Example: Assume that we have a Universal Set T: temperature in Celsius

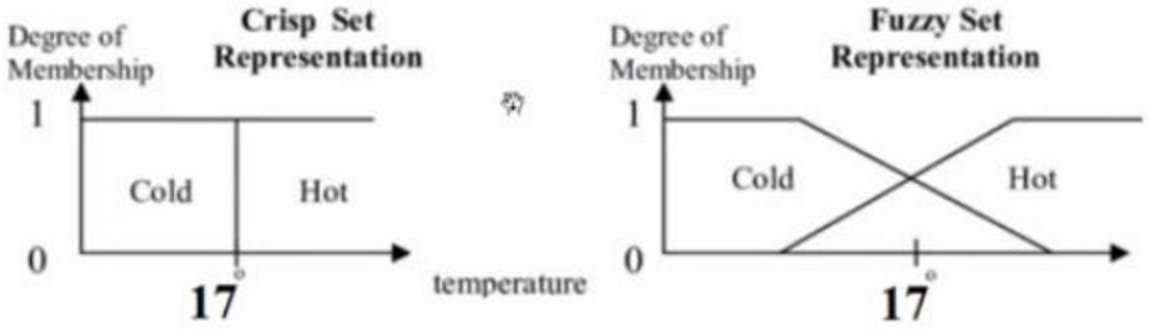
- T ={Cold , Warm , Hot}

  $\Rightarrow$ Cold $= \{t | 7 \leq t \leq 17, t \in T\}$

  $\Rightarrow$ Medium $= \{t | 17 \leq t \leq 28, t \in T\}$

  $\Rightarrow$ Hot $= \{t | 28 \leq t \leq 40, t \in T\}$

Crisp set has a problem of sudden transition from one state to another,

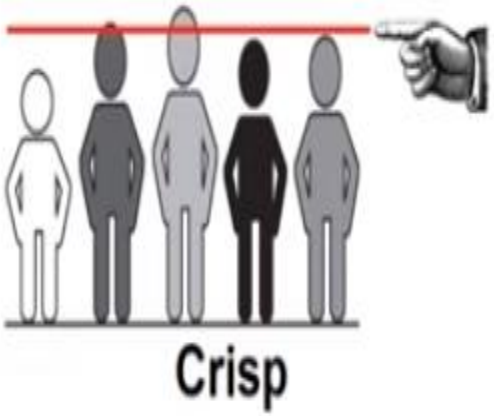


- The basic idea of the fuzzy set theory is that an element belongs to a fuzzy set with a certain degree of membership.
- Hence, a proposition is not either true or false, but may be partly true (or partly false) to any degree.
- This degree is usually taken as a real number in the interval [0,1].

Example:

- Crisp: Is the man tall? We need to draw a line at 180 cm.
- Crisp: Tall men are above 180 cm and not tall men below. Is E tall? the answer is 0
- In contrast, the fuzzy set asks: How tall is D? The answer is the partial membership in the fuzzy set, for example, D is 0.82 tall.

		Degree of membership	
Name	Height (cm)	Crisp	Fuzzy
A	210	1	1.00
B	204	1	1.00
C	197	1	0.98
D	182	1	0.82
E	178	0	0.78
F	142	0	0.24
G	140	0	0.15
H	130	0	0

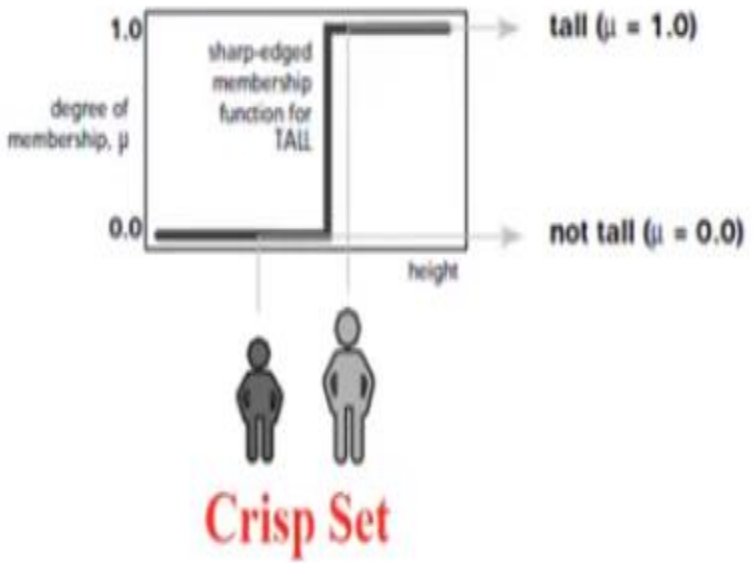


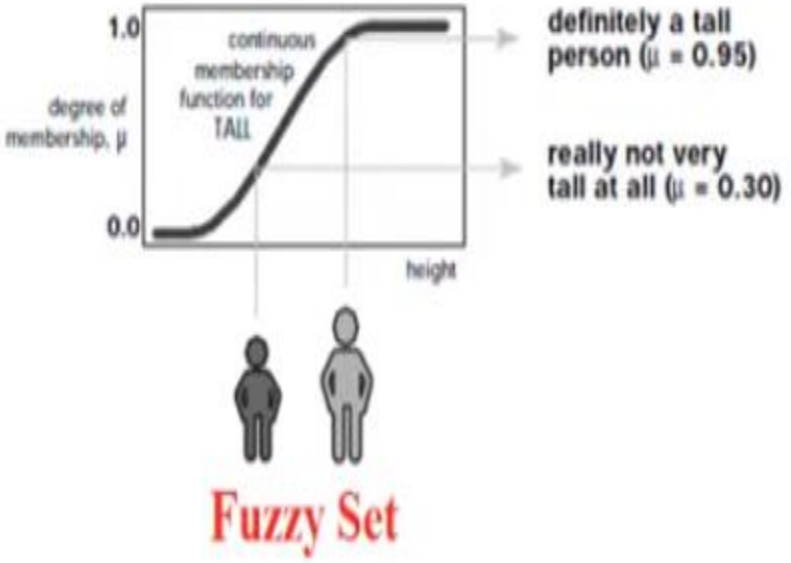
Example:

$$\text{Crisp}: \quad f_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}, \quad \text{where A is a set and } f_A(x) \in \{0, 1\}$$

$$\text{Fuzzy}: \quad \mu_A(x) = \begin{cases} 1, & x \text{ is totally in } A \\ 0 < \mu_A(x) < 1, & x \text{ is partially in } A, \ \mu_A(x) \in [0, 1] \\ 0, & x \text{ is not in } A \end{cases}$$

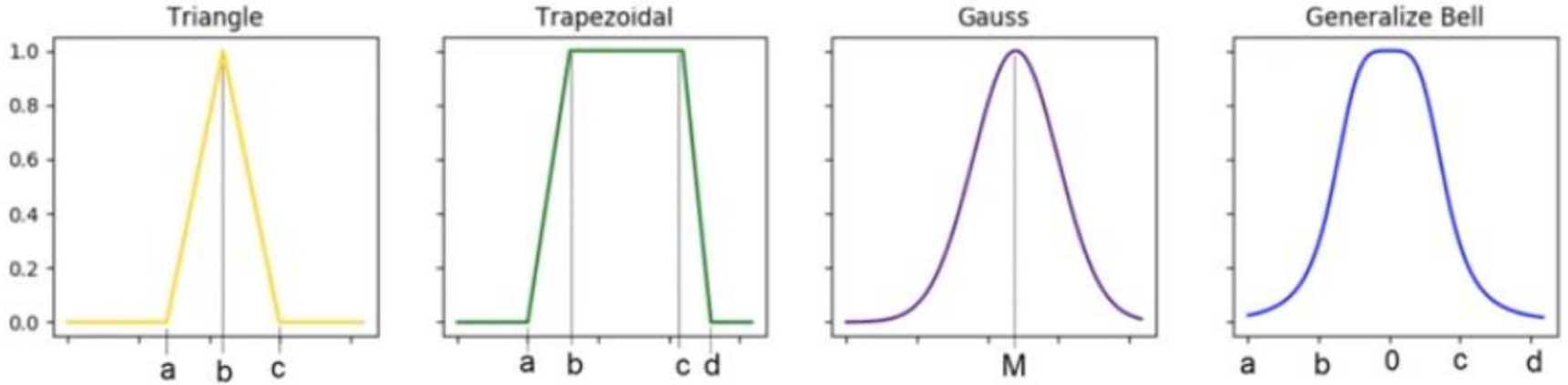
$\mu_A(x)$ is called the membership function of set A.





- **Fuzzification** (or Fuzzifier): Translate input into truth values
- **Rule Evaluation** (or Inference Engine): Compute output truth values
- **Defuzzification** (or Defuzzifier): Transfer truth values into output

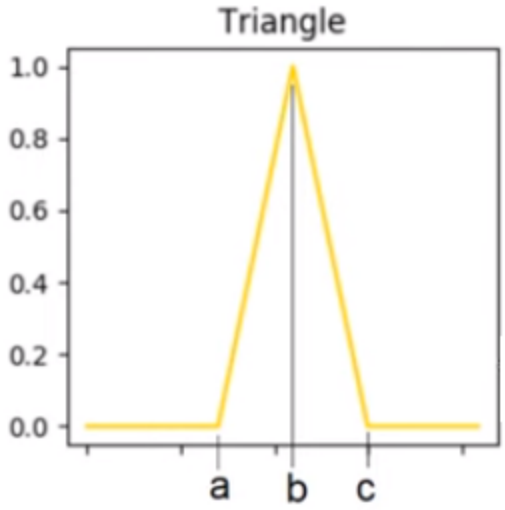
Type of membership functions



- Each fo the above membership functions are also known as fuzzy sets
- We normally choose the type of membership function that suites our application

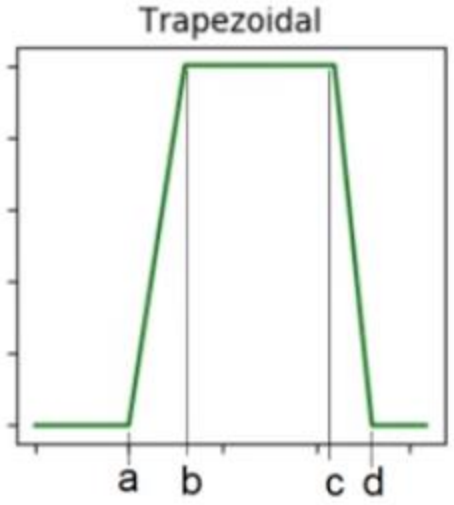
$$\mu_A(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } x \geq c \end{cases}$$

The Triangle membership function is a fuzzy set



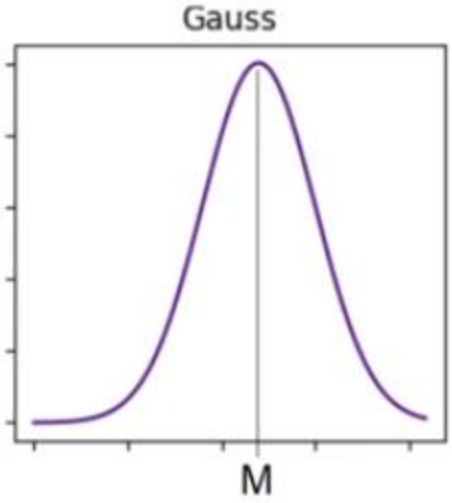
$$\mu_A(x) = \begin{cases} 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

The Trapezoidal membership function is a fuzzy set



$$\mu_A(x) = \exp\left(-\frac{1}{2}\left|\frac{x-M}{\sigma}\right|^2\right)$$

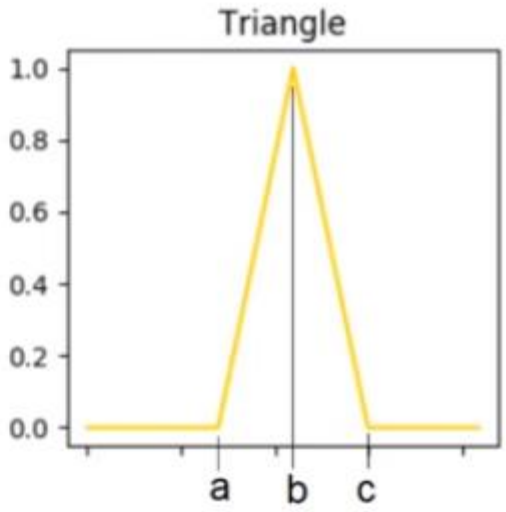
The Gaussian membership function is a fuzzy set



How to Find Membership Function

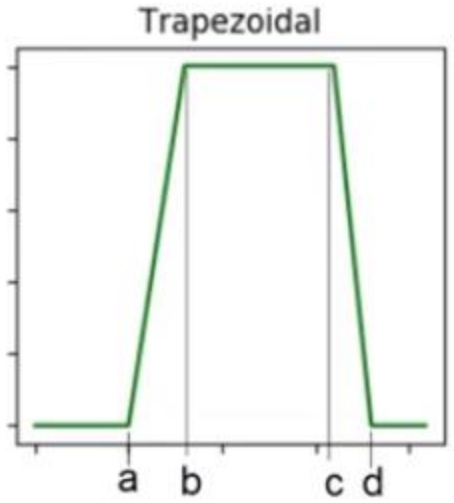
$$\mu_A(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ \frac{c-x}{c-b} & \text{if } b \le x \le c \\ 0 & \text{if } x \ge c \end{cases}$$

$$= \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$



$$\mu_A(x) = \begin{cases} 1 & \text{if } b \le x \\ \frac{d-x}{d-c} & \text{if } c \le x \le d \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$= \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$



Example: **Find Membership Value from the Figure**

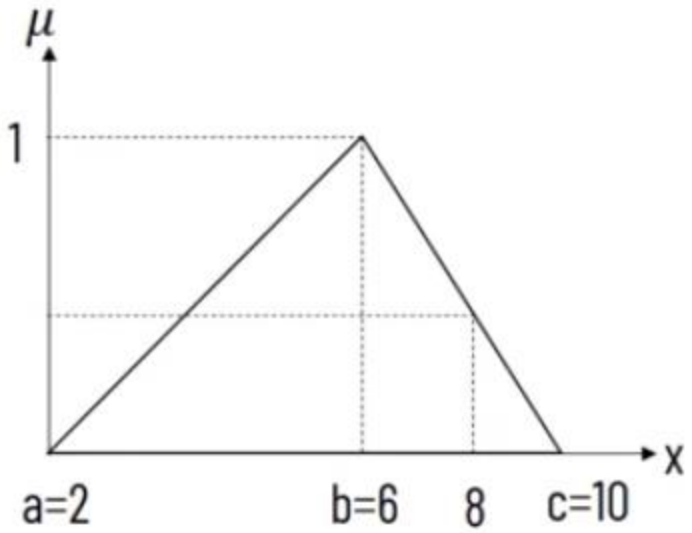
$$\mu_A(x) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

$$= \max\left(\min\left(\frac{x-2}{6-2}, \frac{10-x}{10-6}\right), 0\right)$$

$$= \max\left(\min\left(\frac{8-2}{6-2}, \frac{10-8}{10-6}\right), 0\right)$$

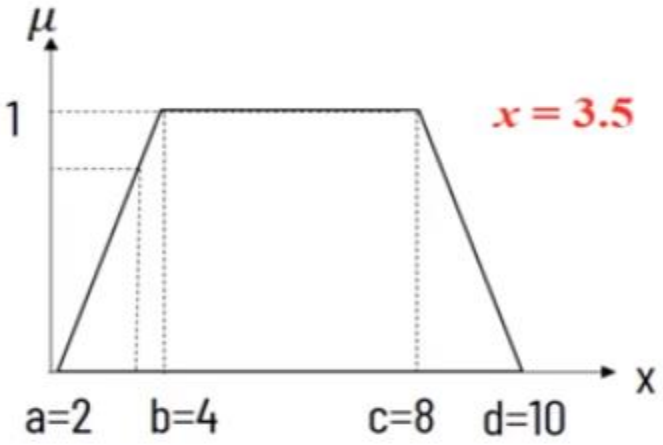
$$= \max\left(\min\left(\frac{3}{2}, \frac{1}{2}\right), 0\right)$$

$$= \max\left(\frac{1}{2}, 0\right) = \frac{1}{2}$$



Example: Find Membership Value from the Figure

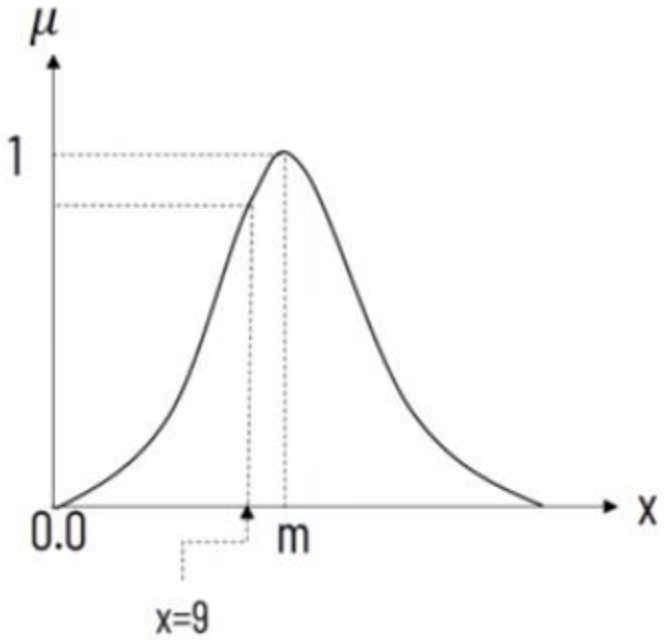
$$\begin{aligned}
\mu_A(x) &= \max\left(\min\left(\tfrac{x-a}{b-a}, 1, \tfrac{d-x}{d-c}\right), 0\right) \\
&= \max\left(\min\left(\tfrac{x-2}{4-2}, 1, \tfrac{10-x}{10-8}\right), 0\right) \\
&= \max\left(\min\left(\tfrac{3.5-2}{4-2}, 1, \tfrac{10-3.5}{10-8}\right), 0\right) \\
&= \max\left(\min\left(\tfrac{1.5}{2}, 1, \tfrac{6.3}{2}\right), 0\right) \\
&= \max(0.75, 0) = 0.75
\end{aligned}$$



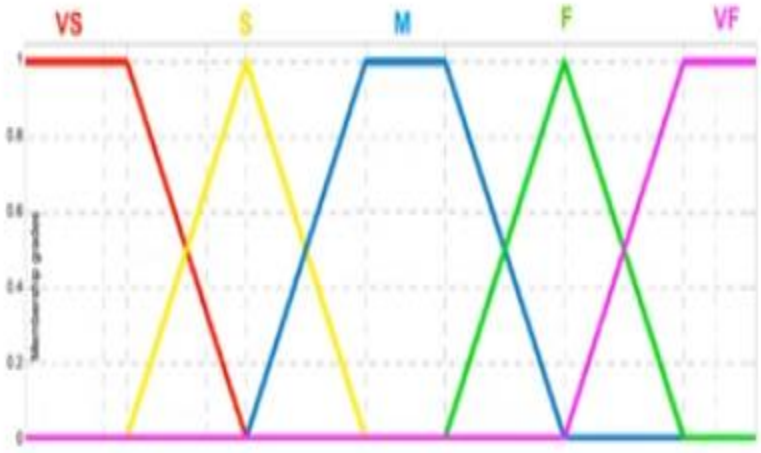
Example: Find Membership Value from the Figure

$$x = 9,\ m = 10,\ \sigma = 3$$

$$\begin{aligned}
\mu_A(x, m, \sigma) &= \exp\left(-\tfrac{1}{2}\left|\tfrac{x-m}{\sigma}\right|^2\right) \\
&= \exp\left(-\tfrac{1}{2}\left|\tfrac{9-10}{3}\right|^2\right) \\
&= 0.946
\end{aligned}$$



Linguistic Variables

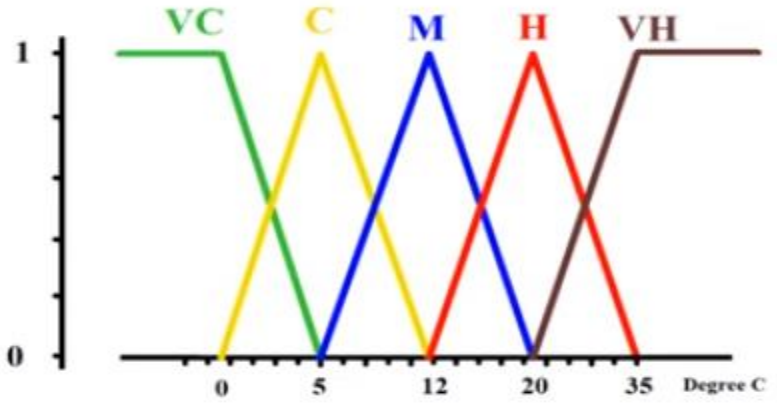


- At the root of fuzzy set theory lies the idea of linguistic variables. A linguistic variable is a fuzzy variable.
- For example, the statement 'X is tall' implies that the linguistic variable X takes the linguistic value T.
- In fuzzy expert systems, linguistic variables are used in fuzzy rules.
- For example, the linguistic variable speed might have the range between 0 and 220km per hour and may include such fuzzy subsets as very slow (VS), slow (S), medium (M), fast (F), and very fast (VF).

Example: temperature

Linguistic Variables

- Very Cold (VC) $\Rightarrow VC \leq 5$
- Cold (C) $\Rightarrow 0 \leq C \leq 12$
- Medium (M) $\Rightarrow 5 \leq C \leq 20$
- Hot (H) $\Rightarrow 12 \leq H \leq 35$
- Very Hot (VH) $\Rightarrow VH \geq 20$

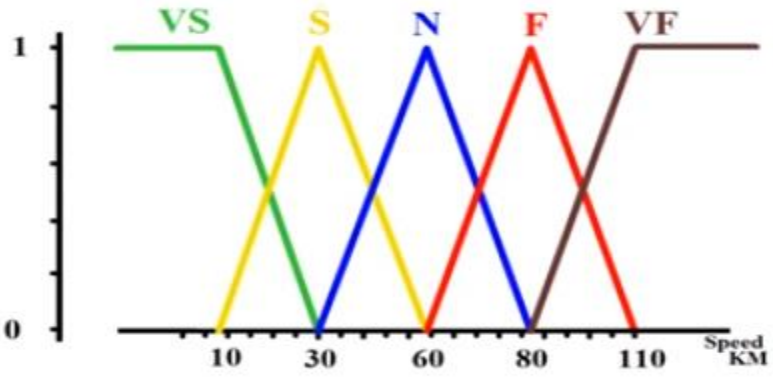


- As the membership function (MF) of Temperature (Very Cold) = 1, the Temperature = 0 or less than 0 and the membership function Cold = 0.
- As VC MF reduces less than 1 and greater than 0, the Temperature increases and the membership function Cold increases as well.
- This is natural as if the cold feeling is reducing, it means that the hot feeling is increasing.

Example: Car Speed

Linguistic Variables

- Very Slow (VS) $\Rightarrow VS \leq 30$
- Slow (S) $\Rightarrow 10 \leq S \leq 60$
- Normal (N) $\Rightarrow 30 \leq N \leq 80$
- Fast (F) $\Rightarrow 60 \leq F \leq 110$
- Very Fast (VF) $\Rightarrow VF \geq 80$



The membership functions VS, S, N, F, and VF are fuzzy sets

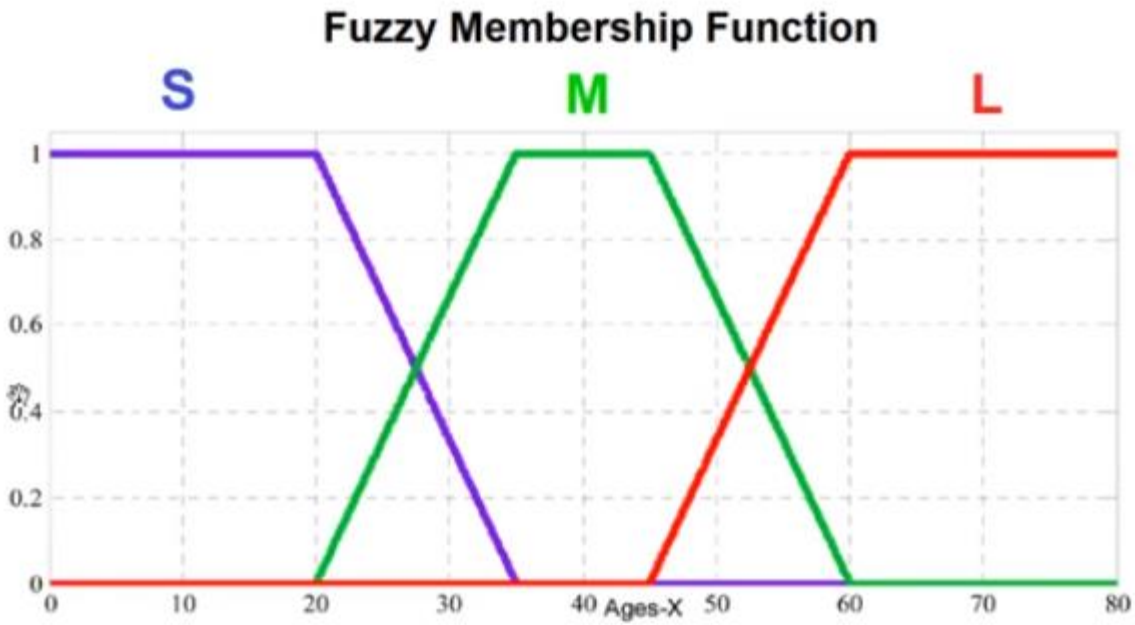
The membership function VS is also a fuzzy set

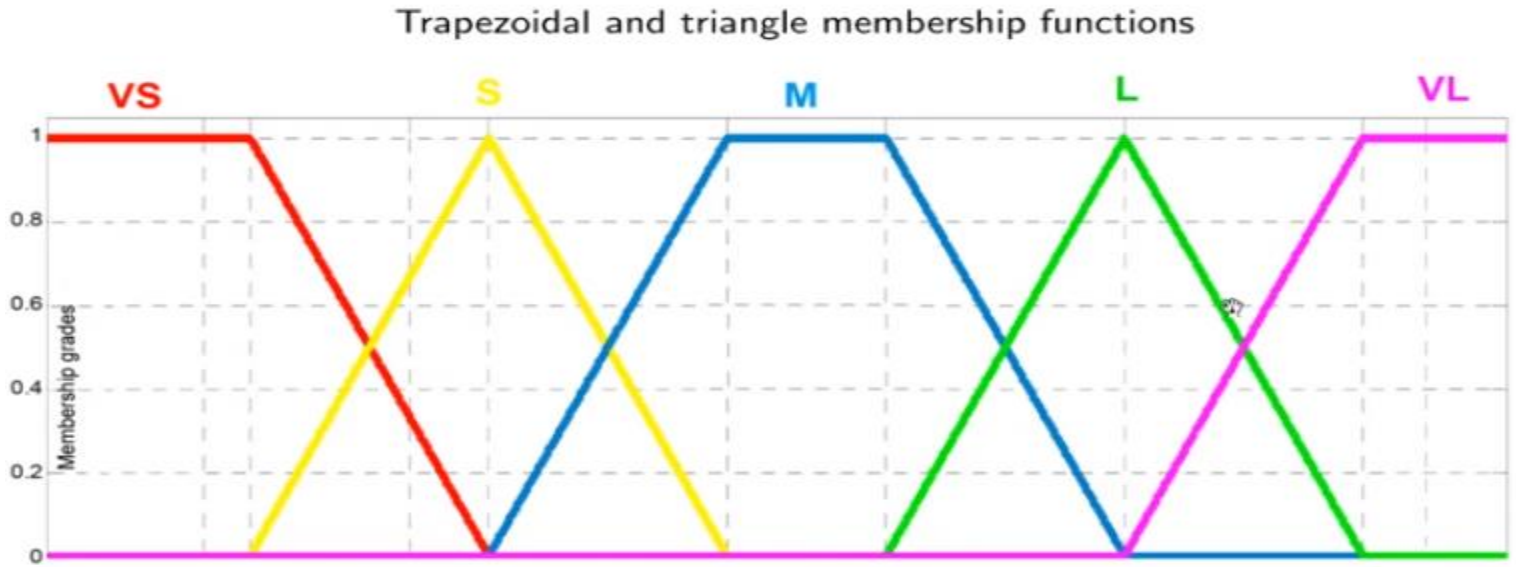
The membership function S is also a fuzzy set, and so on

Fuzzification

- With fuzzification, input variables are assigned degrees of membership in various classes
- A temperature input might be graded according to its degree of coldness, coolness, warmth or heat
- The purpose of fuzzification is to map the inputs from a set of sensors (or features of those sensors) to values from 0 to 1 using a set of input membership functions.

Trapezoidal membership functions





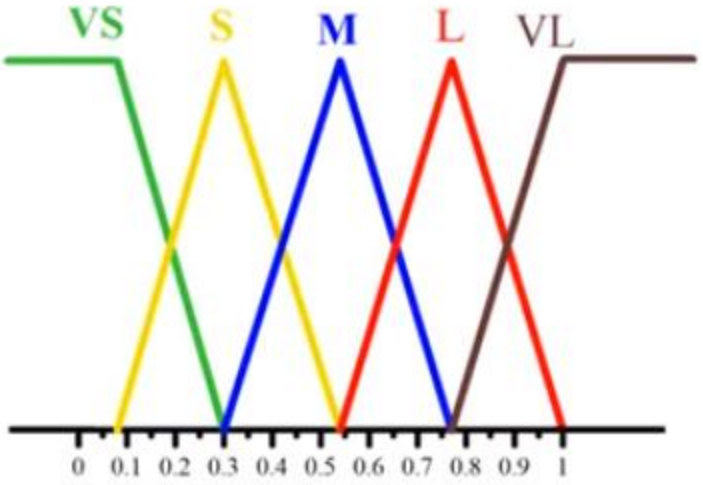
Rule Evaluation

- Inputs are applied to a set of if / then control rules
- e.g. Single Input System,
  IF the temperature is very hot (VH),
  THEN set fan speed very high (VH).
- Multi-input System
  IF (input1 is membership function1) AND/OR (input2 is membership function2) AND/OR ....
  THEN (output is output2 membership function)
- Example:
  IF temperature is high (H) AND humidity is high (H)
  THEN room is hot (H)

Example: if e is absolute error

Let us assume that we aim

if $e \geq 0$ and $e \leq 0.3$	then error is Very Small (VS)
if $e \geq 0.08$ and $e \leq 0.54$	then error is Small (S)
if $e \geq 0.3$ and $e \leq 0.77$	then error is Medium (M)
if $e \geq 0.55$ and $e \leq 1$	then error is Large (L)
if $e \geq 0.77$ and $e \leq 1$	then error is Very Large (VL)



Example of a program creates a fuzzy control system to control the opening of a valve based on the water level. The system has three linguistic variables for water level: 'low', 'medium', and 'high', and three linguistic variables for valve opening: 'small', 'medium', and 'large'. Rules are defined to control the valve opening based on the water level. Finally, the program inputs a water level value and computes the corresponding valve opening.

```
import numpy as np
import skfuzzy as fuzz
from skfuzzy import control as ctrl

# Define fuzzy variables
water_level = ctrl.Antecedent(np.arange(0, 101, 1), 'water_level')
valve_opening = ctrl.Consequent(np.arange(0, 101, 1), 'valve_opening')
```

```
# Define membership functions
water_level['low'] = fuzz.trimf(water_level.universe, [0, 0, 50])
water_level['medium'] = fuzz.trimf(water_level.universe, [0, 50, 100])
water_level['high'] = fuzz.trimf(water_level.universe, [50, 100, 100])

valve_opening['small'] = fuzz.trimf(valve_opening.universe, [0, 0, 50])
valve_opening['medium'] = fuzz.trimf(valve_opening.universe, [0, 50, 100])
valve_opening['large'] = fuzz.trimf(valve_opening.universe, [50, 100, 100])

# Define fuzzy rules
rule1 = ctrl.Rule(water_level['low'], valve_opening['large'])
rule2 = ctrl.Rule(water_level['medium'], valve_opening['medium'])
rule3 = ctrl.Rule(water_level['high'], valve_opening['small'])

# Create fuzzy control system
water_level_ctrl = ctrl.ControlSystem([rule1, rule2, rule3])
water_valve = ctrl.ControlSystemSimulation(water_level_ctrl)

# Input water level
water_valve.input['water_level'] = 70

# Compute the result
water_valve.compute()

# Output valve opening
print("Valve opening:", water_valve.output['valve_opening'])
```

create another example program that demonstrates a fuzzy control system for a simple temperature control system. In this example, we'll have two inputs: "temperature" and "humidity", and one output: "fan_speed". We'll define linguistic variables and membership functions for these variables, create rules to control the fan speed based on temperature and humidity, and then simulate the system with example inputs.

```
import numpy as np
import skfuzzy as fuzz
from skfuzzy import control as ctrl
import matplotlib.pyplot as plt

# Define fuzzy variables
temperature = ctrl.Antecedent(np.arange(0, 101, 1), 'temperature')
humidity = ctrl.Antecedent(np.arange(0, 101, 1), 'humidity')
fan_speed = ctrl.Consequent(np.arange(0, 101, 1), 'fan_speed')

# Define membership functions for temperature
temperature['cold'] = fuzz.trimf(temperature.universe, [0, 0, 50])
temperature['moderate'] = fuzz.trimf(temperature.universe, [0, 50, 100])
temperature['hot'] = fuzz.trimf(temperature.universe, [50, 100, 100])

# Define membership functions for humidity
humidity['dry'] = fuzz.trimf(humidity.universe, [0, 0, 50])
humidity['moderate'] = fuzz.trimf(humidity.universe, [0, 50, 100])
humidity['humid'] = fuzz.trimf(humidity.universe, [50, 100, 100])
```

```python
# Define membership functions for fan speed
fan_speed['low'] = fuzz.trimf(fan_speed.universe, [0, 0, 50])
fan_speed['medium'] = fuzz.trimf(fan_speed.universe, [0, 50, 100])
fan_speed['high'] = fuzz.trimf(fan_speed.universe, [50, 100, 100])

# Define fuzzy rules
rule1 = ctrl.Rule(temperature['cold'] & humidity['dry'], fan_speed['low'])
rule2 = ctrl.Rule(temperature['moderate'] & humidity['moderate'], fan_speed['medium']
rule3 = ctrl.Rule(temperature['hot'] & humidity['humid'], fan_speed['high'])
rule4 = ctrl.Rule(temperature['hot'] & humidity['dry'], fan_speed['high'])

# Create fuzzy control system
fan_ctrl = ctrl.ControlSystem([rule1, rule2, rule3, rule4])
fan_simulation = ctrl.ControlSystemSimulation(fan_ctrl)

# Input crisp values for temperature and humidity
input_temperature = 75  # Example temperature
input_humidity = 40  # Example humidity

# Fuzzification
fan_simulation.input['temperature'] = input_temperature
fan_simulation.input['humidity'] = input_humidity
```

```
# Fuzzification
fan_simulation.input['temperature'] = input_temperature
fan_simulation.input['humidity'] = input_humidity

# Fuzzy inference
fan_simulation.compute()

# Output
print("Input Temperature:", input_temperature)
print("Input Humidity:", input_humidity)
print("Fan Speed:", fan_simulation.output['fan_speed'])
```

In this program:

- The fuzzy variables `**temperature**`, `**humidity**`, and `**fan_speed**` are defined.
- Membership functions are defined for each linguistic variable:
  - `**temperature**` has 'cold', 'moderate', and 'hot' categories.
  - `**humidity**` has 'dry', 'moderate', and 'humid' categories.
  - `**fan_speed**` has 'low', 'medium', and 'high' categories.
- Rules are defined to control the `**fan_speed**` based on `**temperature**` and `**humidity**`.
- A control system is created and then simulated with example input values for `**temperature**` (75) and `**humidity**` (40).
- The program computes the `**fan_speed**` based on the fuzzy control system.
- It then prints the input values and the resulting `**fan_speed**`.
- Finally, it plots the membership functions for `**temperature**`, `**humidity**`, and `**fan_speed**` for visualization.