Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2+1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$

Example: The Integrals of $\sin^2 x$ and $\cos^2 x$

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

DEFINITION: If u is a differentiable function that is never zero, $\int \frac{1}{u}\,du = \ln|u| + C.$

In general $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

EXAMPLE

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1} \qquad u = x^2-5,\ du = 2x\,dx,\ u(0) = -5,\ u(2) = -1$$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

<u>**The Integrals of tan x, cot x, sec x, and csc x**</u>

1- $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} \qquad u = \cos x > 0 \text{ on } (-\pi/2, \pi/2),\ du = -\sin x\,dx$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C. \qquad \text{Reciprocal Rule}$$

2- $$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} \qquad u = \sin x,\ du = \cos x\,dx$$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

3- $$\int \sec x\,dx = \int \sec x\,\frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C \qquad u = \sec x + \tan x,\ du = (\sec x\tan x + \sec^2 x)\,dx$$

4- $$\int \csc x\,dx = \int \csc x\,\frac{(\csc x + \cot x)}{(\csc x + \cot x)}\,dx = \int \frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}\,dx$$

$$\int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C \qquad u = \csc x + \cot x,\ du = (-\csc x\cot x - \csc^2 x)\,dx$$

...egrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

EXAMPLE:

$$\int e^u\,du = e^u + C$$

EXAMPLE :

(a) $\displaystyle\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du$ $u = 3x,\ \frac{1}{3}du = dx,\ u(0) = 0,\ u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$\displaystyle = \frac{1}{3}\int_0^{\ln 8} e^u\,du$

$\displaystyle = \frac{1}{3}e^u\Big]_0^{\ln 8}$

(b) $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2}$ Antiderivative from Example 2c

$= e^1 - e^0 = e - 1$

The integral of a^u

$$\int a^u\,du = \frac{a^u}{\ln a} + C.$$

EXAMPLE :

(a) $\frac{d}{dx}3^x = 3^x \ln 3$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

(d) $\displaystyle\int 2^x\,dx = \frac{2^x}{\ln 2} + C$

(e) $\displaystyle\int 2^{\sin x}\cos x\,dx = \int 2^u\,du = \frac{2^u}{\ln 2} + C$

$\displaystyle = \frac{2^{\sin x}}{\ln 2} + C$

Example :

(a) $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$

(b) $\displaystyle\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx$ $\log_2 x = \frac{\ln x}{\ln 2}$

$\displaystyle = \frac{1}{\ln 2}\int u\,du$ $u = \ln x,\ du = \frac{1}{x}dx$

$\displaystyle = \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$

...tion Formulas

$\displaystyle \int\frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

$\displaystyle \int\frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int\frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE

(a) $\displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2}\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

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$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

Integration Formulas

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)
2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)
3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE

(a) $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

(b) $\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

(c) $\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}}$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

Example :

$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x} + 1}$ Multiply by $(e^x/e^x) = 1$.

$= \int \frac{du}{u^2 + 1}$ Let $u = e^x$, $u^2 = e^{2x}$, $du = e^x\,dx$.

$= \tan^{-1} u + C$ Integrate with respect to u.

$= \tan^{-1}(e^x) + C$ Replace u by e^x.

Example

(a) $\int \frac{dx}{\sqrt{4x - x^2}}$ **(b)** $\int \frac{dx}{4x^2 + 4x + 2}$

Solution

(a) we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2.$$

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x - 2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sin^{-1}\left(\frac{x - 2}{2}\right) + C$$

(b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$