Calculus 2

References:

1. **Maurice Weir, Joel Hass, George B. Thomas,** *Thomas Calculus*, 12th ed. (2012).
2. **G Stephenson** Mathematical Methods for Science Students (1983).
3. Anton Bivens Davis *Calculus* (2002).

.1 L'Hopital's Rule

THEOREM L'Hopital's Rule: Suppose that f(a) = g(a) = 0 or ∞, that f and g are differentiable on an open interval I containing a, and that g' (x) ≠0 on I if x ≠a. Then

$$= \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

inate Form 0/0

LE 1

(a) $\lim_{x\to 0} \frac{3x - \sin x}{x} = \lim_{x\to 0} \frac{3 - \cos x}{1} = \frac{3 - \cos x}{1}\Big|_{x=0} = 2$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

(c) $\lim \frac{\sqrt{1+x} - 1 - x/2}{x^2}$

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THEOREM L'Hopital's Rule: Suppose that f(a) = g(a) = 0 or ∞, that f and g are differentiable on an open interval I containing a, and that g' (x) ≠0 on I if x ≠a. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

Indeterminate Form 0/0

EXAMPLE 1

(a) $\lim_{x\to 0}\frac{3x-\sin x}{x}=\lim_{x\to 0}\frac{3-\cos x}{1}=\frac{3-\cos x}{1}\Big|_{x=0}=2$

(b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$

(c) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-x/2}{x^2}$

$$=\lim_{x\to 0}\frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$

$$=\lim_{x\to 0}\frac{-(1/4)(1+x)^{-3/2}}{2}=-\frac{1}{8}$$

(d) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}$$

$$=\lim_{x\to 0}\frac{\sin x}{6x}$$

$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$$

EXAMPLE 2

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$$

$$=\lim_{x\to 0}\frac{\sin x}{1+2x}=\frac{0}{1}=0.$$

EXAMPLE 3

(a) $\lim_{x\to 0^+}\frac{\sin x}{x^2}$

$$=\lim_{x\to 0^+}\frac{\cos x}{2x}=\infty$$

(b) $\lim_{x\to 0^-}\frac{\sin x}{x^2}$

$$=\lim_{x\to 0^-}\frac{\cos x}{2x}=-\infty$$

Indeterminate Forms ∞\ ∞ , ∞.0 and ∞-∞

EXAMPLE 4: find the limit:

(a) $\lim_{x\to\pi/2}\frac{\sec x}{1+\tan x}$ **(b)** $\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty}\frac{e^x}{x^2}$.

Solution:

(a) $\lim_{x\to(\pi/2)^-}\frac{\sec x}{1+\tan x}$ $\frac{\infty}{\infty}$ from the left

$$=\lim_{x\to(\pi/2)^-}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to(\pi/2)^-}\sin x=1$$

$\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{1/\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}}=0$ $\frac{1/x}{1/\sqrt{x}}=\frac{\sqrt{x}}{x}=\frac{1}{\sqrt{x}}$

$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{e^x}{2x}=\lim_{x\to\infty}\frac{e^x}{2}=\infty$

…PLE 5: Find the limits of these ∞ • 0 forms:

(a) $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$ **(b)** $\lim_{x\to 0^+}\sqrt{x}\ln x$

(a) $\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)=\lim_{h\to 0^+}\left(\frac{1}{h}\sin h\right)=\lim_{h\to 0^+}\frac{\sin h}{h}=1$ $\infty\cdot 0$; Let $h=1/x$.

(b) $\lim_{x\to 0^+}\sqrt{x}\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}$ $\infty\cdot 0$ converted to ∞/∞

(a) $\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Solution:

(a) $\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x}$ $\frac{\infty}{\infty}$ from the left

$$= \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

(b) $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \to \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$ $\frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

(c) $\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$

EXAMPLE 5: Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x \to \infty} \left(x \sin \frac{1}{x}\right)$ **(b)** $\lim_{x \to 0^+} \sqrt{x} \ln x$

(a) $\lim_{x \to \infty} \left(x \sin \frac{1}{x}\right) = \lim_{h \to 0^+} \left(\frac{1}{h} \sin h\right) = \lim_{h \to 0^+} \frac{\sin h}{h} = 1$ $\infty \cdot 0$; Let $h = 1/x$.

(b) $\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/\sqrt{x}}$ $\infty \cdot 0$ converted to ∞/∞

$$= \lim_{x \to 0^+} \frac{1/x}{-1/2x^{3/2}}$$ l'Hôpital's Rule

$$= \lim_{x \to 0^+} (-2\sqrt{x}) = 0$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

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Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$ Common denominator is $x \sin x$.

Then we apply l'Hôpital's Rule to the result:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x}$$ $\frac{0}{0}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x}$$ Still $\frac{0}{0}$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.$$ ■

Indeterminate Powers

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \to 0^+} \ln f(x)$. Since