = \cot(\pi/2) =$

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $\int_{-\pi/4}^{\pi/4} \tan x\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\,dx$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x,\ du = -\sin x\,dx.$
When $x = -\pi/4,\ u = \sqrt{2}/2.$
When $x = \pi/4,\ u = \sqrt{2}/2.$

$$= -\ln|u|\Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$

Integrate, zero width interval

THEOREM:
Let f be continuous on the symmetric interval [-a, a].

(a) If *f* is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

s odd, then $\int_{-a}^{a} f(x)\,dx = 0.$

LE: Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx.$

ince $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval [- 2, 2]

, so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_0^2 (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$

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