Cayley-Hamilton Theorem

Arthur Cayley (16 August 1821 – 26 January 1895) was a British mathematician

Let $\boldsymbol{A}$ be a square $(n \times n)$ matrix with characteristic polynomial

$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots . + c_{n-1}\lambda + c_n$ and $\lambda^n + c_1\lambda^{n-1} + \cdots . + c_{n-1}\lambda + c_n = 0$ then $\boldsymbol{A}^n + c_1\boldsymbol{A}^{n-1} + \cdots . + c_{n-1}\boldsymbol{A} + c_n\boldsymbol{I_n} = 0.$

Example 5: Apply Cayley-Hamilton Theorem on the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}.$$

Solution $A - \lambda I = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix},$

$$|\boldsymbol{A} - \lambda I| = (-\lambda)(3-\lambda) + 2 = \lambda^2 - 3\lambda + 2$$

$(\lambda - 2)(\lambda - 1) = 0 \Rightarrow \lambda = 2, \lambda = 1$

$p(\lambda) = \lambda^2 - 3\lambda + 2$, by Cayley-Hamilton Theorem $\boldsymbol{A}^2 - 3\boldsymbol{A} + 2\boldsymbol{I_2} = 0,$

$$\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ 3 & -9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Exercises

1. Find eigenvalues and eigenvectors for the following matrices

$$\boldsymbol{A}=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}, \boldsymbol{B}=\begin{bmatrix}-4 & -6\\3 & 5\end{bmatrix}, \boldsymbol{C}=\begin{bmatrix}5 & 4\\1 & 2\end{bmatrix}, \boldsymbol{D}=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix},$$

$$\boldsymbol{E}=\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}, \boldsymbol{F}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \boldsymbol{G}=\begin{bmatrix}2 & -4\\-1 & 2\end{bmatrix}, \boldsymbol{H}=\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}, \boldsymbol{K}=\begin{bmatrix}1 & -2\\2 & 1\end{bmatrix},$$

$$\boldsymbol{L}=\begin{bmatrix}5 & -3\\3 & -1\end{bmatrix}, \boldsymbol{M}=\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}.$$

2. Find eigenvalues and eigenvectors for the following matrices:

$$\boldsymbol{A}=\begin{bmatrix}5 & 4 & 2\\4 & 5 & 2\\2 & 2 & 2\end{bmatrix}, \boldsymbol{B}=\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 2\\-2 & 0 & 3\end{bmatrix}, \boldsymbol{C}=\begin{bmatrix}5 & -2 & 2\\4 & -3 & 4\\4 & -6 & 7\end{bmatrix} \text{ and}$$

$$\boldsymbol{D}=\begin{bmatrix}-1 & 0 & 0\\2 & 1 & 0\\7 & -2 & 3\end{bmatrix}.$$

3. Apply Cayley-Hamilton Theorem on the following matrices:

$$\boldsymbol{A}=\begin{bmatrix}6 & -8\\4 & -6\end{bmatrix}, \boldsymbol{B}=\begin{bmatrix}1 & 2 & 3\\0 & 2 & 2\\0 & 0 & -3\end{bmatrix}$$