Pseudo Inverse of a Matrix

The matrix $(\boldsymbol{A^T A})^{-\mathbf{1}}\boldsymbol{A^T}$ is called pseudo inverse of a matrix $\boldsymbol{A}$ and denoted by $pinv(\boldsymbol{A})$. The pseudo inverse can be expressed of a rectangular matrix, or not invertible square matrix.

$$(\boldsymbol{A^T A})^{-\mathbf{1}}\boldsymbol{A^T} = A^{-1}(A^T)^{-1}A^T = A^{-1}I = A^{-1}$$

Example 1: Find $\boldsymbol{A^{-1}}$for the following matrix

$$\mathrm{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix}$$

Solution: we see that A is rectangular matrix that we cannot be compute $\boldsymbol{A^{-1}}$ director. So, we find pseudo inverse as follow :

Firstly find $\mathbf{A^T\,A}$,

$$A^T A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{adj(A^T\boldsymbol{A})}{|A^T\boldsymbol{A}|} = \frac{1}{30}\begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix}$$

$$pinv(\boldsymbol{A}) = (\boldsymbol{A^T A})^{-1}\boldsymbol{A^T} = \frac{1}{30}\begin{bmatrix} 11 & -6 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \frac{1}{30}\begin{bmatrix} 5 & -17 & 4 \\ 0 & 12 & 6 \end{bmatrix}$$

That is $pinv(\boldsymbol{A})$: $\begin{bmatrix} \frac{1}{6} & \frac{-17}{30} & \frac{2}{15} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}$.