تفاضل وتكامل ٢ م...

.1 L'Hopital's Rule

THEOREM L'Hopital's Rule: Suppose that f(a) = g(a) = 0 or ∞, that f and g are differentiable on an open interval I containing a, and that g' (x) ≠0 on I if x ≠a. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

Indeterminate Form 0/0

EXAMPLE 1

(a) $\lim_{x\to 0}\frac{3x-\sin x}{x} = \lim_{x\to 0}\frac{3-\cos x}{1} = \frac{3-\cos x}{1}\Big|_{x=0} = 2$

(b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

(c) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$

$$= \lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$

$$= \lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$

(d) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

$$= \lim_{x\to 0}\frac{1-\cos x}{3x^2}$$

$$= \lim_{x\to 0}\frac{\sin x}{6x}$$

$$= \lim_{x\to 0}\frac{\cos x}{6} = \frac{1}{6}$$

EXAMPLE 2

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$$

$$= \lim_{x\to 0}\frac{\sin x}{1+2x} = \frac{0}{1} = 0.$$

EXAMPLE 3

(a) $\lim_{x\to 0^+}\frac{\sin x}{x^2}$

$$= \lim_{x\to 0^+}\frac{\cos x}{2x} = \infty$$

(b) $\lim_{x\to 0^-}\frac{\sin x}{x^2}$

$$= \lim_{x\to 0^-}\frac{\cos x}{2x} = -\infty$$

Indeterminate Forms ∞\ ∞ , ∞.0 and ∞-∞

EXAMPLE 4: find the limit:

(a) $\lim_{x\to\pi/2}\frac{\sec x}{1+\tan x}$ **(b)** $\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty}\frac{e^x}{x^2}$.

Solution:

(a) $\lim_{x\to(\pi/2)^-}\frac{\sec x}{1+\tan x}$ $\frac{\infty}{\infty}$ from the left

$$= \lim_{x\to(\pi/2)^-}\frac{\sec x\tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-}\sin x = 1$$

m ∞ $\frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}} = 0$ $\frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

n ∞ $\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty$

LE 5: Find the limits of these ∞ • 0 forms:

(a) $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+}\sqrt{x}\ln x$

(a) $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right) = \lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right) = \lim_{h\to 0^+}\frac{\sin h}{h} = 1$ $\infty\cdot 0$; Let $h = 1/x$.

(b) $\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}$ $\infty\cdot 0$ converted to ∞/∞

(a) $\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x}$ **(b)** $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution:

(a) $\lim_{x\to(\pi/2)^-} \frac{\sec x}{1+\tan x}$ $\frac{\infty}{\infty}$ from the left

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

(b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$ $\frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

(c) $\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$

EXAMPLE 5: Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x\to0^+} \sqrt{x} \ln x$

(a) $\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right) = \lim_{h\to0^+} \left(\frac{1}{h} \sin h\right) = \lim_{h\to0^+} \frac{\sin h}{h} = 1$ $\infty \cdot 0$; Let $h = 1/x$.

(b) $\lim_{x\to0^+} \sqrt{x} \ln x = \lim_{x\to0^+} \frac{\ln x}{1/\sqrt{x}}$ $\infty \cdot 0$ converted to ∞/∞

$$= \lim_{x\to0^+} \frac{1/x}{-1/2x^{3/2}}$$ l'Hôpital's Rule

$$= \lim_{x\to0^+} (-2\sqrt{x}) = 0$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to0} \left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

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Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$ Common denominator is $x \sin x$.

Then we apply l'Hôpital's Rule to the result:

$$\lim_{x\to0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to0} \frac{x - \sin x}{x \sin x}$$ $\frac{0}{0}$

$$= \lim_{x\to0} \frac{1 - \cos x}{\sin x + x \cos x}$$ Still $\frac{0}{0}$

$$= \lim_{x\to0} \frac{\sin x}{2\cos x - x \sin x} = \frac{0}{2} = 0.$$ ■

Indeterminate Powers

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x\to0^+} (1+x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1+x)^{1/x}$ and find $\lim_{x\to0^+} \ln f(x)$. Since find $\lim_{x\to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x),$$

l'Hôpital's Rule now applies to give

$$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1}$$

$$= \frac{1}{1} = 1.$$

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{L}.$$

Here a may be either finite or infinite.

Therefore, $\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^1 = e.$

EXAMPLE 8 Find $\lim_{x\to\infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x\to\infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}$$

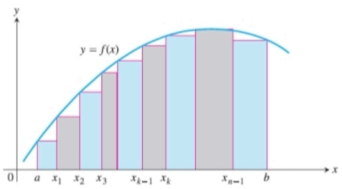
$$= \lim_{x\to\infty} \frac{1/x}{1}$$

$$= \frac{0}{1} = 0.$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■

.2 INTEGRATION:

1) The Definite Integral

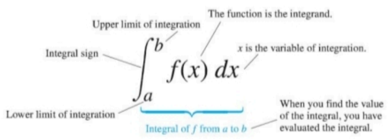


$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right),$$

$\Delta x_k = \Delta x = (b-a)/n$ for all k

$$J = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

$\Delta x = (b-a)/n$

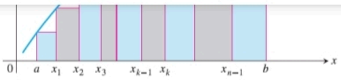


Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ [illegible] A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ A Definition when $f(a)$ exists

[illegible]

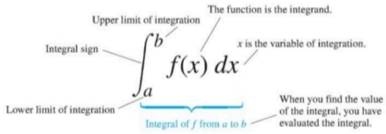
$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right),$$



$\Delta x_k = \Delta x = (b-a)/n$ for all k

$$J = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

$\Delta x = (b-a)/n$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6 $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0$ (Special Case)

EXAMPLE:

Let $\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad$ and $\quad \int_{-1}^{1} h(x)\,dx = 7.$

Then:

1. $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -(-2) = 2$
2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$
   $= 2(5) + 3(7) = 31$
3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

2.1 Integration by Substitution

THEOREM Substitution in Definite Integrals: If g' is continuous on the interval *[a,* b] and *f* is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

SOL:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

(a) $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta\,d\theta = \int_1^0 u\cdot(-du)$

Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$,
$-du = \csc^2\theta\,d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) =$
When $\theta = \pi/2$, $u = \cot(\pi/2) =$

$$= -\int_1^0 u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

EXAMPLE:

Let $\int_{-1}^{1} f(x)\,dx = 5,\quad \int_{1}^{4} f(x)\,dx = -2,\quad$ and $\quad \int_{-1}^{1} h(x)\,dx = 7.$

Then:

1. $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$
2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$
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2.1 Integration by Substitution

THEOREM Substitution in Definite Integrals: If g' is continuous on the interval *[a,* b] and *f* is continuous on the range of $g(x) = u$, then

$$\int_{a}^{b} f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

SOL:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1,\ du = 3x^2\,dx.$
When $x = -1,\ u = (-1)^3 + 1 = 0.$
When $x = 1,\ u = (1)^3 + 1 = 2.$

$$= \int_{0}^{2} \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_{0}^{2}$$

Evaluate the new definite integral.

EXAMPL

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

(a) $\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = \int_{1}^{0} u\cdot(-du)$

Let $u = \cot\theta,\ du = -\csc^2\theta\,d\theta,$
$-du = \csc^2\theta\,d\theta.$
When $\theta = \pi/4,\ u = \cot(\pi/4) =$
When $\theta = \pi/2,\ u = \cot(\pi/2) =$

$$= -\int_{1}^{0} u\,du$$

$$= -\left[\frac{u^2}{2}\right]_{1}^{0}$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $\int_{-\pi/4}^{\pi/4} \tan x\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\,dx$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x,\ du = -\sin x\,dx.$
When $x = -\pi/4,\ u = \sqrt{2}/2.$
When $x = \pi/4,\ u = \sqrt{2}/2.$

$$= -\ln|u|\Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$

Integrate, zero width interval

THEOREM:
Let f be continuous on the symmetric interval [-a, a].

(a) If *f* is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

s odd, then $\int_{-a}^{a} f(x)\,dx = 0.$

LE: Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx.$

ince $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval [- 2, 2]
, so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_{0}^{2}$$

THEOREM:
Let f be continuous on the symmetric interval [-a, a].

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

EXAMPLE: Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

SOL: Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval [- 2, 2]

, so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$
$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$
$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

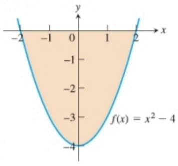
DEFINITION: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b.

$$A=\int_a^b f(x)dx$$

If f(x) is negative then $A=\int_a^b |f(x)|dx$

EXAMPLE

Let $f(x) = x^2 - 4$, compute (a) the definite integral over the interval [-2,2], and (b) the area between the graph and the x-axis over [-2,2].



Solution:

(a) $\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3},$

(b) The area between the graph and the x-axis is $\left|-\frac{32}{3}\right| = \frac{32}{3}$

EXAMPLE: Find the area between the graph $f(x) = x^3 - 2x^2 - x$

SOL: f(x)=0 then $(x^2 - 1)(x - 2) = 0$ that is x=1, -1 and x=2

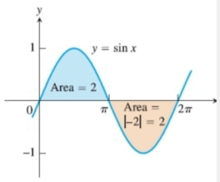
$A = A_1 + A_2 = \int_{-1}^{1} |f(x)|dx + \int_{1}^{2} |f(x)|dx$

$$= \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right] + \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]$$

EXAMPLE: Let the *function* $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.



Solution

(a) The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

(b) To compute the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ we should find the points in which f is intersect x-axis i.e. f(x)=0 this implies to $\sin x=0$ i.e. $x=0$, $x=\pi$ *or* $x=2\pi$
Now subdivide $[0, 2\pi]$ into two pieces: the interval $[0, \pi]$ and the interval $[\pi, 2\pi]$.

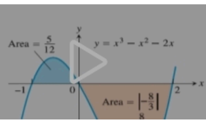
$$\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$
$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

ea $= |2| + |-2| = 4.$

EXAMPLE:

Find the area of the region between the x-axis and the graph of

$f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2$



Solution

First find the zeros of f. $f(x) = x^3 - x^2 - 2x = 0$

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(b) To compute the area between the graph of *f(x)* and the x-axis over $[0, 2\pi]$ we should find the points in which f is intersect x-axis i.e. f(x)=0 this implies to sin x=0 i.e. $x=0$, $x=\pi$ or $x=2\pi$
Now subdivide $[0, 2\pi]$ into two pieces: the interval $[0, \pi]$ and the interval $[\pi, 2\pi]$.

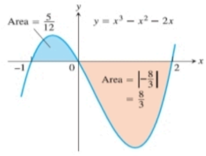
$$\int_0^{\pi} \sin x\, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$
$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

Area $= |2| + |-2| = 4.$

EXAMPLE:

Find the area of the region between the x-axis and the graph of

$\boldsymbol{f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2}$



Solution

First find the zeros of f. $f(x) = x^3 - x^2 - 2x = 0$

$$x(x^2 - x - 2) = 0$$
$$x(x + 1)(x - 2) = 0$$

x = 0, -1, and 2 . The zeros subdivide [-1,2] into two subintervals: [-I, 0], on which f≥ 0, and [0, 2], on which *f*≤*0*. We integrate *f* over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$
$$\int_{0}^{2} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

EXAMPLE: Find $\int_{-1}^{2} |x - 1|dx$

Since $|x - 1| = \begin{cases} x - 1 & x \ge 1 \\ -x + 1 & x < 1 \end{cases}$ then $\int_{-1}^{2} |x - 1|dx = \int_{-1}^{1} (-x + 1)dx + \int_{1}^{2} (x - 1)dx$

.3 Indefinite Integrals and the Substitution Method

Since any two antiderivatives of f differ by a constant, the indefinite integral notation means that for any antiderivative *F* of f,

$$\int f(x)\, dx = F(x) + C,$$

where C is any arbitrary constant.

THEOREM:
The Substitution Rule If u = g(x) is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

tion: Running the Chain Rule Backwards

ifferentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\ldots 1\Big) = u^n \frac{du}{dx}.$$

Therefore $\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n\, du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx} dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5(3x^2 + 1)\, dx.$

Sol: let $u = x^3 + x$ then ...

THEOREM:
The Substitution Rule If u = g(x) is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n\frac{du}{dx}.$$

Therefore $\int u^n\frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx}\,dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx.$

Sol: let $u = x^3 + x$. then $du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$

so that by substitution we have :

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$
$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$
$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u.$$

EXAMPLE:

Find the integral $\int \sqrt{2x + 1}\,dx.$

SOL: let u=2x+1 and n=1/2, $du = \frac{du}{dx}\,dx = 2\,dx$

because of the constant factor 2 is missing from the integral. So we write

$$\int \sqrt{2x + 1}\,dx = \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}_{u}\cdot\underbrace{2\,dx}_{du}$$
$$= \frac{1}{2}\int u^{1/2}\,du \qquad \text{Let } u = 2x + 1,\ du = 2\,dx.$$
$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate with respect to } u.$$
$$= \frac{1}{3}(2x + 1)^{3/2} + C \qquad \text{Substitute } 2x + 1 \text{ for } u.$$

EXAMPLE: Find $\int \sec^2(5t + 1)\cdot 5\,dt.$

SOL: Let $u = 5t + 1$ and $du = 5\,dx$. Then,

$$\int \sec^2(5t + 1)\cdot 5\,dt = \int \sec^2 u\,du \qquad \text{Let } u = 5t + 1,\ du = 5\,dt.$$
$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$
$$= \tan(5t + 1) + C \qquad \text{Substitute } 5t + 1 \text{ for } u.$$

EXAMPLE: $\int \cos(7\theta + 3)\,d\theta.$

SOL: Let $u = 7\theta + 3$ so that $du = 7\,d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7. Then,

$$\int \cos(7\theta + 3)\,d\theta = \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\,d\theta \qquad \text{Place factor 1/7 in front of integral.}$$
$$= \frac{1}{7}\int \cos u\,du \qquad \text{Let } u = 7\theta + 3,\ du = 7\,d\theta.$$
$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$
$$= \frac{1}{7}\sin(7\theta + 3) + C \qquad \text{Substitute } 7\theta + 3 \text{ for } u.$$

LE: $\int x^2\sin(x^3)\,dx = \int \sin(x^3)\cdot x^2\,dx$

$$= \int \sin u\cdot\frac{1}{3}\,du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx.$$
$$= \frac{1}{3}\int \sin u\,du$$
$$= \frac{1}{3}(-\cos u) + C \qquad \text{Integrate.}$$
$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

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EXAMPLE: $\int \cos(7\theta + 3)\, d\theta.$

SOL: Let $u = 7\theta + 3$ so that $du = 7\, d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7. Then,

$$\begin{aligned}
\int \cos(7\theta + 3)\, d\theta &= \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\, d\theta && \text{Place factor 1/7 in front of integral.}\\
&= \frac{1}{7}\int \cos u\, du && \text{Let } u = 7\theta + 3,\ du = 7\, d\theta.\\
&= \frac{1}{7}\sin u + C && \text{Integrate.}\\
&= \frac{1}{7}\sin(7\theta + 3) + C && \text{Substitute } 7\theta + 3 \text{ for } u.
\end{aligned}$$

EXAMPLE:

$$\begin{aligned}
\int x^2 \sin(x^3)\, dx &= \int \sin(x^3)\cdot x^2\, dx\\
&= \int \sin u \cdot \frac{1}{3}\, du && \text{Let } u = x^3,\ du = 3x^2\, dx,\ (1/3)\, du = x^2\, dx.\\
&= \frac{1}{3}\int \sin u\, du\\
&= \frac{1}{3}(-\cos u) + C && \text{Integrate.}\\
&= -\frac{1}{3}\cos(x^3) + C && \text{Replace } u \text{ by } x^3.
\end{aligned}$$

EXAMPLE: Evaluate $\int x\sqrt{2x + 1}\, dx$

SOL: $u = 2x + 1$ to obtain $x = (u - 1)/2$, and find that $x\sqrt{2x + 1}\, dx = \frac{1}{2}(u - 1)\cdot\frac{1}{2}\sqrt{u}\, du.$

The integration now becomes

$$\begin{aligned}
\int x\sqrt{2x + 1}\, dx &= \frac{1}{4}\int (u - 1)\sqrt{u}\, du = \frac{1}{4}\int (u - 1)u^{1/2}\, du && \text{Substitute.}\\
&= \frac{1}{4}\int (u^{3/2} - u^{1/2})\, du && \text{Multiply terms.}\\
&= \frac{1}{4}\left(\frac{2}{5}u^{5/2} \int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} . C\right. && \text{Integrate.}\\
&= \frac{1}{10}(2x + 1)^{5/2} - \frac{1}{6}(2x + 1)^{3/2} + C && \text{Replace } u \text{ by } 2x + 1. \ \blacksquare
\end{aligned}$$

$$\begin{aligned}
\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2 + 1,\ du = 2z\, dz.\\
&= \int u^{-1/3}\, du && \text{In the form } \int u^n\, du\\
&= \frac{u^{2/3}}{2/3} + C && \text{Integrate.}\\
&= \frac{3}{2}u^{2/3} + C\\
&= \frac{3}{2}(z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } z^2 + 1.
\end{aligned}$$

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\begin{aligned}
\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{3u^2\, du}{u} && \text{Let } u = \sqrt[3]{z^2 + 1},\ u^3 = z^2 + 1,\ 3u^2\, du = 2z\, dz.\\
&= 3\int u\, du\\
&= 3\cdot\frac{u^2}{2} + C && \text{Integrate.}\\
&= \frac{3}{2}(z^2 + 1)^{2/3} + C && \text{Replace } u \text{ by } (z^2 + 1)^{1/3}.
\end{aligned}$$

Example: The Integrals of $\sin^2 x$ and $\cos^2 x$

(a) $\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$

Method 2: Substitute $u = \sqrt[3]{z^2+1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u}$$ Let $u = \sqrt[3]{z^2+1}$, $u^3 = z^2+1$, $3u^2\,du = 2z\,dz$.

$$= 3\int u\,du$$

$$= 3\cdot\frac{u^2}{2} + C$$ Integrate.

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $(z^2+1)^{1/3}$.

Example: The Integrals of $\sin^2 x$ and $\cos^2 x$

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$$ $\sin^2 x = \frac{1-\cos 2x}{2}$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$ $\cos^2 x = \frac{1+\cos 2x}{2}$ ■

DEFINITION: If u is a differentiable function that is never zero, $\int \frac{1}{u}\,du = \ln|u| + C.$

In general $$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

EXAMPLE

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1}$$ $u = x^2-5$, $du = 2x\,dx$, $u(0) = -5$, $u(2) = -1$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

<u>The Integrals of tan x, cot x, sec x, and csc x</u>

1- $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$$ $u = \cos x > 0$ on $(-\pi/2, \pi/2)$, $du = -\sin x\,dx$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C.$$ Reciprocal Rule

2- $$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$$ $u = \sin x$, $du = \cos x\,dx$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

3- $$\int \sec x\,dx = \int \sec x\,\frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$ $u = \sec x + \tan x$, $du = (\sec x\tan x + \sec^2 x)\,dx$

4- $$\int \csc x\,dx = \int \csc x\,\frac{(\csc x + \cot x)}{(\csc x + \cot x)}\,dx = \int \frac{\csc^2 x + \csc x\cot x}{\csc x + \cot x}\,dx$$

$$\int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$$ $u = \csc x + \cot x$, $du = (-\csc x\cot x - \csc^2 x)\,dx$

...egrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

EXAMPLE:

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$$\int e^u\,du = e^u + C$$

EXAMPLE :

(a) $\displaystyle\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du$ $\qquad u = 3x,\ \frac{1}{3}du = dx,\ u(0) = 0,$ $u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du$$

$$= \frac{1}{3}e^u\Big]_0^{\ln 8}$$

(b) $\displaystyle\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2}$ $\qquad$ Antiderivative from Example 2c

$$= e^1 - e^0 = e - 1$$

The integral of a^u

$$\int a^u\,du = \frac{a^u}{\ln a} + C.$$

EXAMPLE :

(a) $\dfrac{d}{dx}3^x = 3^x \ln 3$

(b) $\dfrac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\dfrac{d}{dx}(-x) = -3^{-x}\ln 3$

(c) $\dfrac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\dfrac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

(d) $\displaystyle\int 2^x\,dx = \frac{2^x}{\ln 2} + C$

(e) $\displaystyle\int 2^{\sin x}\cos x\,dx = \int 2^u\,du = \frac{2^u}{\ln 2} + C$

$$= \frac{2^{\sin x}}{\ln 2} + C$$

Example :

(a) $\dfrac{d}{dx}\log_{10}(3x + 1) = \dfrac{1}{\ln 10}\cdot\dfrac{1}{3x + 1}\dfrac{d}{dx}(3x + 1) = \dfrac{3}{(\ln 10)(3x + 1)}$

(b) $\displaystyle\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx$ $\qquad \log_2 x = \frac{\ln x}{\ln 2}$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x,\ du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

tion Formulas

$\displaystyle \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ $\qquad$ (Valid for $u^2 < a^2$)

$\displaystyle \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ $\qquad$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ $\qquad$ (Valid for $|u| > a > 0$)

EXAMPLE

(a) $\displaystyle\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1}x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

Integration Formulas

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

(c) $$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}}$$
$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$
$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$
$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

Example :

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x} + 1}$$ Multiply by $(e^x/e^x) = 1$.

$$= \int \frac{du}{u^2 + 1}$$ Let $u = e^x$, $u^2 = e^{2x}$, $du = e^x\,dx$.

$$= \tan^{-1} u + C$$ Integrate with respect to u.

$$= \tan^{-1}(e^x) + C$$ Replace u by e^x.

Example

(a) $\displaystyle\int \frac{dx}{\sqrt{4x - x^2}}$ **(b)** $\displaystyle\int \frac{dx}{4x^2 + 4x + 2}$

Solution

(a) we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2.$$

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x - 2)^2}}$$
$$= \int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$
$$= \sin^{-1}\left(\frac{x - 2}{2}\right) + C$$

(b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$