Positive Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive definite if the scalar $(v^T\ \boldsymbol{A}\ v)$ is positive for every nonzero column vector v of n real numbers. However, a Hermitian matrix $\boldsymbol{A}$ is said to be positive definite if the scalar

$(v^H\ \boldsymbol{A}\ v)$ is real and positive for all nonzero column vector v of n complex numbers.

Example 8: Find if the following matrices are positive definite:

$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, and { $\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **HW**}.

Solution: The identity matrix $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is positive definite because a real matrix, it is symmetric, and for any non-zero column vector v with real entries a and b, $v^T\ \boldsymbol{I}\ v = [a \quad b] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^2 + b^2$ is positive but as a complex matrix, for any non-zero column vector v with complex entries a and b,

$v^H\ \boldsymbol{I}\ v = [a^* \ \ b^*] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^*a + b^*b = |a|^2 + |b|^2$ is positive and one of a and b is not zero.

The symmetric matrix $\boldsymbol{B} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite because a real matrix, and for any nonzero column vector v with <u>real</u> entries a , b and c,

$v^T\ \boldsymbol{B}\ v = [a \quad b \quad c] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} =$

$[(2a-b) \quad (-a+2b-c) \quad (-b+2c)] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 2a^2 - ab - ab +$
$2b^2 - bc - bc + 2c^2 = a^2 + (a^2 - 2ab + b^2) + (b^2 - 2ac + c^2) +$
$c^2 = a^2 + (a-b)^2 + (b-c)^2 + c^2$ is positive.

Other way to knowing the matrix is positive definite or not illustrated in the following definition

Definition

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive definite if all the eigenvalues of the matrix $\boldsymbol{A}$ is positive.

Example 9: The following matrix $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not positive definite since its eigenvalues is 1 and -3

Positive Semi Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive semi definite if the scalar $(v^T \ \boldsymbol{A} \ v \geq 0)$ (i.e. non- negative) for every nonzero column vector v of n real numbers but If $\boldsymbol{A}$ is complex matrix then $\boldsymbol{A}$ is said to be positive semi definite if the scalar $(v^H \ \boldsymbol{A} \ v \geq 0)$. For examples, the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is positive semi definite because a real matrix, it is symmetric, and for any nonzero column vector v with real entries a and b,

$$v^T \boldsymbol{A} \, v = [a \quad b] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \geq 0$$

Other way to knowing the matrix is positive semi definite or not illustrated in the following definition

Definition

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be positive semi definite if all the eigenvalues $\boldsymbol{\lambda}$ of the matrix $\boldsymbol{A}$ is as $\boldsymbol{\lambda} \geq \boldsymbol{0}$.

Negative Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be negative definite if the scalar $(v^T \ \boldsymbol{A} \ v)$ is negative for every nonzero column vector v of n real numbers. However, a Hermitian matrix $\boldsymbol{A}$ is said to be negative definite if the scalar

$(v^H \ \boldsymbol{A} \ v)$ is real and negative for all nonzero column vector v of n complex numbers. For examples, the matrix $\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is negative definite because a real matrix, it is symmetric, and for any nonzero column vector v with real entries a , b and c,, v^T $\boldsymbol{A}$ $v =$ $[a \quad b \quad c]\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} < 0.$

Other way the eigenvalue of $\boldsymbol{A}$ is $-3, -2$ and -1, i.e., all $\boldsymbol{\lambda} < \boldsymbol{0}$.

Negative Semi Definite Matrices

A symmetric $(n \times n)$ real matrix $\boldsymbol{A}$ is said to be negative semi definite if the scalar $(v^T\ \boldsymbol{A}\ v \leq 0)$ (i.e. non- positive) for every nonzero column vector v of n real numbers but If $\boldsymbol{A}$ is complex matrix then $\boldsymbol{A}$ is said to be negative semi definite if the scalar $(v^H\ \boldsymbol{A}\ v \leq 0)$. For examples, the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ is negative semi definite because a real matrix, it is symmetric, and for any non-zero column vector v with real entries a and b, v^T $\boldsymbol{A}$ $v = [a \quad b]\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \leq 0.$

Notes

- The example1 above $\boldsymbol{B}$ shows that a matrix in which some elements are negative may still be positive definite. Conversely, a matrix whose entries are positive is not necessary positive definite and the following example as, $\boldsymbol{C} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ for which $[-1 \quad 1]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = [1 \quad -1]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2 < 0.$
- For any real invertible matrix $\boldsymbol{A}$, the $\boldsymbol{A^T A}$ is a positive definite.
- A symmetric matrix is a positive definite $\leftrightarrow$ all eigenvalues are positive.
- A symmetric matrix is a negative definite $\leftrightarrow$ all eigenvalues are negative.

Exercises:

Classify the following matrices as positive definite, negative definite, positive semi definite or negative semi definite:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -4 & -6 \\ -3 & -5 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

$$\boldsymbol{E} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{F} = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}, \boldsymbol{H} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \boldsymbol{K} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$\boldsymbol{L} = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}, \boldsymbol{M} = \begin{bmatrix} 5 & 4 & 1 \\ 4 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \boldsymbol{N} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 2 \\ -2 & 0 & 3 \end{bmatrix}, \boldsymbol{Q} = \begin{bmatrix} -5 & -2 \\ -4 & -1 \\ -3 & 0 \end{bmatrix}.$$