

Calculus 2

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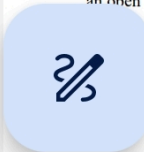
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.1 L'Hopital's Rule

THEOREM L'Hopital's Rule: Suppose that $f(a) = g(a) = 0$ or ∞ , that f and g are differentiable on an open interval I containing a , and that $g'(x) \neq 0$ on I if $x \neq a$. Then



$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

Indeterminate Form 0/0

LE 1

(a) $\lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \lim_{x \rightarrow 0} \frac{3 - \cos x}{1} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$

(b) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

(c) $\lim_{x \rightarrow 2} \frac{\sqrt{1+x} - 1 - x/2}{x-2}$

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EXAMPLE 1

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(c) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$
 $= \lim_{x \rightarrow 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$
 $= \lim_{x \rightarrow 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$

(d) $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$
 $= \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2}$
 $= \lim_{x \rightarrow 0} \frac{\sin x}{6x}$
 $= \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6}$

EXAMPLE 2

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0$$

EXAMPLE 3

(a) $\lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = \infty$

(b) $\lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} = \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = -\infty$

Indeterminate Forms ∞/∞ , $\infty \cdot 0$ and $\infty - \infty$

EXAMPLE 4: find the limit:

(a) $\lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x}$ (b) $\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}}$ (c) $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$

Solution:

(a) $\lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{1 + \tan x} = \frac{\infty}{\infty}$ from the left
 $= \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow (\pi/2)^-} \sin x = 1$

(b) $\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0$ $\frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

(c) $\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$



EXAMPLE 5: Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right)$ (b) $\lim_{x \rightarrow 0^+} \sqrt{x} \ln x$

(a) $\lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) = \lim_{h \rightarrow 0^+} \left(\frac{1}{h} \sin h \right) = \lim_{h \rightarrow 0^+} \frac{\sin h}{h} = 1$ $\infty \cdot 0$; Let $h = 1/x$

(b) $\lim_{x \rightarrow 0^+} \sqrt{x} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/\sqrt{x}}$ $\infty \cdot 0$ converted to ∞/∞

(a) $\lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x}$ (b) $\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}}$ (c) $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$

Solution:

(a) $\lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{1 + \tan x} \quad \frac{\infty}{\infty} \text{ from the left}$
 $= \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow (\pi/2)^-} \sin x = 1$
 (b) $\lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0 \quad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$
 (c) $\lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$

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 (b) $\lim_{x \rightarrow 0^+} \sqrt{x} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/\sqrt{x}} \quad \infty \cdot 0 \text{ converted to } \infty/\infty$
 $= \lim_{x \rightarrow 0^+} \frac{1/x}{-1/2x^{3/2}} \quad \text{l'Hôpital's Rule}$
 $= \lim_{x \rightarrow 0^+} (-2\sqrt{x}) = 0$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

Solution If $x \rightarrow 0^+$, then $\sin x \rightarrow 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow \infty - \infty.$$

Similarly, if $x \rightarrow 0^-$, then $\sin x \rightarrow 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow -\infty - (-\infty) = -\infty + \infty.$$

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Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \quad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} \quad \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} \quad \text{Still } \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. \quad \blacksquare \end{aligned}$$

Indeterminate Powers

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x \rightarrow 0^+} (1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞ . We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \rightarrow 0^+} \ln f(x)$. Since

