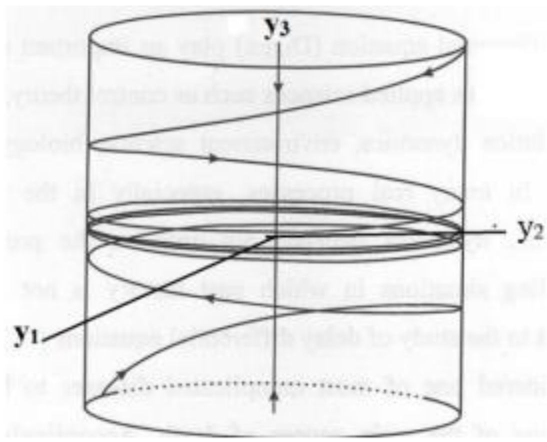
**Example 3.17.** Sketch the phase portrait of the system

$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} X$, the eigenvalues are $\pm i, -1$ then $J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

| | | | | |
|---|---|---|---|---|
| $J =$ | 0 | −1 | 0 | $\rightarrow \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \; \dot{y}_3 = -y_3$ |
| | 1 | 0 | 0 | |
| | 0 | 0 | −1 | |

And the eigenvector are $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$




**Example 3.18.** Now consider $X' = AX$ where

$A = \begin{bmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{bmatrix}$. The characteristic equation is

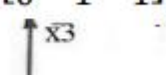
$-\lambda^3 + 0.8\lambda^2 - 0.81\lambda + 1.01 = 0$, which we have kindly factored for yc

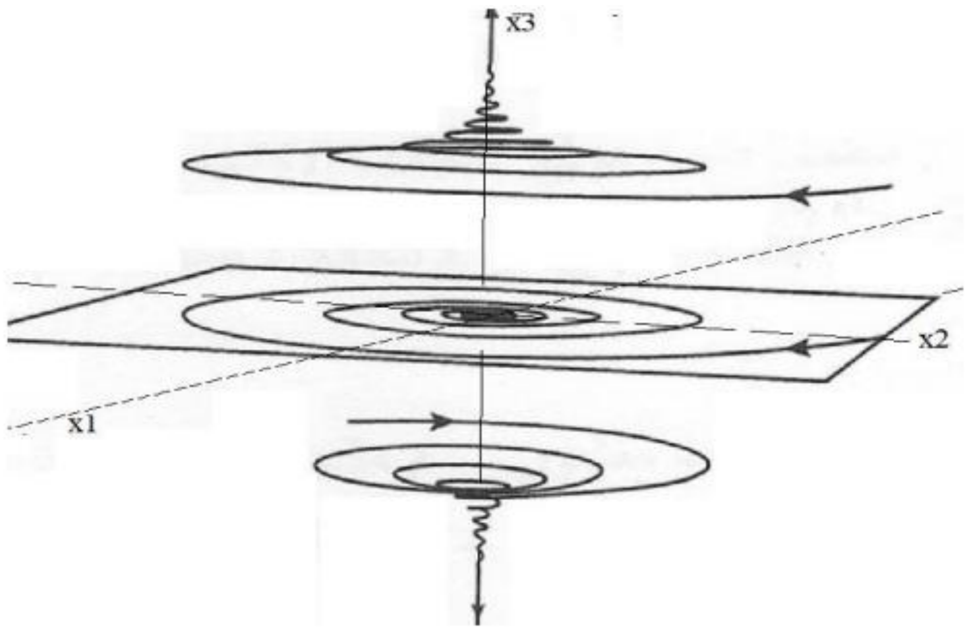
$(\lambda^2 + 0.2\lambda + 1.01)(1 - \lambda) = 0$. Therefore the eigenvalues are the roots o

equation, which are 1 and $-0.1 \pm i$. $J = \begin{bmatrix} -0.1 & -1 & 0 \\ 1 & -0.1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Solving $(A - (-0.1 + i)I)V = 0$ yields the eigenvector

$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ then $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$




x3
x2
x1

Case (d.)(equal eigenvalue) $J = \begin{bmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$J \quad = \quad D \quad + \quad C$$

$$DC = CD \rightarrow e^{Jt} = e^{Dt+Ct} = e^{Dt}e^{Ct}, \quad e^{Dt} = e^{\lambda_0 t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, e^{Ct} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{Jt} = e^{\lambda_0 t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} = e^{\lambda_0 t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.16)$$

**Example 2.17.** Sketch the trajectory of $\dot{X} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} X, \lambda_1 = \lambda_2 = \lambda_3 = 2$

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\dot{Y} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} Y$ that achieves $Y_0 = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}, c > 0$

Ans.

| | |
|---|---|
| $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ | $\rightarrow \begin{bmatrix} \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y_2 \\ y_3 \end{bmatrix},$ <br> $\dot{y}_2 = 2y_2 + y_3, \ \ \dot{y}_3 = 2y_3$ <br> $\dot{y}_1 = 2y_1 + y_2$ |

we can solve first $\dot{y}_3$ and then $\dot{y}_2$ and use the result to find $\dot{y}_1$, or we can use directly (2.15) and theorem 2.5.1 we get

$$Y(t) = e^{Jt}Y_0 = e^{\lambda_0 t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$$

$$y_1 = \left(bt + \frac{ct^2}{2}\right)e^{2t}, \qquad y_2 = (b + ct)e^{2t}, \qquad y_3 = ce^{2t}$$

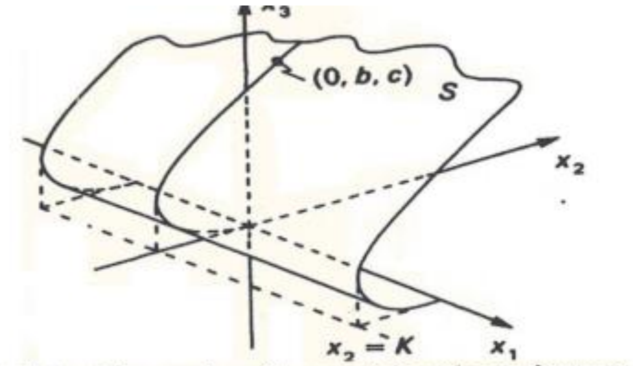
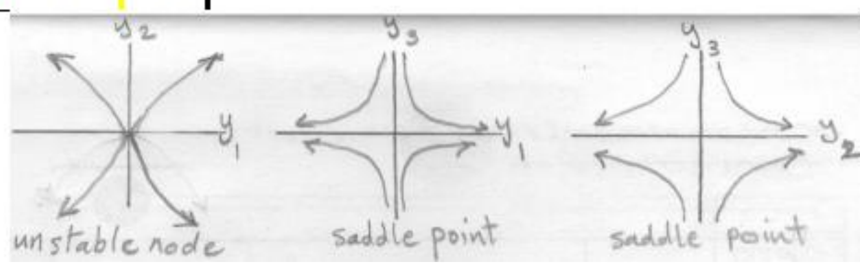



Fig. 2.14. The surface $S$ containing the trajectory $\{\phi_t(0, b, c) | t \in \mathbb{R}\}$ of (2.107), obtained by translating the curve given by (2.109) in the $x_1$-direction.

**Example 3.18**. Sketch the phase portrait of the system

$$\dot{X} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix} X, \text{ the eigenvalues are } 2, 1, -1 \text{ then } J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \rightarrow \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ \dot{y}_3 = -y_3$$




And the eigenvector are $V_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ then $M = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$