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Lectures (4-5) for MSc

Boundary Conditions for an Electric Field

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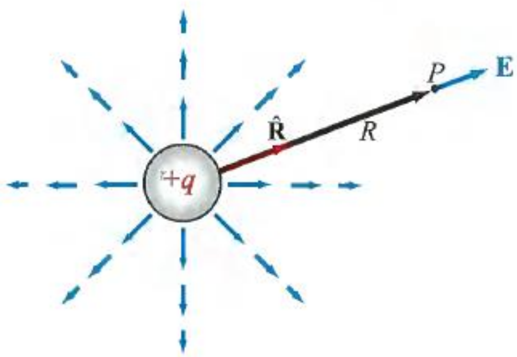
Electric Field due to Charges:

The electric field is defined as the electric force per unit charge. Field direction is the direction of the force it will exert on a positive test charge. The electric field is directed radially outward from a positive charge and radially inward from a negative point charge.

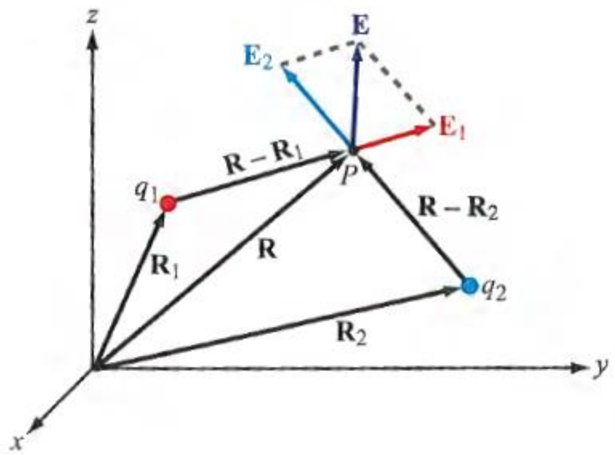
- **Electric Field due to Single Point Charges:**



Fig(1)

$$E = \hat{R}\frac{q}{4\pi\varepsilon R^2} \qquad \left(\frac{V}{m}\right), \ldots\ldots (1)$$

- **Electric Field due to Two Point Charges:**



Fig(2)

The electric field (E) at point P due to two charges is equal to the vector sum of E_1 and E_2. The electric field at P due to q_1 alone is

$$E_1 = \frac{q_1(R - R_1)}{4\pi\varepsilon|R - R_1|^3} \qquad \left(\frac{V}{m}\right), \ldots\ldots (2)$$

Similarly, the electric field at *P* due to q_2 alone is

$$E_2 = \frac{q_2(R - R_2)}{4\pi\varepsilon|R - R_2|^3} \qquad \left(\frac{V}{m}\right), \ldots\ldots (3)$$

The electric field obeys the principle of linear superposition.

Hence, the total electric field Eat *P* due to q_1 and q_2 is

$$E = E_1 + E_2$$

$$E = q_1 \frac{(R - R_1)}{4\pi\varepsilon|R - R_1|^3} + q_2 \frac{(R - R_2)}{4\pi\varepsilon|R - R_2|^3} \qquad \left(\frac{V}{m}\right) \ldots\ldots (4)$$

- **Electric Field due to Multiple Point Charges:**

Generalized eq.4 for N charges get:

$$E = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} q_i \frac{(R - R_i)}{|R - R_i|^3} \qquad \left(\frac{V}{m}\right), \ldots\ldots (5)$$

Example: Electric Field Due to Two Point Charges, the two point charges with q_1= 2×10^{-5}C and q_2 = - 4×10^{-5}C are located in free space at points with Cartesian coordinates (1,3,-1) and (-3,1,- 2), respectively.

Find a) the electric field **E** at (3,1,-2) and

b) the force **F** on a charge q_3=8×10^{-5}C located at that point. All distances are in meters.

Solution: (a)

From eq.(4), the electric field E withe $\varepsilon = \varepsilon_0$ (free space) is

$$E = \frac{q_1(R - R_1)}{4\pi\varepsilon_0|R - R_1|^3} + \frac{q_2(R - R_2)}{4\pi\varepsilon_0|R - R_2|^3} \qquad \left(\frac{V}{m}\right),$$

$$E = \frac{1}{4\pi\varepsilon_0}\left(q_1 \frac{(R - R_1)}{|R - R_1|^3} + q_2 \frac{(R - R_2)}{|R - R_2|^3}\right)$$

The vectors R_1, R_2 *and* R *are*:

$$R_1 = \hat{\imath} + 3\hat{\jmath} - \hat{k}$$
$$R_2 = -3\hat{\imath} + \hat{\jmath} - 2\hat{k}$$
$$R = 3\hat{\imath} + \hat{\jmath} - 2\hat{k}$$

$$\boldsymbol{R - R_1 = 2\hat{\imath} - 2\hat{\jmath} - \hat{k}}$$
$$\boldsymbol{R - R_2 = 6\hat{\imath}}$$
$$E = \frac{1}{4\pi\varepsilon_0}\left(2\frac{\left(2\hat{\imath} - 2\hat{\jmath} - \hat{k}\right)}{27} - 4\frac{6\hat{\imath}}{216}\right) \times 10^{-5}$$
$$E = \frac{\hat{\imath} - 4\hat{\jmath} - 2\hat{k}}{108\pi\varepsilon_0} \times 10^{-5} \qquad \left(\frac{V}{m}\right),$$

b) $F = q_3 E = 8 \times 10^{-5} \times \frac{\hat{\imath}-4\hat{\jmath}-2\hat{k}}{108\pi\varepsilon_0} \times 10^{-5} \qquad (N),$

$$\boldsymbol{F} = \frac{2\hat{\imath}-8\hat{\jmath}-4\hat{k}}{27\pi\varepsilon_0} \times 10^{-10} \qquad (N),$$

Gauss's Law:

Here we will use Maxwell's equations to confirm the electric field expressions contained in Coulomb's law, and suggest alternative techniques for evaluating electric fields induced by an electric charge:

$$\nabla . D = \rho_v \quad .. (6) \quad \text{Differential form of Gauss's law}$$

When solving electromagnetic problems, we often go back and forth between equations in differential and integral form, depending on which of the two happens to be the more applicable or convenient to use. To convert eq.(6) into integral form, we multiply both sides by dv and evaluate their integrals over an arbitrary volume v:

$$\int_v \nabla . D dv = \int_v \rho_v dv = Q \quad ..(7)$$

Here, Q is the total charge enclosed in the volume v.
Now States that the volume integral of the divergence of any vector over a volume v equals the total outward flux of that vector through the surface S enclosing v. Thus, for the vector **D:**

$$\int_v \nabla . D dv = \oint_S D . ds = Q \quad \ldots(8)$$

Comparison between eq7 and eq8 get:

$$\oint_S D . ds = Q \quad \ldots..(9) \quad \text{Integral form of gauss's law}$$

Poisson's Equation: With **D** = $\boldsymbol{\varepsilon E}$, the differential form of Gauss's law given by the eq(6) $\nabla . D = \rho_v$ can be written as:

$$\boldsymbol{\nabla}.\boldsymbol{E} = \frac{\boldsymbol{\rho_v}}{\boldsymbol{\varepsilon}} \quad \ldots (10)$$

Electric Field as a Function of Electric Potential φ

$$\mathrm{E} = -\nabla\varphi$$

So eq10 become:

$$\nabla.E = \nabla.(\nabla\varphi) = -\frac{\rho_v}{\varepsilon} \qquad (11)$$

The Laplacian of a scalar function φ:

$$\nabla^2\varphi = \nabla.(\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

Eq. (11) can be cast in the abbreviated form:

$$\nabla^2\varphi = -\frac{\rho_v}{\varepsilon} \quad \textbf{(Poisson's equation)} \quad (12)$$

This is known as *Poisson's equation.* For a volume $\acute{v}$ containing a volume charge density distribution ρ_v, the solution expressed by:

$$\varphi = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\rho_v}{R'}dv'$$

If the medium under consideration does not contain a charge, so eq.12, reduced to:

$$\nabla^2\varphi = 0 \quad \textbf{(Laplace's equation)} \quad (13)$$

Laplace equations are useful in determining the electrostatic voltage φ in regions with boundaries where φ is known as the area between capacitor plates with the specified voltage difference across them.

Conductors and Dielectrics:

The electromagnetic *constitutive parameters* of a material medium are its electrical permittivity ***ε*, magnetic permeability *μ* and conductivity *σ*.**

The conductivity of a material is a measure of how easily electrons move through the material under the influence of an externally applied electric field.

Materials are classified as conductors (metals) or insulators (dielectrics) according to their amounts of conductivity. A conductor contains a large number of loosely connected electrons in the outer shells of its atoms. In the absence of an external electric field, these free electrons move in random directions and at varying speeds. Their random motion produces zero average current through the conductor. However, when an external electric field is applied, electrons migrate from one atom to another in the direction opposite to the direction of the external field their movement leads to the conduction current:

$$\boldsymbol{J} = \boldsymbol{\sigma E} \qquad ..\left(\frac{A}{m^2}\right), \quad .. \ (14)$$

Where σ represent the material's conductivity with units of siemen per meter (S/m) or ($\Omega^{-1}\cdot m^{-1}$).

In dielectric materials, called insulating materials, the electrons are tightly bound to atoms, so much so that it is very difficult to separate them under the influence of an electric field. Hence, no large conduction current could flow through it.

- ***A perfect dielectric is a material with*** $\boldsymbol{\sigma = 0}$.
- ***In contrast, a perfect conductor is a material with*** $\boldsymbol{\sigma = \infty}$. ***Some materials, called superconductors, exhibit such a behavior.***

The conductivity σ of most metals is in the range from 10^6 to10^7 S/m, compared with 10^{-10} to 10^{-17} S/m for good insulators. The following table shows the conductivity of some known material at room temperature (20°C).

Table Conductivity of some common materials at 20 °C.

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

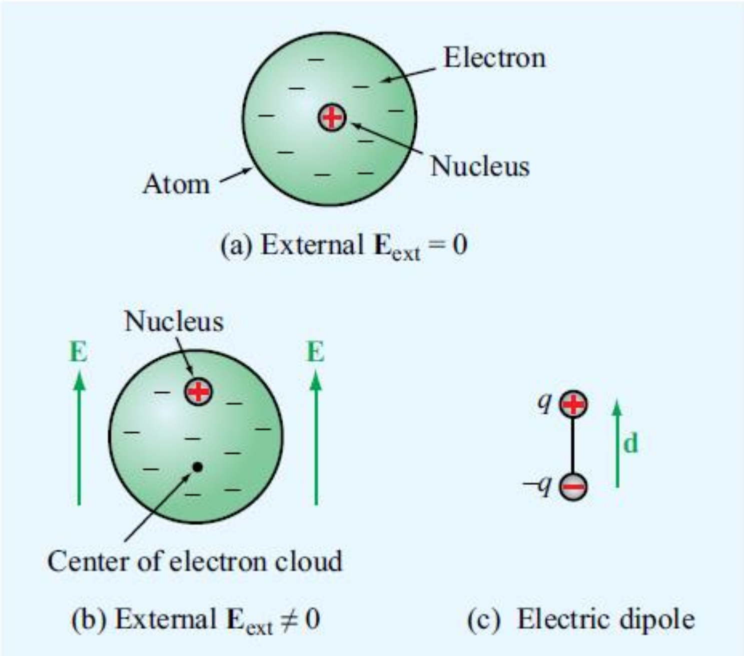
A class of materials called ***semiconductors*** allow for conduction currents even though their conductivities are much smaller than those of metals.
In view of eq(14) : $\mathbf{J} = \boldsymbol{\sigma}\mathbf{E}$,

- In a perfect dielectric with $\sigma = 0 \longrightarrow$ J=0 regardless of E,
- In a perfect conductor with $\boldsymbol{\sigma} = \infty \rightarrow \boldsymbol{E} = \frac{\boldsymbol{J}}{\boldsymbol{\sigma}} = \mathbf{0}$ regardless of $\boldsymbol{J}$.

That is: **Perfect dielectric: $\boldsymbol{J}$ =0; Perfect conductor: E=0**

The fundamental difference between a ***conductor*** and a dielectric is that electrons in the outermost atomic shells of conductor are only ***weakly tied*** to atoms and hence can ***freely migrate*** through the material, whereas in ***dielectric*** they are ***strongly bound*** to the atom.

In the absence of an electric field, the electrons in so-called ***nonpolar*** **(**has no separation of charges**)** molecules form a symmetrical cloud around the nucleus, with the center of the cloud coinciding with the nucleus see fig(3) below:



Fig(3): In the absence of an external electric field **E**, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance *d*.

The electric field generated by the positively charged nucleus attracts and holds the electron cloud around it, and the mutual repulsion of the electron clouds of adjacent atoms shapes its form.

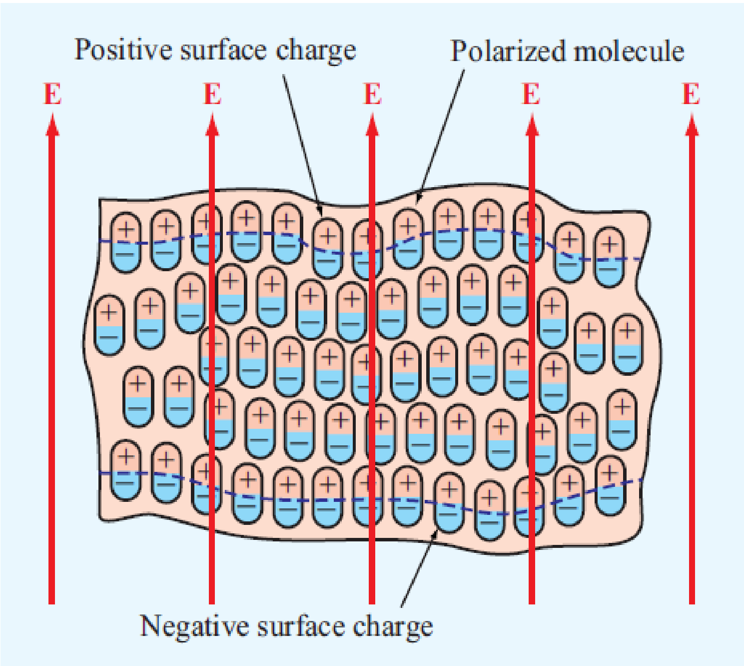
- When a conductor is subjected to an externally applied electric field, the most loosely bound electrons in each atom can jump from one atom to the next, thereby setting up an electric current.

- In a dielectric, however, an externally applied electric field **E** cannot effect mass migration of charges since none are able to move freely. Instead, the external electric filed (**E)** will ***polarize*** the atoms or molecules in the material by moving the center of the electron cloud away from the nucleus **Fig(3:b)**.

The polarized atom or molecule may be represented by an electric dipole consisting of charges **+q** in the nucleus and **−q** at the center of the electron cloud **Fig(3:c)**. ***Each such dipole sets up a small electric field, pointing from the positively charged nucleus to the center of the equally but negatively charged electron cloud.*** **This *induced* electric field, called a *polarization* field.**

Generally ***polarization*** field is weaker than and opposite in direction to, **E**. Consequently, the net electric field present in the dielectric material is smaller than **E**.

Within a block of dielectric material subject to a uniform external field, the dipoles align themselves linearly, as shown in **Fig(4)**. Along the upper and lower edges of the material, the dipole arrangement exhibits positive and negative surface charge densities, respectively.



Fig(4): A dielectric medium polarized by an external electric field **E**.

Polarization Field: Whereas in free space $\mathbf{D} = \boldsymbol{\varepsilon}_0\mathbf{E}$, the presence of microscopic dipoles in a dielectric material alters that relationship to:

$$\mathbf{D} = \boldsymbol{\varepsilon}_0\mathbf{E} + \mathbf{P} \quad \dots(\mathbf{15})$$

Where **P** is called the ***electric polarization field,*** it explains the polarizing properties of the material.

The polarization field is produced by the electric field **E** and depends on the material properties. A dielectric medium is said to be ***linear*** if the magnitude of the induced polarization field **P** is directly proportional to the magnitude of **E**, and ***isotropic*** if **P** and **E** are in the same direction.

Some crystals allow more polarization to take place along certain directions, such as the crystal axes, than along others. In such ***anisotropic*** dielectrics, **E** and **P** may have different directions. A medium is said to be ***homogeneous*** if its constitutive parameters (***ε, μ,* and *σ***) are constant throughout the medium. For such media **P** is directly proportional to **E** and is expressed as:

$$\mathbf{P} = \boldsymbol{\chi}_\mathbf{e}\boldsymbol{\varepsilon}_0\mathbf{E} \quad \dots(\mathbf{16})$$

Where $\boldsymbol{\chi}_\mathbf{e}$ is called the ***electric susceptibility*** of the material.

Inserting eq(16) into eq(15) get:

$$\mathbf{D} = \boldsymbol{\varepsilon}_0\mathbf{E} + \boldsymbol{\chi}_\mathbf{e}\boldsymbol{\varepsilon}_0\mathbf{E} = (\mathbf{1} + \boldsymbol{\chi}_\mathbf{e})\boldsymbol{\varepsilon}_0\mathbf{E} = \boldsymbol{\varepsilon}\mathbf{E} \quad \dots.(\mathbf{17})$$

Where $\boldsymbol{\varepsilon}$ represents the permittivity of the material and is expressed as follows:

$$\boldsymbol{\varepsilon} = (\mathbf{1} + \boldsymbol{\chi}_\mathbf{e})\boldsymbol{\varepsilon}_0 \quad \dots..(\mathbf{18})$$

It is often convenient to characterize the permittivity of a material relative to that of free space, $\boldsymbol{\varepsilon}_0$; this is accommodated by the relative permittivity $\boldsymbol{\varepsilon}_\mathbf{r} = \frac{\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}_0}$**.** Values of $\boldsymbol{\varepsilon}_\mathbf{r}$ are listed in **following Table** for a few common materials. In free space $\boldsymbol{\varepsilon}_\mathbf{r}$=1, and ***for most conductors*** $\boldsymbol{\varepsilon_r}$**≈1**. The dielectric constant of air is approximately 1.0006 at sea level.

$$\boldsymbol{\varepsilon}_\mathbf{r} = \frac{\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}_0} = (\mathbf{1} + \boldsymbol{\chi}_\mathbf{e}) \quad \dots(\mathbf{19}).$$

Table Relative permittivity (dielectric constant) and dielectric strength of common materials.

Material	Relative Permittivity, ϵ_r	Dielectric Strength, E_{ds} (MV/m)
Air (at sea level)	1.0006	3
Petroleum oil	2.1	12
Polystyrene	2.6	20
Glass	4.5–10	25–40
Quartz	3.8–5	30
Bakelite	5	20
Mica	5.4–6	200

$\epsilon = \epsilon_r\epsilon_0$ and $\epsilon_0 = 8.854 \times 10^{-12}$ F/m.

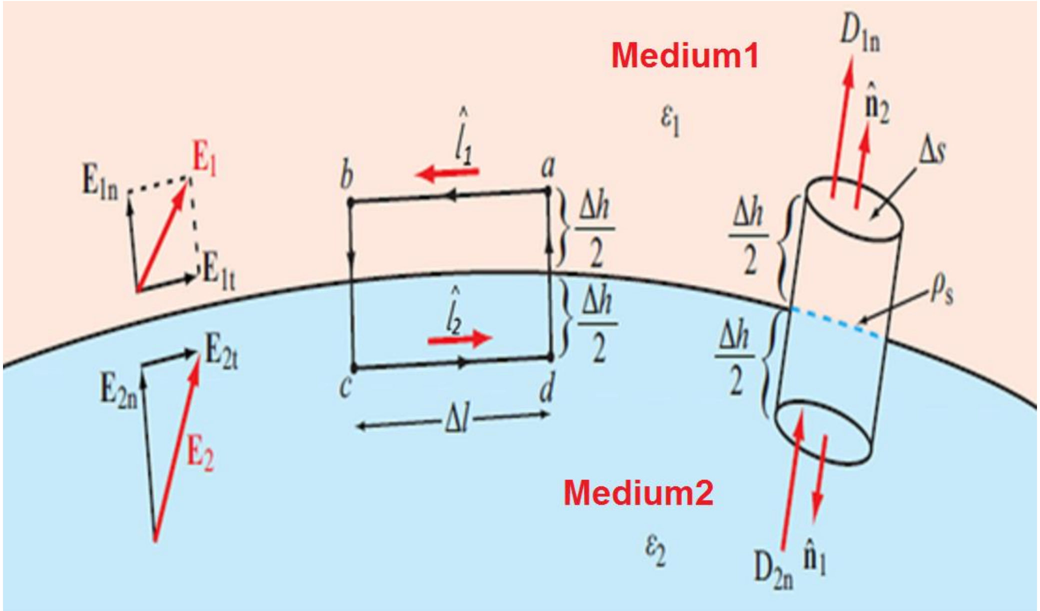
Interface Boundary Conditions for electromagnetic fields:

- **Interface Boundary Conditions for an Electric field (E):**

A vector field is said to be spatially ***continuous*** if it does not exhibit abrupt changes in either ***magnitude or direction*** as a function of position. Even though the electric field may be continuous in adjoining dissimilar media, it may well be discontinuous at the boundary between them. The boundary conditions specify how the components of fields tangential and normal to an interface between two media relate across the interface.

Here we derive a general set of boundary conditions for **E**, **D**, and **J**, applicable at the interface between *any* two dissimilar media, be they two dielectrics or a conductor and a dielectric.

Fig(5) shows an interface between medium1 with permittivity $\boldsymbol{\varepsilon_1}$ and medium2 with permittivity $\boldsymbol{\varepsilon_2}$**.** In the general case, the interface may contain a surface charge density $\boldsymbol{\rho_s}$ (unrelated to the dielectric polarization charge density).



Fig(5): Interface between two media.

we decompose $\mathbf{E}_1$ and $\mathbf{E}_2$ into components tangential $\boldsymbol{E_t}$ and normal $\boldsymbol{E_n}$ to the boundary see **Fig(5)**:

$$\boldsymbol{E_1 = E_{1t} + E_{1n}} \qquad ..(\mathbf{20}\boldsymbol{a})$$

$$\boldsymbol{E_2 = E_{2t} + E_{2n}} \qquad ..(\mathbf{20}\boldsymbol{b})$$

- To derive the boundary conditions for the tangential components of **E** and **D**, we consider the closed rectangular loop ***abcda*** shown in **Fig(5)** and apply the conservative property of the electric field ***(the line integral of the electrostatic field* E *around any closed contour C is zero)***:

$$\oint_C \boldsymbol{E.dl} = \mathbf{0} \qquad (\boldsymbol{electrostatics}) \qquad \ldots(\mathbf{21})$$

Which states that the line integral of the electrostatic field around a closed path is always zero. By letting $\Delta h \to 0$, the contributions to the line integral by segments *bc* and *da* vanish. Hence:

$$\oint_C \boldsymbol{E.dl} = \int_a^b \boldsymbol{E_1.\hat{l}_1 dl} + \int_c^d \boldsymbol{E_2.\hat{l}_2\, dl} = \mathbf{0} \qquad \ldots(\mathbf{22})$$

Or

$$\oint_C \boldsymbol{E.dl} = \boldsymbol{E_1.\hat{l}_1 \Delta l + E_2.\hat{l}_2 \Delta l} = \mathbf{0} \qquad \ldots.(\mathbf{23})$$

Where $\hat{l}_1 \boldsymbol{and}\, \hat{l}_2$ and are unit vectors along segments ***ab*** and ***cd***, and **E**$_1$ and **E**$_2$ are the electric fields in media 1 and 2. Noting that $\hat{\boldsymbol{l}}_1 = -\hat{\boldsymbol{l}}_2$, so eq23 become:

$$\boldsymbol{E_1 . \hat{l}_1 \Delta l - E_2 . \hat{l}_1 \Delta l = (E_1 . \hat{l}_1 - E_2 . \hat{l}_1) \Delta l = 0}$$

So get:

$$\boldsymbol{(E_1 - E_2) . \hat{l}_1 = 0}$$

$$\therefore \quad \boldsymbol{E_{1t} - E_{2t} = 0} \qquad \dots (\boldsymbol{24})$$

In other words, the component of $\boldsymbol{E_1}$ along $\hat{\boldsymbol{l}}_1$ equals that of $\boldsymbol{E_2}$ along $\hat{\boldsymbol{l}}_1$, for all $\hat{\boldsymbol{l}}_1$ tangential to the boundary, hence,

$$\boldsymbol{E_{1t} = E_{2t}} \qquad \left(\frac{\boldsymbol{V}}{\boldsymbol{m}}\right) \qquad \dots (\boldsymbol{25})$$

Thus, the tangential component of the electric field is continuous across the boundary between any two media.

Now decompose D1 and D2 into tangential components ($\boldsymbol{D_{1t} = \varepsilon_1 E_{1t}}$ **and** $\boldsymbol{D_{2t} = \varepsilon_2 E_{2t}}$**),** and substitute in eq(25) we get the boundary condition on the tangential component of the electric flux density is:

$$\frac{\boldsymbol{D_{1t}}}{\boldsymbol{\varepsilon_1}} = \frac{\boldsymbol{D_{2t}}}{\boldsymbol{\varepsilon_2}} \qquad \dots\dots (\boldsymbol{26})$$

- Next we apply Gauss's law to determine boundary conditions on the normal components of **E** and **D.** According to Gauss's law, the total outward flux of **D** through the three surfaces of the **small cylinder shown in Fig(5) must equal the total charge enclosed in the cylinder**. By letting the cylinder's height $\Delta h \to 0$, the contribution to the total flux through the side surface goes to zero. Also, even if each of the two media happens to contain free charge densities, the only charge remaining in the collapsed cylinder is that distributed on the boundary. Thus, $Q = \rho_s \Delta s$ and:

$$\oint_S \boldsymbol{D . ds} = \int_{\boldsymbol{top}} \boldsymbol{D_1 . \hat{n}_2\, ds} + \int_{\boldsymbol{bottom}} \boldsymbol{D_2 . \hat{n}_1\, ds = \rho_s \Delta s} \qquad \dots (\boldsymbol{27})$$

Or

$$\boldsymbol{D_1 . \hat{n}_2 \Delta s + D_2 . \hat{n}_1 \Delta s = \rho_s \Delta s} \qquad (\boldsymbol{28})$$

Where $\hat{\boldsymbol{n}}_1$ **and** $\hat{\boldsymbol{n}}_2$ are the outward normal unit vectors of the bottom and top surfaces, respectively. It is important to remember that ***the normal unit vector at the surface of any medium is always defined to be in the***

outward direction away from that medium. Since $\hat{\boldsymbol{n}}_1 = -\hat{\boldsymbol{n}}_2$, eq(28) simplifies to:

$$\boldsymbol{D}_1.\hat{\boldsymbol{n}}_2\Delta s + \boldsymbol{D}_2.\hat{\boldsymbol{n}}_1\Delta s = (\boldsymbol{D}_1.\hat{\boldsymbol{n}}_2 - \boldsymbol{D}_2.\hat{\boldsymbol{n}}_2)\Delta s = \rho_s\Delta s$$

So get:

$$\hat{n}_2.(\ \boldsymbol{D}_1 - \ \boldsymbol{D}_2) = \rho_s \qquad (C/m^2)\ \ldots(29)$$

If $\boldsymbol{D}_{1n}$ **and** $\boldsymbol{D}_{2n}$ denote as the normal components of $\boldsymbol{D}_1$ **and** $\boldsymbol{D}_2$ along $\hat{\boldsymbol{n}}_2$ we get:

$$\boldsymbol{D}_{1n} - \ \boldsymbol{D}_{2n} = \rho_s \qquad \left(\frac{C}{m^2}\right), \quad \ldots(30)$$

The normal component of D changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.

Since, $\boldsymbol{D}_{1n} = \varepsilon_1\boldsymbol{E}_{1n}\ \boldsymbol{and}\ \boldsymbol{D}_{2n} = \varepsilon_2\boldsymbol{E}_{2n}$, the corresponding boundary conditions for **E-filed, eq.30** become:

$$\varepsilon_1\boldsymbol{E}_{1n} - \ \varepsilon_2\boldsymbol{E}_{2n} = \rho_s \qquad (C/m^2)\ \ldots(31)$$

In summary:

- The conservative property of **E**: $\nabla \times \boldsymbol{E} = \boldsymbol{0} \leftrightarrow \oint_C \ \boldsymbol{E}.\boldsymbol{dl} = \boldsymbol{0}$

This led to the result that **E** has a **continuous** tangential component across a boundary.

- The divergence property of **D:** $\nabla.\boldsymbol{D} = \rho_v \leftrightarrow \oint_S \ \boldsymbol{D}.\boldsymbol{ds} = \boldsymbol{Q}$

This led to the result that the normal component of **D** changes by ρ_s across the boundary.

A summary of the Boundary Conditions that apply to the interface between any two media and a dielectric medium and a conductor medium are shown in the following table.

Table Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ϵ_1	Medium 2 Conductor
Tangential E	$E_{1t} = E_{2t}$	$E_{1t} = E_{2t} = 0$	
Tangential D	$D_{1t}/\epsilon_1 = D_{2t}/\epsilon_2$	$D_{1t} = D_{2t} = 0$	
Normal E	$\epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\epsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

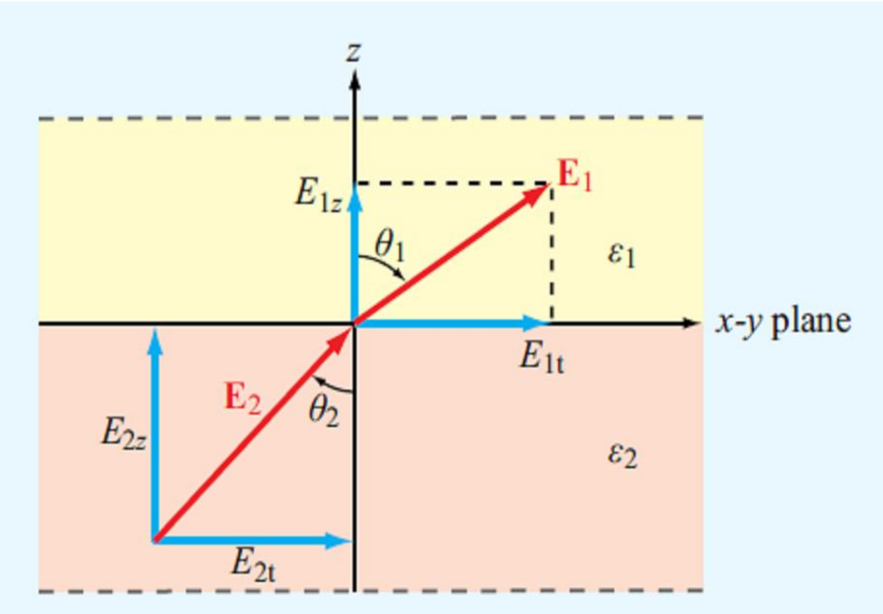
Application of Boundary Conditions

Example: The *x–y* plane is a charge-free boundary separating two dielectric media with permittivities $\boldsymbol{\varepsilon_1}$ and $\boldsymbol{\varepsilon_2}$, as shown in **Fig(6)**. If the electric field in medium1 is:

$$\boldsymbol{E_1} = \boldsymbol{\hat{\imath}E_{1x}} + \boldsymbol{\hat{\jmath}E_{1y}} + \boldsymbol{\hat{k}E_{1z}}$$

Find **a)** the electric field $\boldsymbol{E_2}$ in medium2 and

b) the angles $\boldsymbol{\theta 1}$ and $\boldsymbol{\theta 2}$.



Fig(6): Application of boundary conditions at the interface between two dielectric media

Solution:

a) Let $\boldsymbol{E_2} = \boldsymbol{\hat{\imath}E_{2x}} + \boldsymbol{\hat{\jmath}E_{2y}} + \boldsymbol{\hat{k}E_{2z}}$

- Tangential components ($\boldsymbol{E_{2x}}$ and $\boldsymbol{E_{2y}}$)
- Normal component ($\boldsymbol{E_{2z}}$)

Our task is to find the components of $\boldsymbol{E_2}$ in terms of the given components of $\boldsymbol{E_1}$. The normal to the boundary is $\hat{\boldsymbol{k}}$. Hence, the *x* and *y* components of the fields are tangential to the boundary and the *z* components are normal to the boundary. At a charge-free interface ($\boldsymbol{\rho_s}$ = 0), the tangential components of **E** and the normal components of **D** are continuous. Consequently:

$\hat{\imath}$

Tangential components**:** $\boldsymbol{E_{2t} = E_{1t} \rightarrow \hat{\imath}E_{2x} + \hat{\jmath}\, E_{2y} = \hat{\imath}\, E_{1x} + \hat{\jmath}E_{1y}}$

So:

$$\boldsymbol{E_{2x} = E_{1x}} \quad \dots(\mathbf{1})$$

$$\boldsymbol{E_{2y} = E_{1y}} \quad ..(\mathbf{2})$$

And normal component **:**

$$\boldsymbol{D_{2n} = D_{1n} \rightarrow D_{2z} = D_{1z} \qquad or \qquad \varepsilon_2 E_{2z} = \varepsilon_1 E_{1z}}$$

$$\therefore \ \boldsymbol{E_{2z} = \frac{\varepsilon_1}{\varepsilon_2} E_{1z}} \quad \dots(\mathbf{3})$$

Hence: $$\boldsymbol{E_2 = \hat{\imath}E_{1x} + \hat{\jmath}E_{1y} + \hat{k}\,\frac{\varepsilon_1}{\varepsilon_2}E_{1z}} \quad ..(\mathbf{4})$$

b) The tangential components of $\boldsymbol{E_1}$ ***and*** $\boldsymbol{E_2}$ are $\boldsymbol{E_{1t} = \sqrt{E_{1x}^2 + E_{1y}^2}}$ **and** $\boldsymbol{E_{2t} = \sqrt{E_{2x}^2 + E_{2y}^2}}$**.**

The angles: $\boldsymbol{\theta}$**1 and** $\boldsymbol{\theta}$**2** are then given by:

$$\boldsymbol{tan\theta_1 = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}}}$$

$$\boldsymbol{tan\theta_2 = \frac{E_{2t}}{E_{2z}} = \frac{\sqrt{E_{2x}^2 + E_{2y}^2}}{E_{2z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{\frac{\varepsilon_1}{\varepsilon_2}E_{1z}}}$$

$$\boldsymbol{\frac{tan\theta_1}{tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}} \quad \dots(\mathbf{5})$$

Summary **Interface Boundary conditions for an electric field Laws:**

$$\boldsymbol{E_{1t} = E_{2t}}$$

$$\boldsymbol{\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}}$$

$$\boldsymbol{D_{1n} - D_{2n} = \rho_s}$$

$$\boldsymbol{\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s}$$

References

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