الفصل الدراسي الثاني

Theory of Differential Equations

Chapter one: Systems of differential equations

Introduction:

First Order Differential Equations

$$y' = f(t,y) \qquad \text{(DE)}$$

$$y' = f(t,y), \quad y(t_0) = y_0. \qquad \text{(IDE)}$$

$$\dot{X}(t) = F\big(t, X(t)\big) \dots\dots\dots\dots \quad (1)$$

Definition 1. Let $F(t,X)$ be real valued function with Domain $D \subseteq R^n$ a vector function $X(t)$ is said to be a solution of equation (1) if it satisfies equation (1).

1.1. Existence and uniqueness theorem

Theorem 1. If $f_i(t,X)$ is continuous on open domain $D1 \subset D$ so for any $(t_0, X_0) \in D1$ there is a solution $X(t), t \in I$ such that $X(t_0) = X_0,\ t_0 \in I$.

Theorem 2. If $f_i(t,X)$ and $\frac{\partial f_i(t,X)}{\partial x_i}$ continuous in an open domain $D1 \subset D$ so for any $(t_0, X_0) \in D1$ there is a unique solution $X(t), t \in I$ such that $X(t_0) = X_0,\ t_0 \in I$.

1.2. Introduction

$$Y'(t) = F(t,Y)$$

$$\begin{aligned} y_1' &= f_1(t, y_1, y_2, \dots, y_n) \\ y_2' &= f_2(t, y_1, y_2, \dots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, y_2, \dots, y_n) \end{aligned} \qquad (1.1)$$

Linear differential system

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \dots + a_{1n}(t)y_n + h_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \dots + a_{2n}(t)y_n + h_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \dots + a_{nn}(t)y_n + h_n(t) \end{aligned} \qquad (1.2)$$

A differential equation in standard form (1.2) is *homogeneous* if $h_i(t) = 0, i = 1,2,\dots,n$. Now, the homogeneous linear system with constant coefficients

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n \end{aligned} \qquad (1.3)$$

The(scalar) vector $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is said vector valued function if $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

then the system (1.3) can be written as

$$\dot{Y}(t) = AY(t) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} \quad (1.4)$$

Theorem 1. Let $X(t)$ and $Y(t)$ be two solutions of (1.4). Then
(a) $cX(t)$ is a solution, for any constant c, and (b) $X(t) + Y(t)$ is again a solution.
It is clear that $A(cX) = cAX = c\dot{X} = (c\dot{X})$

1- لتحويل نظام 2×2 الى معادلة واحدة من الرتبة الثانية نتبع مايلي

Example 1. Convert $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, to one equation

$$y_1' = y_1 + y_2, \qquad y_2' = y_1 - y_2$$

$$y_1 = y_2' + y_2 \rightarrow y_1' = y_2'' + y'_2 \rightarrow y_2'' = y_1' - y_2' = y_1 + y_2 - y'_2$$

$$= y_2' + y_2 + y_2 - y_2' \rightarrow y_2'' = 2y_2 \ or \ y'' - 2y = 0$$

2- لتحويل معادلة واحدة من الرتبة الثانية الى نظام 2×2 نتبع مايلي

Example 2. Convert $y'' - 2y = 0$ to a system

Let $y_1 = y, \ y_2 = y' \rightarrow y_1' = y_2, \ y_2' = 2y_1 \rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$,

ملاحظة: نلاحظ في مثال 2 لم يرجع الى النظام الاصلي ولكن النظامين لهما نفس المعادلة المميزة.

Definition. A set of vectors $X_1, X_2, \ldots, X_n$ in V is said to be linearly dependent if one of these vectors is a linear combination of the others. That is a set of vectors $X_1, X_2, \ldots, X_n$ is said to be linearly dependent if there exist constants $c_1, c_2, \ldots, c_n$, not all zero such that $c_1X_1 + c_2X_2 + \ldots + c_nX_n = 0$.
If all $c_1, c_2, \ldots, c_n = 0$ then $X_1, X_2, \ldots, X_n$ is said linearly independent.
Example 3. Show that e^t, e^{2t}, e^{3t} are linearly independent while $e^t, 2e^t, 3e^t$ are linearly dependent.

$$c_1e^t + c_2e^{2t} + c_3e^{3t} = 0. \quad (1)$$

$$e^t[c_1 + c_2e^t + c_3e^{2t}] = 0, \ \ e^t \neq 0 \rightarrow$$

$$c_1 + c_2e^t + c_3e^{2t} = 0. \quad (2)$$

Differentiate $c_2e^t + 2c_3e^{2t} = 0 \rightarrow e^t[c_2 + 2c_3e^t] = 0 \rightarrow$

$$c_2 + 2c_3e^t = 0\,. \quad (3)$$

Differentiate $2c_3e^t = 0 \rightarrow c_3 = 0$, put it in (3) $c_2 = 0$, from (2) $\rightarrow c_1 = 0$, So that e^t, e^{2t}, e^{3t} are linearly independent. To see $e^t, 2e^t, 3e^t$ are linearly independent.

$c_1e^t + 2c_2e^t + 3c_3e^t = 0 \rightarrow\ e^t[c_1 + 2c_2 + 3c_3] = 0 \rightarrow c_1 + 2c_2 + 3c_3 = 0 \rightarrow c_1 = -2c_2 - 3c_3.$

اي ان كل ثابت يعتمد على الباقين

Example 4. Let $V = R^3$ and let $X_1, X_2,$ and X_3 be the vectors

$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, X_3 = \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}$

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} = 0.$$

$$c_1 + c_2 + 3c_3 = 0 \qquad (1)$$

$$-c_1 + 2c_2 = 0 \qquad (2)$$

$$c_1 + 3c_2 + 5c_3 = 0 \qquad (3)$$

From (1),(3) we get $-2c_1 + 4c_2 = 0 \rightarrow -c_1 + 2c_2 = 0 \rightarrow c_1 = 2c_2$, linearly dependent, has infinitely many solutions

Example 5. Let $V = R^2$ and let $X_1, X_2,$ be the vectors

$$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

$det \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = -4 \neq 0$ then $X_1, X_2,$ linearly independent

1.2 The eigenvalue-eigenvector method

of finding solutions

Our goal is to find n linearly independent solutions $X_1(t), X_2(t),, \dots, X_n(t)$. Now, recall that both the first-order and second-order linear homogeneous scalar equations have exponential functions as solutions. This suggests that we try

$$\dot{X} = AX, \ \ X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \ \ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad (1)$$

Let $X(t) = e^{\lambda t}V$ where V is a constant vector, to see when X be a solution of (1).

$$\dot{X}(t) = \lambda e^{\lambda t}V = e^{\lambda t}\lambda V \text{ and } AX = Ae^{\lambda t}V = e^{\lambda t}AV$$

So X is a solution of (1) if and only if $e^{\lambda t}\lambda V = e^{\lambda t}\lambda V$ that is

$$AV = \lambda V \qquad (2)$$

Thus $X(t) = e^{\lambda t}V$ is a solution of (1) if and only if (2) holds.

Definition. A nonzero vector V satisfying (2) is called an eigenvector of A

with eigenvalue λ.

Remark if $V = 0$ then (2) is trivial (not acceptable)

From (2) we get $AV - \lambda V = 0 \rightarrow$

$$(A - \lambda I)V = 0 \qquad (3)$$

So if V is eigenvector then $V \neq 0$ then $det(A - \lambda I) = 0$ that is

$$det\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} = 0 \qquad (4)$$

The characteristic polynomial of the matrix A *and* λ is said the eigenvalue of A.

First: Real distinct eigenvalues:

Theorem 1. Any n eigenvectors $V_1, V_2, , \ldots, V_n$ of A with distinct eigenvalues $\lambda_1, \lambda_2, , \ldots, \lambda_n$ respectively, are linearly independent.

Proof: By induction we have $V_1, V_2, , \ldots, V_n$ nonzero eigenvector and $\lambda_1, \lambda_2, \ldots, \lambda_n$ not equal eigenvalue($\lambda_i \neq \lambda_j$),

1. if $n = 1$ the theorem is true,

2. Suppose it is true when $n = k$ that is

$c_1V_1 + c_2V_2 + \ldots + c_kV_k = 0$ and $c_1 = c_2 = \cdots = c_k = 0 \qquad (a)$

3. To see the statement is true when $n = k + 1$ then

$$c_1V_1 + c_2V_2 + \ldots + c_kV_k + c_{k+1}V_{k+1} = 0 \qquad (b)$$

$$c_1AV_1 + c_2AV_2 + \ldots + c_kAV_k + c_{k+1}AV_{k+1} = 0$$

$$c_1\lambda_1V_1 + c_2\lambda_2V_2 + \ldots + c_k\lambda_kV_k + c_{k+1}\lambda_{k+1}V_{k+1} = 0 \quad (c)$$

Multiplying (b) by λ_1 and subtract from (c) we get

$$c_2(\lambda_1 - \lambda_2)V_2 + \ldots + c_k(\lambda_1 - \lambda_k)V_k + c_{k+1}(\lambda_1 - \lambda_{k+1})V_{k+1} = 0 \quad (d)$$

Since $V_2, V_3, , \ldots, V_{k+1}$ are k Linearly independent then $c_{k+1}(\lambda_1 - \lambda_{k+1}) = 0$

And $\lambda_1 \neq \lambda_{k+1} \rightarrow c_{k+1} = 0$ hence $c_1 = c_2 = \cdots = c_k = c_{k+1} = 0$.

Example 1. Find all solutions of the equation

$$\dot{X} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} X$$

Solution. The characteristic polynomial of the matrix A from (4) is

$$det\begin{bmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{bmatrix} = 0$$

$$= -(1+\lambda)(1-\lambda)(2-\lambda) + 2 + 12 - 8(2-\lambda) + (1-\lambda) - 3(1+\lambda)$$

$$= (1-\lambda)(\lambda-3)(\lambda+2).$$

Thus the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = -2$.

(i) $\lambda_1 = 1$: We find the corresponding eigenvector $V_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$ from (3)

$$(A - \lambda_1 I)V_1 = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} = 0$$

This implies that $-v_{21} + 4v_{31} = 0, 3v_{11} + v_{21} - v_{31} = 0, 2v_{11} + v_{21} - 2v_{31} = 0$

Solving these equations we get $v_{21} = 4v_{31}, v_{11} = -v_{31}$. Let $v_{31} = 1$ then

$v_{21} = 4, v_{11} = -1$ then $V_1 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$

$$X_1(t) = e^{\lambda_1 t} V_1 = e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

(ii) $\lambda_2 = 3$: We find the corresponding eigenvector $V_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ from (3)

$$(A - \lambda_2 I)V_2 = \begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix} = 0$$

This implies that $-2v_{12} - v_{22} + 4v_{32} = 0, 3v_{12} - v_{22} - v_{32} = 0, 2v_{12} + v_{22} - 4v_{32} = 0$

Solving these equations we get $v_{12} = v_{32}, v_{22} = 2v_{32}$. Let $v_{32} = 1$ then

$v_{12} = 1, v_{22} = 2$ then $V_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$X_2(t) = e^{\lambda_2 t} V_2 = e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(iii) $\lambda_3 = -2$: We find the corresponding eigenvector $V_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$ from (3)

$$(A - \lambda_3 I)V_3 = \begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix} = 0$$

This implies that $3v_{13} - v_{23} + 4v_{33} = 0, 3v_{13} + 4v_{23} - v_{33} = 0, 2v_{13} + v_{23} + v_{33} = 0$

Solving these equations we get $v_{13} = -v_{33}, v_{23} = v_{33}$. Let $v_{33} = 1$ then

$v_{13} = -1, v_{23} = 1$ then $V_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$$X_3(t) = e^{\lambda_3 t} V_3 = e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

The general solution is

$$X(t) = c_1 X_1 + c_2 X_2 + c_3 X_3 = c_1 e^{t} \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$X(t) = \begin{bmatrix} -c_1 e^{t} + c_2 e^{3t} - c_3 e^{-2t} \\ -4c_1 e^{t} + 2c_2 e^{3t} + c_3 e^{-2t} \\ c_1 e^{t} + c_2 e^{3t} + c_3 e^{-2t} \end{bmatrix}$$

or$X(t) = \begin{bmatrix} -e^{t} & e^{3t} & -e^{-2t} \\ -4e^{t} & 2e^{3t} & e^{-2t} \\ e^{t} & e^{3t} & e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \Phi(t)C, \Phi(t)$ is said fundamental matrix

Example 2. Solve the initial-value problem $\dot{X} = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} X, X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution. The characteristic polynomial of the matrix A by (4) is

$$det(A - \lambda I) = 0 \rightarrow det \begin{bmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{bmatrix} = 0 \rightarrow \lambda^2 - 2\lambda - 35 = 0$$

$$\rightarrow (\lambda - 7)(\lambda + 5) = 0 \rightarrow \lambda_1 = 7, \lambda_2 = -5$$

(i) $\lambda_1 = 7$ to find the corresponding eigenvector $(A - \lambda_1 I)V_1 = 0, V_1 = \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow$

$$(A - \lambda I)V = 0$$

(ii)

Example 2. Solve the initial-value problem $\dot{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix} X,$

$\det A = 0$ لان الصفوف معتمدة على بعضها والاعمدة متشابهه

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

اذن عدد القيم الذاتية $\lambda = 0$ عددها 4 والقيمة الخامسة $\lambda \neq 0$ اذن يوجد اربع متجهات ذاتية خاصة

$\lambda = 0$ ويوجد متجه واحد للقيمة $\lambda \neq 0$ ولمعرفة هذه المتجهات نستخدم (3)

$$(A-\lambda I)V=\begin{bmatrix}1&1&1&1&1\\2&2&2&2&2\\3&3&3&3&3\\4&4&4&4&4\\5&5&5&5&5\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix}=0\rightarrow a+b+c+d+e=0$$

للسهولة نجعل لكل متجه ثلاثة اعداد اصفار لينتج

$$V_1: b,c,d=0\rightarrow e=-a, V_2: a,c,d=0\rightarrow e=-b, V_3: a,b,d=0\rightarrow e=-c,$$
$$V_4: a,b,c=0\rightarrow e=-d,$$

$$V_1=\begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix}, V_2=\begin{bmatrix}0\\1\\0\\0\\-1\end{bmatrix}, V_3=\begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix}, V_4=\begin{bmatrix}0\\0\\0\\1\\-1\end{bmatrix}$$

اما بالنسبة للقيمة الذاتية غير الصفرية

Let $V_5=\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\rightarrow AV_5=\lambda V_5\rightarrow\begin{bmatrix}1&1&1&1&1\\2&2&2&2&2\\3&3&3&3&3\\4&4&4&4&4\\5&5&5&5&5\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}=\lambda\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\rightarrow\begin{bmatrix}15\\30\\45\\60\\75\end{bmatrix}=\lambda\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\rightarrow$

$15\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}=\lambda\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}\rightarrow 15=\lambda$

Home Work

1- Find the solution of

a- $\dot{X}=\begin{bmatrix}6&-3\\2&1\end{bmatrix}X,$ b- $\dot{X}=\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}X$

c- $\dot{X}=\begin{bmatrix}1&1\\4&1\end{bmatrix}X,\ X(0)=\begin{pmatrix}2\\3\end{pmatrix}$ d- $\dot{X}=\begin{bmatrix}1&-1&0\\1&2&1\\1&10&2\end{bmatrix}X,\ X(0)=\begin{pmatrix}-1\\-4\\13\end{pmatrix}$

Second: Complex eigenvalue

If $\lambda=a+ib$ is a complex eigenvalue of A with eigenvector $V=V_1+i\,V_2$, then $X(t)=e^{\lambda t}V$ is a complex-valued solution of the differential equation

$$\dot{X}=AX. \qquad (1)$$

This complex-valued solution gives rise to two real-valued solutions, as we now show.

Lemma 1. Let $x(t)=Y(t)+iZ(t)$ be a complex-valued solution of (1). Then, both $y(t)$ and $z(t)$ are real-valued solutions of (1).

$$X(t)=e^{\lambda t}V=e^{(a+ib)t}(V_1+i\,V_2)=e^{at}(\cos bt+i\sin bt)(V_1+iV_2)$$

$$= e^{at}[(V_1 \cos bt - V_2 \sin bt) + i(V_1 \sin bt + V_2 \cos bt)]$$

$$Y(t) = e^{at}(V_1 \cos bt - V_2 \sin bt)$$

$$Z(t) = e^{at}(V_1 \sin bt + V_2 \cos bt)$$

are two real-valued solutions of (1). Moreover, these two solutions must be linearly independent solution.

Example 3 Solve the system $\dot{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} X,\ X(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The characteristic polynomial of the matrix A from (4) is

$$det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = 0$$

$$= (1-\lambda)^3 + (1-\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 + 1 - \lambda = (1-\lambda)(\lambda^2 - 2\lambda + 2) = 0.$$

Thus the eigenvalues of A are $\lambda_1 = 1$, $\lambda_{2,3} = 1 \pm i$.

(i) $\lambda_1 = 1$: We find the corresponding eigenvector $V_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ from (3)

$$(A - \lambda_1 I)V_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

This implies that $c = 0, b = 0$. Let $a = 1$ then $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$X_1(t) = e^{\lambda_1 t} V_1 = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(ii) $\lambda_2 = 1 + i$: We find the corresponding eigenvector $V_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ from (3)

$$(A - \lambda_2 I)V_2 = \begin{bmatrix} -i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

This implies that $-ia = 0 \to a = 0, -ib - c = 0, b - ic = 0 \to b = ic$. Let $c = 1$ then $b = i \to \ V_2 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$X_2(t) = e^{\lambda_2 t} V_2 = e^{(1+i)t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = e^t e^{it} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$X_2(t) = e^t(\cos t + i\sin t)\left(\begin{bmatrix}0\\0\\1\end{bmatrix} + i\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$$

$$X_2(t) = e^t[\cos t\begin{bmatrix}0\\0\\1\end{bmatrix} - \sin t\begin{bmatrix}0\\1\\0\end{bmatrix} + i[\cos t\begin{bmatrix}0\\1\\0\end{bmatrix} + \sin t\begin{bmatrix}0\\0\\1\end{bmatrix}]]$$

$$X_2(t) = e^t[\cos t\begin{bmatrix}0\\0\\1\end{bmatrix} - \sin t\begin{bmatrix}0\\1\\0\end{bmatrix}] = e^t\begin{bmatrix}0\\-\sin t\\\cos t\end{bmatrix} \text{ and}$$

$$X_3(t) = e^t[\cos t\begin{bmatrix}0\\1\\0\end{bmatrix} + \sin t\begin{bmatrix}0\\0\\1\end{bmatrix}] = e^t\begin{bmatrix}0\\\cos t\\\sin t\end{bmatrix}$$

$$X(t) = c_1X_1 + c_2X_2 + c_3X_3 = c_1e^t\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2e^t\begin{bmatrix}0\\-\sin t\\\cos t\end{bmatrix} + c_3e^t\begin{bmatrix}0\\\cos t\\\sin t\end{bmatrix}$$

$$X(t) = e^t\begin{bmatrix}c_1\\-c_2\sin t + c_3\cos t\\c_2\cos \mathrm{t} + c_3\sin t\end{bmatrix}$$

When $t = 0,\ X(0) = \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{bmatrix}c_1\\c_3\\c_2\end{bmatrix},\ X(t) = e^t\begin{bmatrix}1\\-\sin t + \cos t\\\cos \mathrm{t} + \sin t\end{bmatrix}.$

Home work

1- Find the solution of

a- $\dot{X} = \begin{bmatrix}-3 & 2\\-1 & -1\end{bmatrix}X,$ b- $\dot{X} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\-2 & 0 & -1\end{bmatrix}X$

c- $\dot{X} = \begin{bmatrix}1 & -1\\5 & -3\end{bmatrix}X,\ X(0) = \begin{pmatrix}1\\2\end{pmatrix}$ d- $\dot{X} = \begin{bmatrix}-3 & 0 & 2\\1 & -1 & 0\\-2 & -1 & 0\end{bmatrix}X,\ X(0) = \begin{pmatrix}0\\-1\\-2\end{pmatrix}$

Third: Equal roots

If the eigenvalue λ_i with multiplicity k then the other linear independent eigenvector can be obtain from the equation

$$(A - \lambda_i I)^k V = 0 \qquad (5)$$

Or we can use

$$(A - \lambda_1 I)V_2 = V_1,\ (A - \lambda_1 I)V_3 = V_2, \dots,\ (A - \lambda_1 I)V_k = V_{k-1}, \qquad (6)$$

And the solution is

$$X_2(t) = e^{\lambda_1 t}[V_2 + t(A - \lambda_1 I)V_2 + \frac{t^2}{2}(A - \lambda_1 I)^2V_2 + \cdots$$
$$+ \frac{t^{k-1}}{(k-1)!}(A - \lambda_1 I)^{k-1}V_2 \qquad (7)$$

Example 1. Find three linearly independent solutions of the differential equation $\dot{X} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} X$,

The characteristic polynomial of the matrix A from (4) is

$$det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$\Longrightarrow (1-\lambda)^2(2-\lambda) = 0 \Longrightarrow \lambda_1 = 1$, with multiplicity two $(k = 2)$, $\lambda_3 = 2$ with multiplicity one,

(i) $\lambda_1 = 1$: We find the corresponding eigenvector $V_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ from (3)

$$(A - \lambda_1 I)V_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

This implies that $b = 0, c = 0$. Let $a = 1$ then $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$X_1(t) = e^{\lambda_1 t} V_1 = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

From (5) or (6) we get $V_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$(A - \lambda_1 I)V_2 = V_1 \Longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow b = 1, c = 0, \ a \text{ arbitrary}$$

$$V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda_1 I)^2 V_2 = 0 \Longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$\Longrightarrow c = 0, \ a, b$ arbitrary $V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

from (7) we get

$$X_2(t) = e^{\lambda_1 t}[V_2 + t(A - \lambda_1 I)V_2] = e^t \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right] =$$

$$= e^t \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] = e^t \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

(iii) $\lambda_3 = 2$: We find the corresponding eigenvector $V_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ from (3)

$$(A - \lambda_3 I)V_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

This implies that $-a + b = 0, -b = 0 \Longrightarrow a = 0, c$ is arbitrary. Let $c = 1$ then

$V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$X_3(t) = e^{\lambda_3 t} V_3 = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 2. Solve the initial-value problem $\dot{X} = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} X, X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Solution. The characteristic polynomial of the matrix A by (4) is

$$det(A - \lambda I) = 0 \ \rightarrow \ det \begin{bmatrix} 2 - \lambda & 0 \\ 4 & 2 - \lambda \end{bmatrix} = 0 \rightarrow (\lambda - 2)^2 = 0 \rightarrow \ \lambda_1 = 2$$

Is eigenvalue of multiplicity 2.

(i) $\lambda_1 = 2$ to find the corresponding eigenvector $(A - \lambda_1 I)V_1 = 0, V_1 = \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow$

$(A - 2I) \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Longrightarrow \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \Longrightarrow a = 0$, let $b = 1$ then $V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$X_1 = e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

to find the second vector $V_2 = \begin{bmatrix} a \\ b \end{bmatrix}$ from (3) $\rightarrow \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow 4a = 1 \Longrightarrow a =$

$\frac{1}{4}$, $\quad V_2 = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$ from (7) we get

$$X_2(t) = e^{\lambda_1 t}[V_2 + t(A - \lambda_1 I)V_2] = e^{2t} \left[\begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} \right] = e^{2t} \begin{bmatrix} \frac{1}{4} \\ t \end{bmatrix}$$

$$X(t) = c_1 X_1 + c_2 X_2 = c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \frac{1}{4} \\ t \end{bmatrix} \Longrightarrow X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} c_2 \\ c_1 \end{bmatrix}$$

$$c_1 = 2, \ \ c_2 = 4$$

$$X(t) = 2e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4e^{2t} \begin{bmatrix} \frac{1}{4} \\ t \end{bmatrix} = e^{2t} \begin{bmatrix} 1 \\ 2 + 4t \end{bmatrix}$$

Example 3. Solve the initial-value problem $\dot{X} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} X,\ X(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

The characteristic polynomial of the matrix A from (4) is

$$det\begin{bmatrix} 2-\lambda & 1 & 3 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = 0$$

$\Longrightarrow (2-\lambda)^3 = 0 \Longrightarrow \lambda_1 = 2$, with multiplicity 3 ($k = 3$),

(i) $\lambda_1 = 1$: We find the corresponding eigenvector $V_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ from (3)

$$(A - \lambda_1 I)V_1 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

This implies that $b + 3c = 0, c = 0 \rightarrow b = 0$. Let $a = 1$ then $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$X_1(t) = e^{\lambda_1 t}V_1 = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

From (5) or (6) we get $V_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$(A - \lambda_1 I)V_2 = V_1 \Longrightarrow \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow b + 3c = 1, c = 0, b = 1, a$

arbitrary $V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, since this is the second eigenvalue then by (7)

$$X_2(t) = e^{\lambda_1 t}[V_2 + t(A - \lambda_1 I)V_2] = e^{2t}[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t[\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}]] = e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

$(A - \lambda_1 I)V_3 = V_2 \Longrightarrow \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow b + 3c = 0, -c = 1, c = -1,\ b =$

$3, a$ arbitrary $V_3 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$, since this is the third eigenvalue then by (7)

$$X_3(t) = e^{\lambda_1 t}[V_3 + t(A-\lambda_1 I)V_3 + \frac{t^2}{2}(A-\lambda_1 I)^2 V_3$$

$$= e^{2t}[\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} \frac{t^2}{2} \\ 3+t \\ -1 \end{bmatrix}$$

$$X(t) = c_1X_1 + c_2X_2 + c_3X_3 = e^{2t}[c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} \frac{t^2}{2} \\ 3+t \\ -1 \end{bmatrix}]$$

$$X(0) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 + 3c_3 \\ -c_3 \end{bmatrix}, c_1 = 1, c_3 = -1, c_2 = 5$$

$$X(t) = e^{2t}\begin{bmatrix} 1 + 5t - \frac{t^2}{2} \\ 2 - t \\ 1 \end{bmatrix}$$

Theorem 2 (Cayley-Hamilton Theorem) Every $n \times n$ constant matrix satisfies its characteristic equation.

Theorem 2 (Cayley-Hamilton). Let $p(\lambda) = p_0 + p_1\lambda + \ldots + (-1)^n p_n \lambda^n$ be the characteristic polynomial of A. Then,

$$p(A) = p_0 + p_1 A + \ldots + (-1)^n p_n A^n = 0.$$

Example let $A = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ then $p(\lambda) = \lambda^2 + 4\lambda - 1 = 0$ its characteristic equation so $p(A) = A^2 + 4A - I = 0$

Home work

1- Find the solution of

a- $\dot{X} = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} X,$ b- $\dot{X} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X$

c- $\dot{X} = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} X,\ X(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ d- $\dot{X} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} X,\ X(0) = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$

1.3 Fundamental matrix solutions $\Phi(t)$; and exponential matrix e^{At}

$$\dot{X} = AX \qquad (1)$$

Definition 2. An n × n matrix function **Φ** is said to be a fundamental matrix for the vector differential equation (1) provided **Φ** is a solution of the matrix equation (1) on I, often

$$\boldsymbol{\Phi}(\mathbf{t}) = [X_1\ X_2\ \ldots\ X_n] \to X(t) = \boldsymbol{\Phi}(\mathbf{t})\boldsymbol{C} \qquad (2)$$

Definition 3. An n × n matrix function $\boldsymbol{e^{At}}$ is said to be a exponential matrix for the vector differential equation (1) provided

$$X(t) = e^{A(t-t_0)}C \qquad (3)$$

Example 1. Find a fundamental matrix solution of the system of differential

equations

The independent solutions are $X_1 = e^t\begin{bmatrix}1\\0\\-1\end{bmatrix}, X_2 = e^{2t}\begin{bmatrix}-2\\1\\0\end{bmatrix}, X_3 = e^{3t}\begin{bmatrix}0\\1\\-1\end{bmatrix}$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, V_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}, V_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix}, V_3 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$\boldsymbol{\Phi}(\mathbf{t}) = \begin{bmatrix}e^t & -2e^{2t} & 0\\0 & e^{2t} & e^{3t}\\-e^t & 0 & -e^{3t}\end{bmatrix},$$

$$\begin{bmatrix}a-1 & b & c\\d & e-1 & f\\g & h & i-1\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix} = 0 \to a-c = 1, d-f = 0, g-i = -1$$

$$\begin{bmatrix}a-2 & b & c\\d & e-2 & f\\g & h & i-2\end{bmatrix}\begin{bmatrix}-2\\1\\0\end{bmatrix} = 0 \to -2a+b = -4, -2d+e = 2, -2g+h = 0,$$

$$\begin{bmatrix}a-3 & b & c\\d & e-3 & f\\g & h & i-3\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix} = 0 \to b-c = 0, e-f = 3, h-i = -3,$$

$$\to b-2c = -2, b = c = 2, a = 3, -f+2d = 1, f = d = 1, e = 4, g-h = 2, g = -2, h = -4, i = -1$$

$$MJ = AM \to \begin{bmatrix}1 & -2 & 0\\0 & 1 & 1\\-1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix} = \begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}\begin{bmatrix}1 & -2 & 0\\0 & 1 & 1\\-1 & 0 & -1\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -4 & 0\\0 & 2 & 3\\-1 & 0 & -3\end{bmatrix} = \begin{bmatrix}a-c & -2a+b & b-c\\d-f & -2d+e & e-f\\g-i & -2g+h & h-i\end{bmatrix}$$

Theorem 3. Let $\boldsymbol{\Phi}(\mathbf{t})$ be a fundamental matrix solution of the differential equation

$$\dot{X} = AX \qquad (1)$$

Then, $e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0) \qquad (4)$

In other words, the product of any fundamental matrix solution of (I) with its inverse at $t = 0$ must yield e^{At}.

Lemma 2. A matrix $\Phi(\mathrm{t})$ is a fundamental matrix solution of (1) if and only if $\dot{\Phi}(\mathrm{t}) = A\Phi(\mathrm{t})$ and $\det\Phi(0) \neq 0$.

Proof of Lemma: Let $X_1(t)X_2(t) \ldots X_n(t)$ be linearly independent solution of (1).
Let $\mathbf{\Phi(t)} = [X_1(t)\ X_2(t) \ldots X_n(t)]$ then $\mathbf{\Phi(t)}$ is Fundamental solution iff
$\dot{\mathbf{\Phi}}(\mathbf{t}) = [\dot{X}_1(t)\ \dot{X}_2(t)\ \ldots\ \dot{X}_n(t)] = [AX_1(t)\quad AX_2(t)\quad \cdots\quad AX_n(t)] = A[X_1(t)\ X_2(t)\ \ldots\ X_n(t)] = \boldsymbol{A\Phi}(\mathbf{t})$ and

$$\mathbf{\Phi(t)} = [e^{\lambda_1 t}V_1\ e^{\lambda_2 t}V_2 \ldots e^{\lambda_n t}V_n] \Longrightarrow \mathbf{\Phi(0)} = [V_1\ V_2 \ldots V_n]$$

Since $V_1\ V_2 \ldots V_n$ are eigenvectors so they are linearly independent then $\det\mathbf{\Phi(0)} \neq 0$. □

Lemma 3. The matrix-valued function $e^{At} = I + At + A^2\frac{t^2}{2} + \cdots$ (5) is a fundamental matrix solution of (1).

Proof: $\frac{d}{dt}e^{At} = A + A^2t + A^3\frac{t^2}{2} + \cdots = A\left(I + At + A^2\frac{t^2}{2} + \cdots\right) = Ae^{At}$ so e^{At} is a solution of (1),$\det(e^{A0}) = \det(e^0) = \det(I) = 1 \neq 0$
So by Lemma 2 e^{At} is fundamental matrix solution. □

Lemma 4. Let $\Phi(\mathrm{t})$ be a fundamental matrix solution of (1). Then, $\Psi(\mathrm{t}) = \Phi(\mathrm{t})C$ is also a fundamental matrix solution of (1) provided C is constant nonsingular matrix ($\det C \neq 0$).

Proof: Let $\Psi(\mathrm{t}) = \Phi(\mathrm{t})C \to \Psi'(\mathrm{t}) = \Phi'(\mathrm{t})C, \Psi'(\mathrm{t}) = A\Phi(\mathrm{t})C = A\Psi(\mathrm{t})$,
Then $\Psi(\mathrm{t})$ is a solution of (1)
$\det\Psi(\mathrm{t}) = \det\Phi(\mathrm{t})C = \det\Phi(\mathrm{t})\det C \to \det\Psi(0) = \det\Phi(0)\det C \neq 0$
Then $\Psi(\mathrm{t})$ is a fundamental matrix □

Proof of Theorem3**:** Let $\Phi(\mathrm{t})$ be fundamental matrix, by Lemma 3
e^{At} is also a fundamental matrix**,** then by Lemma 4, $e^{At} = \Phi(\mathrm{t})C$ (6)
Let $t = 0$ in (6) $I = \Phi(0)C \to C = \Phi^{-1}(0) \to e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0)$. □

$$e^{A(t-t_0)} = \Phi(\mathrm{t})\Phi^{-1}(t_0) \qquad (7)$$

Example 2. Find e^{At} if $\dot{X} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix} X$ and use it to solve the system

Solution. Our first step is to find 3 linearly independent solutions of the system:
$\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 5$ and $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, V_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ their corresponding

eigenvalues, then $\Phi(t)=\begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}$ is FMS from (6)

$$e^{At}=\Phi(t)\Phi^{-1}(0)=\begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}^{-1}=$$

$$\begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}\frac{1}{4}\begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}=\begin{bmatrix} e^{t} & e^{3t} & e^{5t} \\ 0 & 2e^{3t} & 2e^{5t} \\ 0 & 0 & 2e^{5t} \end{bmatrix}\begin{bmatrix} 1 & \frac{-1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$=\begin{bmatrix} e^{t} & \frac{-1}{2}e^{t}+\frac{1}{2}e^{3t} & \frac{-1}{2}e^{3t}+\frac{1}{2}e^{5t} \\ 0 & e^{3t} & e^{5t} \\ 0 & 0 & e^{5t} \end{bmatrix}$$

Example 3 Find e^{At} and Use it to solve $\dot{X}=\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}X$,

Ans. The matrix A is lower triangular so $\lambda_1=2=\lambda_2,\lambda_3=3$ and $V_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},X_1=$ $e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},V_2=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},X_2=e^{2t}\begin{bmatrix} 1 \\ t \\ -1 \end{bmatrix},V_3=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},X_3=e^{3t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

$\Phi(t)=\begin{bmatrix} 0 & e^{2t} & 0 \\ e^{2t} & te^{2t} & 0 \\ 0 & -e^{2t} & e^{3t} \end{bmatrix}$ is FMS

$$e^{At}=\Phi(t)\Phi^{-1}(0)=\begin{bmatrix} 0 & e^{2t} & 0 \\ e^{2t} & te^{2t} & 0 \\ 0 & -e^{2t} & e^{3t} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} 0 & e^{2t} & 0 \\ e^{2t} & te^{2t} & 0 \\ 0 & -e^{2t} & e^{3t} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}=\begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ e^{3t}-e^{2t} & 0 & e^{3t} \end{bmatrix}$$

$$X(t)=e^{At}C=\begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ e^{3t}-e^{2t} & 0 & e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} c_1e^{2t} \\ c_1te^{2t}+c_2e^{2t} \\ c_1(e^{3t}-e^{2t})+c_3e^{3t} \end{bmatrix}$$

طريقة ثانية لايجاد e^{At}

$$e^{At}=I+At+A^2\frac{t^2}{2}+\cdots=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}+\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}t+\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}^2\frac{t^2}{2!}+\cdots$$

$$=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}+\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}t+\begin{bmatrix} 4 & 0 & 0 \\ 4 & 4 & 0 \\ 5 & 0 & 9 \end{bmatrix}\frac{t^2}{2!}+\cdots$$

$$=\begin{bmatrix} 1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots & 0 & 0 \\ t+\frac{4t^2}{2}+\frac{12t^3}{3!}+\cdots & 1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots & 0 \\ t+\frac{5t^2}{2}+\frac{19t^3}{3!}+\cdots & 0 & 1+3t+\frac{(3t)^2}{2!}+\frac{(3t)^3}{3!}+\cdots \end{bmatrix}$$

$$=\begin{bmatrix} e^{2t} & 0 & 0 \\ te^{2t} & e^{2t} & 0 \\ e^{3t}-e^{2t} & 0 & e^{3t} \end{bmatrix}$$

Properties of e^{At}

1- if A is diagonal $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ then $e^{At}=\begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$

2- if A is upper (or lower)triangular $A=\begin{bmatrix} 2 & a \\ 0 & 3 \end{bmatrix} \to e^{At}=\begin{bmatrix} e^{2t} & -ae^{2t}+ae^{3t} \\ 0 & e^{3t} \end{bmatrix}$

$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix}$ then $e^{At}=\begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix}$

1.4 The nonhomogeneous equation; variation of parameters

Let the matrix $\Phi(\mathrm{t})=[X_1(t)\ \ X_2(t)\ \cdots\ \ X_n(t)]$ be FMS of the homogenous system

$$\dot{\mathrm{X}}(\mathrm{t})=AX(t) \qquad (1)$$

Then the system

$$\dot{\mathrm{X}}(\mathrm{t})=AX(t)+H(t) \qquad (2)$$

Is the nonhomogenous system,

Theorem 4 Let $\Phi(\mathrm{t})$ be FMS and e^{At} be exponential matrix then the general solution satisfying $X(t_0)=X_0$ of (2) is

$$X(t)=e^{A(t-t_0)}X_0+e^{At}\int_{t_0}^{t}e^{-As}H(s)\,ds$$

Proof: We have to seek a solution in the form

$X(t)=\Phi(\mathrm{t})U(t).$ (3)

$$U(t)=\Phi^{-1}(\mathrm{t})X(t) \qquad (4)$$

Differentiating (3) we get $\dot{X}(t)=\dot{\Phi}(\mathrm{t})U(t)+\Phi(\mathrm{t})\dot{U}(t)$,

$$AX(t)+H(t)=\dot{\Phi}(\mathrm{t})U(t)+\Phi(\mathrm{t})\dot{U}(t)=A\Phi(\mathrm{t})U(t)+\Phi(\mathrm{t})\dot{U}(t)$$

$$=AX(t)+\Phi(\mathrm{t})\dot{U}(t),$$

$$H(t)=\Phi(\mathrm{t})\dot{U}(t)\to\dot{U}(t)=\Phi^{-1}(\mathrm{t})H(t)$$

Integrating this expression between t_0 and t gives

$$U(t) - U(t_0) = \int_{t_0}^{t} \Phi^{-1}(s)H(s)\, ds$$

$$U(t) = \Phi^{-1}(t_0)X(t_0) + \int_{t_0}^{t} \Phi^{-1}(s)H(s)\, ds$$

$$\Phi(t)U(t) = \Phi(t)\Phi^{-1}(t_0)X_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)H(s)\, ds$$

$$X(t) = \Phi(t)\Phi^{-1}(t_0)X_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)H(s)\, ds \qquad (5)$$

$$X(t) = e^{A(t-t_0)}X_0 + e^{A(t-t_0)}\Phi(t_0)\int_{t_0}^{t} e^{-A(s-t_0)}\Phi^{-1}(t_0)H(s)\, ds$$

$$X(t) = e^{A(t-t_0)}X_0 + e^{At}\int_{t_0}^{t} e^{-As}H(s)\, ds \qquad (6)$$

طريقة اخرى للبرهان

Multiply (2) by $e^{-At} \rightarrow e^{-At}\dot{X}(t) = e^{-At}AX(t) + e^{-At}H(t)$

$e^{-At}\dot{X}(t) - e^{-At}AX(t) = e^{-At}H(t) \rightarrow e^{-At}\dot{X}(t) - Ae^{-At}X(t) = e^{-At}H(t)$

$\Longrightarrow e^{-At}X'(t) + (e^{-At})'X(t) = e^{-At}H(t) \Longrightarrow (e^{-At}X(t))' = e^{-At}H(t)$

Integrating this expression between t_0 and t gives

$$e^{-At}X(t) - e^{-At_0}X(t_0) = \int_{t_0}^{t} e^{-As}H(s)ds$$

$$e^{-At}X(t) = e^{-At_0}X(t_0) + \int_{t_0}^{t} e^{-As}H(s)ds$$

$$X(t) = e^{A(t-t_0)}X_0 + e^{At}\int_{t_0}^{t} e^{-As}H(s)\, ds\,.$$

Example 1. Solve the initial-value problem

$$\dot{X} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ e^{t}\cos 2t \end{bmatrix}, \qquad X(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

في البداية نحل النظام المتجانس $\dot{X} = AX$ وذلك باستخراج القيم الذاتية

$$\det\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda + 5) = 0 \rightarrow \lambda_1 = 1, \lambda_{2,3} = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$$1.\ \lambda_1 = 1 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \rightarrow 2a - 2c = 0, 3a + 2b = 0, c = a, b = -\frac{3}{2}a$$

$$V_1 = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, X_1 = e^t\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^t \\ -3e^t \\ 2e^t \end{bmatrix}$$

$$2.\ \lambda = 1 + 2i \rightarrow \begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \rightarrow -2ia = 0 \rightarrow a = 0,$$

$$2a - 2ib - 2c = 0,\ \ ib + c = 0 \rightarrow V = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix}$$

$$X = e^{(1+2i)t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -i \end{bmatrix}\right) = e^t(\cos 2t + i \sin 2t)\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - i\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= e^t[\cos 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i(-\cos 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})]$$

$$X_2 = e^t[\cos 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}], X_3 = e^t[-\cos 2t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin 2t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}]$$

$$X_2 = e^t\begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}, X_3 = e^t\begin{bmatrix} 0 \\ \sin 2t \\ -\cos 2t \end{bmatrix}$$

$$\Phi(\mathrm{t}) = \begin{bmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos 2t & e^t\sin 2t \\ 2e^t & e^t\sin 2t & -e^t\cos 2t \end{bmatrix}, \Phi(0) = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\Phi^{-1}(0) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0) = \begin{bmatrix} 2e^t & 0 & 0 \\ -3e^t & e^t\cos 2t & e^t\sin 2t \\ 2e^t & e^t\sin 2t & -e^t\cos 2t \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t\cos 2t + e^t\sin 2t & e^t\cos 2t & -e^t\sin 2t \\ e^t + \frac{3}{2}e^t\cos 2t - e^t\sin 2t & e^t\sin 2t & e^t\cos 2t \end{bmatrix}$$

Then by (6) we get

$$X(t) = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t \cos 2t + e^t \sin 2t & e^t \cos 2t & -e^t \sin 2t \\ e^t + \frac{3}{2}e^t \cos 2t - e^t \sin 2t & e^t \sin 2t & e^t \cos 2t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$+ \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t \cos 2t + e^t \sin 2t & e^t \cos 2t & -e^t \sin 2t \\ e^t + \frac{3}{2}e^t \cos 2t - e^t \sin 2t & e^t \sin 2t & e^t \cos 2t \end{bmatrix} \int_0^t \begin{bmatrix} e^s & 0 & 0 \\ -\frac{3}{2}e^s + \frac{3}{2}e^t \cos 2s + e^s \sin 2s & e^s \cos 2s & -e^s \sin 2s \\ e^s + \frac{3}{2}e^s \cos 2s - e^s \sin 2s & e^s \sin 2s & e^s \cos 2s \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ e^s \cos 2s \end{bmatrix} ds$$

$$X(t) = \begin{bmatrix} e^t \\ e^t \cos 2t - e^t \sin 2t \\ e^t \cos 2t + e^t \sin 2t \end{bmatrix}$$

$$+ \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t + \frac{3}{2}e^t \cos 2t + e^t \sin 2t & e^t \cos 2t & -e^t \sin 2t \\ e^t + \frac{3}{2}e^t \cos 2t - e^t \sin 2t & e^t \sin 2t & e^t \cos 2t \end{bmatrix} \int_0^t \begin{bmatrix} 0 \\ -e^{2s} \cos 2s \sin 2s \\ e^{2s} \cos^2 2s \end{bmatrix} ds$$

$$X(t) =$$

Example 2 Solve the initial-value problem $\dot{X} = \begin{bmatrix} 3 & -4 \\ 0 & 3 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t,\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\det(A - \lambda I) = 0 \rightarrow \det \begin{bmatrix} 3-\lambda & -4 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 3,$$

$$\lambda_1 = 3 \rightarrow (A - 3I)V_1 = 0 \rightarrow \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \rightarrow b = 0, V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X_1 = e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 3 \rightarrow (A - 3I)V_2 = V_1 \rightarrow \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, -4b = 1, b = \frac{-1}{4}, V_2 = \begin{bmatrix} 0 \\ \frac{-1}{4} \end{bmatrix}$$

$$X_2 = e^{3t} \left[\begin{bmatrix} 0 \\ \frac{-1}{4} \end{bmatrix} + t \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{-1}{4} \end{bmatrix} \right] = e^{3t} \begin{bmatrix} t \\ \frac{-1}{4} \end{bmatrix}$$

$$\Phi(\mathrm{t}) = e^{3t} \begin{bmatrix} 1 & t \\ 0 & \frac{-1}{4} \end{bmatrix} \rightarrow \Phi^{-1}(0) = -4 \begin{bmatrix} \frac{-1}{4} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$$

$$e^{At} = \Phi(\mathrm{t})\Phi^{-1}(0) = e^{3t} \begin{bmatrix} 1 & t \\ 0 & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} = e^{3t} \begin{bmatrix} 1 & -4t \\ 0 & 1 \end{bmatrix}$$

Then by (6) we get

$$X(\mathrm{t}) = e^{3t} \begin{bmatrix} 1 & -4t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 & -4t \\ 0 & 1 \end{bmatrix} \int_0^t e^{3s} \begin{bmatrix} 1 & -4s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} ds$$

$$= e^{3t} \begin{bmatrix} -4t \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 & -4t \\ 0 & 1 \end{bmatrix} \int_0^t \begin{bmatrix} e^{3s} \\ 0 \end{bmatrix} ds$$

$$= e^{3t}\begin{bmatrix} -4t + \frac{1}{3}[e^{3t} - 1] \\ 1 \end{bmatrix}$$

Homework

1. Solve the initial-value problem $\dot{X} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} X + \begin{bmatrix} \sin t \\ \tan t \end{bmatrix}$, $X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. Solve the initial-value problem $\dot{X} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$, $X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3.13 Solving systems by Laplace transforms

$$\dot{\mathrm{X}}(\mathrm{t}) = AX(t) + H(t), \quad X(0) = X_0 \qquad (1)$$

$$\mathbf{X}(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \mathcal{L}\{\mathbf{x}(t)\} = \begin{bmatrix} \int_0^\infty e^{-st} x_1(t)dt \\ \vdots \\ \int_0^\infty e^{-st} x_n(t)dt \end{bmatrix} \qquad (2)$$

$$\mathbf{F}(s) = \begin{pmatrix} F_1(s) \\ \vdots \\ F_n(s) \end{pmatrix} = \mathcal{L}\{\mathbf{f}(t)\} = \begin{pmatrix} \int_0^\infty e^{-st} f_1(t)dt \\ \vdots \\ \int_0^\infty e^{-st} f_n(t)dt \end{pmatrix} \qquad (3)$$

Taking Laplace transforms of both sides of (1) gives

$$\mathcal{L}\{\dot{\mathrm{X}}(t)\} = \mathcal{L}\{AX(t) + H\} = A\mathcal{L}\{X(t)\} + \mathcal{L}\{H\} \rightarrow$$

$$\begin{bmatrix} \mathcal{L}\{\dot{x}_1(t)\} \\ \vdots \\ \mathcal{L}\{\dot{x}_n(t)\} \end{bmatrix} = A \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix} + \begin{bmatrix} \mathcal{L}\{h_1(t)\} \\ \vdots \\ \mathcal{L}\{h_n(t)\} \end{bmatrix}$$

$$\begin{bmatrix} s\mathcal{L}\{x_1(t)\} - x_1(0) \\ \vdots \\ \mathcal{L}\{x_n(t)\} - x_n(0) \end{bmatrix} = A \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix} + \begin{bmatrix} \mathcal{L}\{h_1(t)\} \\ \vdots \\ \mathcal{L}\{h_n(t)\} \end{bmatrix} \qquad (4)$$

Example 1. Solve the initial-value problem

$$\dot{\boldsymbol{X}} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \boldsymbol{X} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \ \boldsymbol{X}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution. Taking Laplace transforms of both sides of the differential equation gives

$$\begin{bmatrix} s\mathcal{L}\{x_1(t)\} - 2 \\ s\mathcal{L}\{x_2(t)\} - 1 \end{bmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{L}\{x_1(t)\} \\ \mathcal{L}\{x_2(t)\} \end{pmatrix} + \frac{1}{s-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or

$$(s-1)\mathcal{L}\{x_1(t)\} - 4\mathcal{L}\{x_2(t)\} = 2 + \tfrac{1}{s-1} \qquad (s-1)X_1(s) - 4X_2(s) = 2 + \tfrac{1}{s-1}$$

$$-\mathcal{L}\{x_1(t)\} + (s-1)\mathcal{L}\{x_2(t)\} = 1 + \tfrac{1}{s-1}. \quad -X_1(s) + (s-1)X_2(s) = 1 + \tfrac{1}{s-1}.$$

$$((s-1)^2-4)\mathcal{L}\{x_1(t)\} = 2(s-1)+5+\frac{4}{s-1}$$

$$((s-1)^2-4)\mathcal{L}\{x_1(t)\} = \frac{2s-2}{(s-3)(s+1)(s-1)} + \frac{5s-1}{}$$

The solution of these equations is

$$\mathcal{L}\{x_1(t)\} = \frac{2}{s-3}+\frac{1}{s^2-1},\ \mathcal{L}\{x_2(t)\} = \frac{1}{s-3}+\frac{s}{(s-1)(s+1)(s-3)}$$

Now,

$$\frac{2}{s-3} = 2\mathcal{L}\{e^{3t}\}, \qquad \frac{1}{s^2-1} = \mathcal{L}\{\sinh t\} = \mathcal{L}\left\{\frac{e^t-e^{-t}}{2}\right\}$$

$$\mathcal{L}\{x_1(t)\} = 2\mathcal{L}\{e^{3t}\} + \mathcal{L}\left\{\frac{e^t-e^{-t}}{2}\right\} = \mathcal{L}\left\{2e^{3t}+\frac{e^t-e^{-t}}{2}\right\}$$

$$x_1(t) = 2e^{3t}+\frac{e^t-e^{-t}}{2}$$

$$\frac{s}{(s-1)(s+1)(s-3)} = \frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s-3}$$

$$s = A(s^2+2s-3)+B(s^2-4s+3)+C(s^2-1)$$

$$A+B+C=0, 2A-4B=1, -3A+3B-C=0$$

$$A=-\tfrac{1}{4}, B=-\tfrac{1}{8}, C=\tfrac{3}{8},$$

$$\mathcal{L}\{x_2(t)\} = \mathcal{L}\{e^{3t}\} - \frac{1}{4}\mathcal{L}\{e^t\} - \frac{1}{8}\mathcal{L}\{e^{-t}\} + \frac{3}{8}\mathcal{L}\{e^{3t}\}$$

$$x_2(t) = \frac{11}{8}e^{3t} - \frac{1}{4}e^t - \frac{1}{8}e^{-t}$$

Homework

1. $\dot{\mathbf{x}} = \begin{pmatrix}1 & -1\\ 5 & -3\end{pmatrix}\mathbf{x},\ \mathbf{x}(0) = \begin{pmatrix}1\\2\end{pmatrix}$
2. $\dot{x} = \begin{pmatrix}3 & -2\\ 2 & -2\end{pmatrix}\mathbf{x} + \begin{pmatrix}t\\3e^t\end{pmatrix},\ \mathbf{x}(0) = \begin{pmatrix}2\\1\end{pmatrix}$
3. $\dot{\mathbf{x}} = \begin{pmatrix}1 & 1\\ 4 & 1\end{pmatrix}\mathbf{x} + \begin{pmatrix}2\\-1\end{pmatrix}e^t,\ \mathbf{x}(0) = \begin{pmatrix}0\\0\end{pmatrix}$
4. $\dot{\mathrm{x}} = \begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}x + \begin{pmatrix}\sin t\\ \tan t\end{pmatrix},\ x(0) = \begin{pmatrix}-1\\0\end{pmatrix}$

الفصل الدراسي الثاني

Theory of Differential Equations

Chapter 2; Qualitative theory of differential equations

2.1 Introduction

$$y'(t) = f(t, y(t)) \qquad \text{(DE)}$$

$$\dot{X}(t) = F\big(t, X(t)\big) \dots\dots\dots\dots \quad (1)$$

$$y'(t) = f(y(t)) \qquad (ADE)$$

$$\dot{X}(t) = F\big(X(t)\big) \dots\dots\dots\dots \quad (1')$$

An Equation is autonomous if f do not depend explicitly on t, like (ADE) or $(1')$
While equation (DE) & (1) are nonautonomous.

Definition 1. (Equilibrium points) of (1).

A points c_i are said to be equilibrium (critical; fixed; accumulation) points of equation autonomous equation if $f(c_i) = 0$.

Example 1. Find the equilibrium points of $y' = 3y^2 - 2y - 5$

$$3y^2 - 2y - 5 = 0 \rightarrow (3y - 5)(y + 1) = 0 \rightarrow y = \frac{5}{3} = c_1, y = -1 = c_2.$$

Example 2. 1. Find the equilibrium points of $y' = e^y,\ y' = y^2 + 1$

$e^y > 0 \neq 0,\ y^2 + 1 > 0 \neq 0$ so there is no critical point in these equations.

2. $y' = \sin y, \quad y' = y^2 - e^{y-1}$

For the system the critical points are (c_1, c_2)

Example 3 Find the equilibrium points of $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^2 - 4y^2 \\ y^2 - 2x + 2y + 5 \end{bmatrix}$,

$$x^2 - 4y^2 = 0 \rightarrow x = 2y \ \&\ x = -2y$$

If $x = 2y \rightarrow y^2 - 4y + 2y + 5 = 0 \rightarrow y^2 - 2y + 5 = 0 \rightarrow y = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$ ignore

If $x = -2y \rightarrow y^2 + 6y + 5 = 0 \rightarrow (y + 5)(y + 1) = 0 \rightarrow y = -5, x = 10, y = -1, x = 2 \rightarrow (10, -5),\ (2, -1)$ or $\begin{bmatrix} 10 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Example 4 Find the equilibrium points of $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (x-1)(y-1) \\ (x+1)(y+1) \end{bmatrix}$,

Home work

1. $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - x^2 - 2xy \\ 2y - 2y^2 - 3xy \end{bmatrix}$,

2. $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} ax - bxy \\ -cx + dxy \\ z + x^2 + y^2 \end{bmatrix}$,

2.2. Stability of linear systems

$$\dot{X}(t) = F\big(X(t)\big) \dots\dots\dots\dots \quad (1')$$

Definition 1. The solution $X = \boldsymbol{\varphi}(t)$ of (1') is stable if every solution $\boldsymbol{\psi}(t)$ of (1') which starts sufficiently close to $\boldsymbol{\varphi}(t)$ at $t = 0$ must remain close to $\boldsymbol{\varphi}(t)$ for all future time t. The solution $\boldsymbol{\varphi}(t)$ is unstable if there exists at least one solution $\boldsymbol{\psi}(t)$ of (1') which starts near $\boldsymbol{\varphi}(t)$ at $t = 0$ but which does not remain close to $\boldsymbol{\varphi}(t)$ for all future time. More precisely, the solution $\boldsymbol{\varphi}(t)$ is stable if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $|\varphi_i(t) - \psi_i(t)| < \varepsilon$ if $|\varphi_i(0) - \psi_i(0)| < \delta(\varepsilon)$, $i = 1,2,\dots,n$. for every solution $\boldsymbol{\psi}(t)$ of (1').

$$\dot{X}(t) = AX \dots\dots\dots\dots \quad (2)$$

Theorem 1. (a) Every solution $X = \boldsymbol{\varphi}(t)$ of (1') is stable if all the eigenvalues of A have negative real part.

(b) Every solution $X = \boldsymbol{\varphi}(t)$ of (2) is unstable if at least one eigenvalue of A has positive real part.

(c) Suppose that all the eigenvalues of A which are purely imaginary then every solution $X = \boldsymbol{\varphi}(t)$ of (1') is stable

Definition 2. Let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector with n components. The numbers $x_1, x_2, \cdots, x_n$ may be real or complex. We define the length of X, denoted by $\|X\|$ as $\|X\| = \max\{x_1, x_2, \cdots, x_n\}$.

For example, if $X = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

then $\|X\| = 3$ and if $X = \begin{bmatrix} 1+2i \\ 2 \\ -1 \end{bmatrix}$ then $\|X\| = 5$.

Definition 3. A solution $X = \boldsymbol{\varphi}(t)$ of (2.1') is asymptotically stable if it is stable, and if every solution $\boldsymbol{\psi}(t)$ which starts sufficiently close to $\boldsymbol{\varphi}(t)$ must approach $\boldsymbol{\psi}(t)$ as t approaches infinity. In particular, an equilibrium solution $X(t) = X_0$ of (1') is asymptotically stable if every solution $\boldsymbol{\psi}(t)$ of (1') which starts sufficiently close to X_0 at time $t = 0$ not only remains close to X_0 for all future time, but ultimately approaches X_0 as t approaches infinity.

Example 1. Determine whether each solution $X(t)$ of the system

$\dot{X} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -1 & 2 \\ -3 & -2 & -1 \end{bmatrix} X$ is stable, asymptotically stable, or unstable.

To find the eigenvalue

$$det \begin{bmatrix} -1-\lambda & 0 & 0 \\ -2 & -1-\lambda & 2 \\ -3 & -2 & -1-\lambda \end{bmatrix} = 0 \rightarrow -(1+\lambda)^3 - 4(1+\lambda) = 0 \rightarrow$$

$-(1+\lambda)[(1+\lambda)^2 + 4] = 0 \rightarrow\rightarrow -(1+\lambda)[\lambda^2 + 2\lambda + 5] \rightarrow \lambda = -1, \lambda = -1 \pm 2i$

By theorem 1 the solution is asymptotically stable

Example 2. Determine whether each solution $X(t)$ of the system

$$\dot{X} = \begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix} X$$

$$det \begin{bmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{bmatrix} = 0 \rightarrow \lambda^2 - 2\lambda - 24 = 0 \rightarrow \lambda = 6, \lambda = -4$$

By theorem 1 the solution is unstable

Example 3. Determine whether each solution $X(t)$ of the system

$$\dot{X} = \begin{bmatrix} 0 & -8 \\ 2 & 0 \end{bmatrix} X$$

$$det \begin{bmatrix} -\lambda & -8 \\ 2 & -\lambda \end{bmatrix} = 0 \rightarrow \lambda^2 + 16 = 0 \rightarrow \lambda = 4i, \lambda = -4i$$

By theorem 1 the solution is stable

Example 4. Determine whether each solution $X(t)$ of the system

$\dot{X} = \begin{bmatrix} 2 & -3 & 0 \\ 0 & -6 & -2 \\ -6 & 0 & -3 \end{bmatrix} X$ is stable, asymptotically stable, or unstable.

To find the eigenvalue

$$det \begin{bmatrix} 2-\lambda & -3 & 0 \\ 0 & -6-\lambda & -2 \\ -6 & 0 & -3-\lambda \end{bmatrix} = 0 \rightarrow -\lambda^2(\lambda+7) = 0 \rightarrow$$

$$\lambda = 0, \lambda = 0, \lambda = -7$$

By theorem 1 the solution is unstable

Homework

1. $\dot{X} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} X$, 2. $\dot{X} = \begin{bmatrix} -5 & 3 \\ -1 & 1 \end{bmatrix} X$, 3. $\dot{X} = \begin{bmatrix} -7 & 1 & -6 \\ 10 & -4 & 12 \\ 2 & -1 & 1 \end{bmatrix} X$,

4. $\dot{X} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} X$

2.3 Linear and Nonlinear System

2.3.1 Linear Changes of Variable

$$\dot{X} = AX \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad (2.1)$$

We use the linear change of variable $\quad X = MY \quad$ (2.2)

where $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, M is nonsingular matrix, then $\dot{X} = M\dot{Y} = AX = AMY$

$$\dot{Y} = M^{-1}AMY \Longrightarrow J = M^{-1}AM$$

$$\dot{Y} = JY \qquad (2.3)$$

Definition 1 We say that matrix J is similar to matrix A if there is nonsingular matrix M such that $J = M^{-1}AM \qquad$ (2.4)

Example 1. The change of variable $x_1 = y_1 + y_2, x_2 = y_1 - y_2$ transform the system $\dot{x}_1 = x_2,\ \dot{x}_2 = x_1$ to the system……………..

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow M^{-1} = \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$J = M^{-1}AM = \tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ by (2.3)}$$

$$\dot{Y} = JY = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \dot{y}_1 = y_1,\ \dot{y}_2 = -y_2$$

Example 2. The change of variable $x_1 = y_2, x_2 = y_1,\ x_3 = -y_2 + y_3$ transform the system $\dot{x}_1 = x_2,\ \dot{x}_2 = x_3, \dot{x}_3 = x_1$ to the system……………..

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, M^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J = M^{-1}AM = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\dot{Y} = JY = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow \dot{y}_1 = -y_2 + y_3,\ \dot{y}_2 = y_1, \dot{y}_3 = y_1 + y_2$$

Definition 2 We say that matrix J is said Jordan form of A if it is similar to matrix A and $M = [V_1 V_2 \cdots V_n], \Rightarrow J = M^{-1}AM$

Theorem 2 Let A be a real $2\times\ 2$ matrix, then there is a real, nonsingular matrix M such that $J=M^{-1}AM$ is one of the types:

(a) If A has distinct real eigenvalue then $J=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}, \lambda_1>\lambda_2$;

(b) If A is diagonal and has equal eigenvalue then $J=\begin{bmatrix}\lambda_0 & 0\\ 0 & \lambda_0\end{bmatrix}$;

(c) If A is nondiagonal and has equal eigenvalue then $J=\begin{bmatrix}\lambda_0 & 1\\ 0 & \lambda_0\end{bmatrix}$;

(d) If A has complex eigenvalue $\lambda=\alpha\pm i\beta$ then $J=\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}$

Example 3 Find the Jordan forms of each of the following matrices:

$(a)\ A_1=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}, (b)\ A_2=\begin{bmatrix}2 & 1\\ -2 & 4\end{bmatrix},\ \ (c)\ A_3=\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix},\ \ (d) A_4=\begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}$

$(a)\ \det\begin{bmatrix}1-\lambda & 2\\ 1 & 1-\lambda\end{bmatrix}=0 \Longrightarrow \lambda^2-2\lambda-1=0\to\lambda=1\pm\sqrt{2}\to$

$J=\begin{bmatrix}1+\sqrt{2} & 0\\ 0 & 1-\sqrt{2}\end{bmatrix}, \to\ \lambda_{1,2}$ are real distinct.

$(b)\ \det\begin{bmatrix}2-\lambda & 1\\ -2 & 4-\lambda\end{bmatrix}=0 \Longrightarrow \lambda^2-6\lambda+10=0\to\lambda=3\pm i=\alpha\pm i\beta\to$

$J=\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}=\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}, \to\ \lambda_{1,2}$ are complex.

$(c)\ \det\begin{bmatrix}3-\lambda & -1\\ 1 & 1-\lambda\end{bmatrix}=0 \Longrightarrow \lambda^2-4\lambda+4=0\to\lambda_{1,2}=2\to$A is nondiagonal

$J=\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}, \to\ \lambda_{1,2}$ are equal and A nondiagonal.

$(d)\ \lambda_{1,2}=-3\to\ \lambda_{1,2}$ are equal, A is diagonal then $J=\begin{bmatrix}-3 & 0\\ 0 & -3\end{bmatrix}$.

Remark 1: If A has complex eigenvalue then $M=\begin{pmatrix}a_{11}-\alpha & -\beta\\ a_{21} & 0\end{pmatrix}$ (2.5)

Example 4. Find a matrix M which converts each of the matrices in Example 3 into their appropriate Jordan forms.

$M_1=[V_1\ \ V_2], V_1=\begin{bmatrix}\sqrt{2}\\ 1\end{bmatrix}, V_2=\begin{bmatrix}-\sqrt{2}\\ 1\end{bmatrix}\to M_1=\begin{bmatrix}\sqrt{2} & -\sqrt{2}\\ 1 & 1\end{bmatrix}$

$$M^{-1}AM=J\to\begin{bmatrix}\sqrt{2} & -\sqrt{2}\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & -\sqrt{2}\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1+\sqrt{2} & 0\\ 0 & 1-\sqrt{2}\end{bmatrix}$$

$M_2=\begin{pmatrix}a_{11}-\alpha & -\beta\\ a_{21} & 0\end{pmatrix}=\begin{pmatrix}-1 & -1\\ -2 & 0\end{pmatrix}$

$\begin{bmatrix}-1 & -1\\ -2 & 0\end{bmatrix}^{-1}\begin{bmatrix}2 & 1\\ -2 & 4\end{bmatrix}\begin{bmatrix}-1 & -1\\ -2 & 0\end{bmatrix}=\begin{bmatrix}3 & -1\\ 1 & 3\end{bmatrix}$

$$M_3=[V_1\ \ V_2], V_1=\begin{bmatrix}1\\ 1\end{bmatrix}, (A-\lambda I)V_2=V_1\Longrightarrow\ V_2=\begin{bmatrix}2\\ 1\end{bmatrix}\to M_3=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$$

$$M^{-1}AM$$

$$= J \to \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$M_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2.3 Phase Portraits for Canonical Systems in Plane:

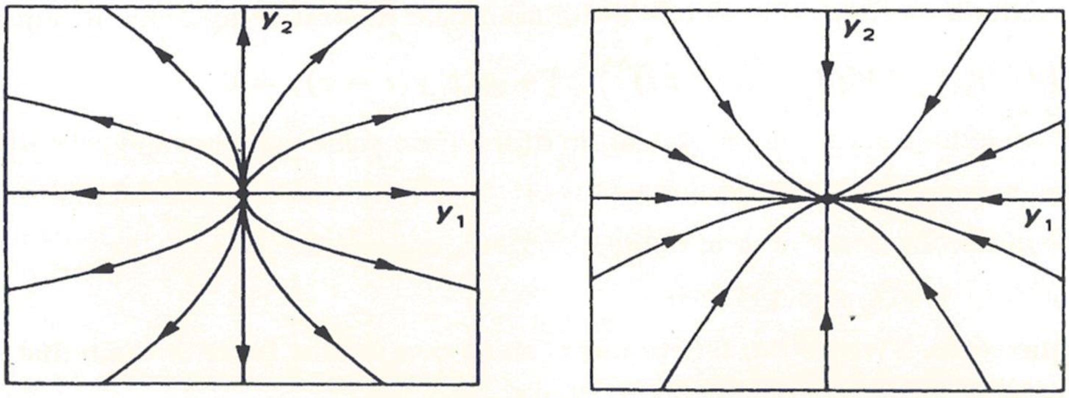
Definition 3: A linear system $\dot{X} = AX$ is said to be simple if the matrix A is non-singular, (i.e. $\det(A) \neq 0$ and A has non-zero eigenvalues).

(a) Real, distinct eigenvalues

$$\dot{Y} = JY \Rightarrow \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \Rightarrow \dot{y}_1 = \lambda_1 y_1, \;\; \dot{y}_2 = \lambda_2 y_2,$$

$$y_1 = e^{\lambda_1 t}, \;\; y_2 = e^{\lambda_2 t}, \qquad (2.6)$$

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{dy_2}{dy_1} = \frac{\lambda_2 y_2}{\lambda_1 y_1}, \frac{dy_2}{y_2} = \frac{\lambda_2}{\lambda_1}\frac{dy_1}{y_1} \Rightarrow \ln y_2 = \frac{\lambda_2}{\lambda_1}\ln y_1 + \ln c \Rightarrow y_2 = c y_1^{\frac{\lambda_2}{\lambda_1}} \; (2.7)$$



(a) (b)

Fig. 2.1. Real distinct eigenvalues of the same sign give rise to nodes: (a) unstable ($\lambda_1 > \lambda_2 > 0$); (b) stable ($\lambda_2 < \lambda_1 < 0$).

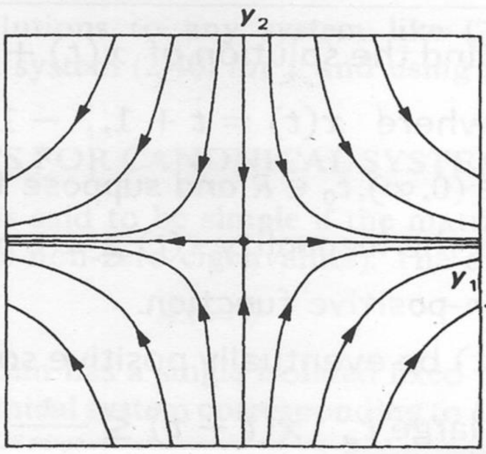


Fig. 2.2. Real eigenvalues of opposite sign (c) ($\lambda_2 < 0 < \lambda_1$) give rise to saddle points.

(b) Equal eigenvalues

If $J = A$ is diagonal, the canonical system has solutions given by Theorem 2-b with $\lambda_1 = \lambda_2 = \lambda_0$. Thus (2.7) corresponds to a special node $y_2 = c y_1$, called a star node (stable if $\lambda_0 < 0$; unstable if $\lambda_0 > 0$), in which the non-trivial trajectories are all radial straight lines (as shown in Fig. 2.3).

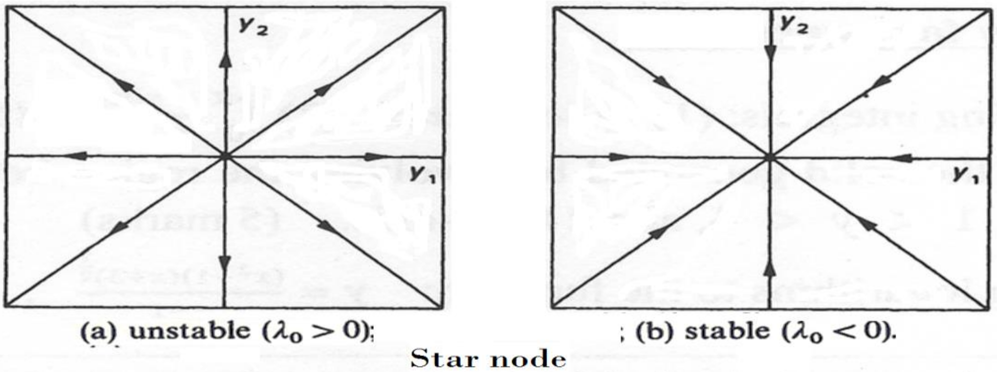


Fig. 2.3. Equal eigenvalues ($\lambda_1 = \lambda_2 = \lambda_0$) give rise to star nodes: (a) unstable; (b) stable; when A is diagonal.

(c) Equal eigenvalues, A is non-diagonal, $\lambda_1 = \lambda_2 = \lambda_0$ hence

$$J = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}, y_1 = (c_1 + c_2 t)e^{\lambda_0 t}, \quad y_2 = c_2 e^{\lambda_0 t}, \qquad (2.8)$$

$$\frac{dy_2}{dy_1} = \frac{\lambda_0 y_2}{\lambda_0 y_1 + y_2} \to \; y_1 = c_1 y_2 + \frac{y_2}{\lambda_0} \ln y_2, \qquad (2.9)$$

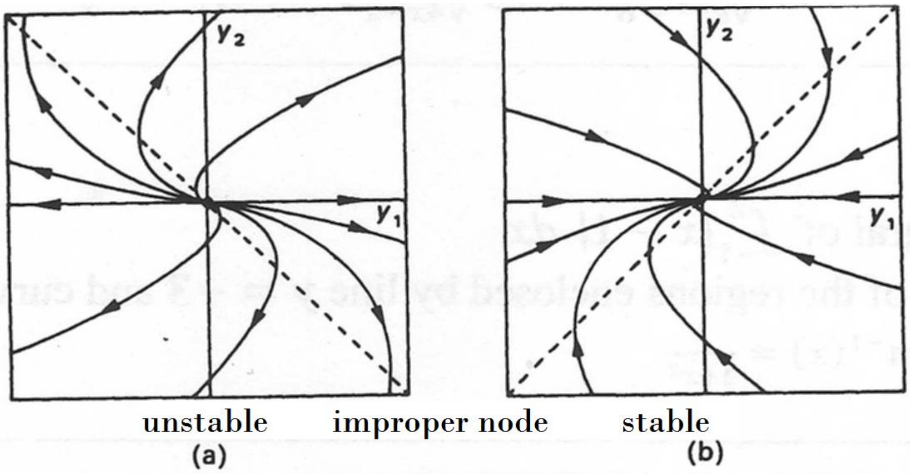


Fig. 2.4. When A is not diagonal, equal eigenvalues indicate that the origin is an improper node: (a) unstable ($\lambda_0 > 0$); (b) stable ($\lambda_0 < 0$).

(d) Complex eigenvalues

$$J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \lambda_{1,2} = \alpha \pm i\beta, \;\; \dot{y}_1 = \alpha y_1 - \beta y_2, \dot{y}_2 = \beta y_1 + \alpha y_2$$

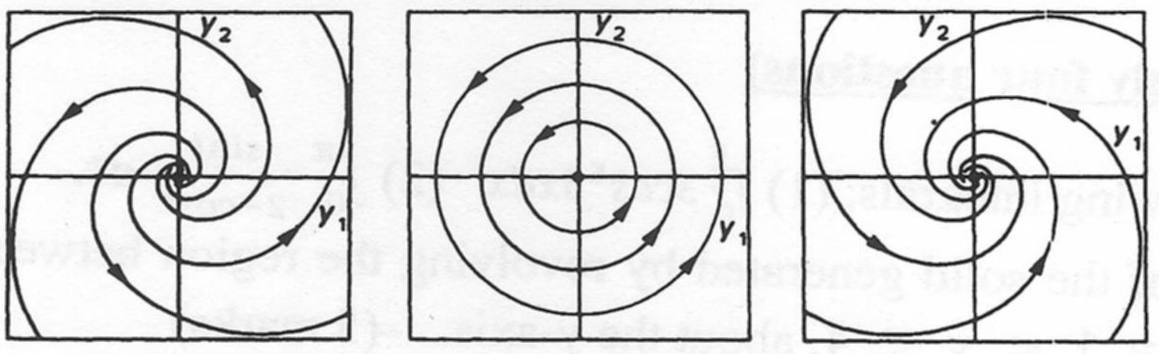
Using polar coordinate's $r^2 = y_1^2 + y_2^2$, $\tan\theta = \frac{y_2}{y_1}$

$$\to \dot{r} = \alpha r, \qquad \dot{\theta} = \beta \qquad (2.10)$$

$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \beta t + \theta_0 \qquad (2.11)$$

if $\alpha < 0 \to$ spiral(focus)stable, if $\alpha > 0 \to$ spiral unstable,

if $\alpha = 0 \to$ centre (stable)



(a) spiral unst. centers (b) spiral stable (c)

Fig. 2.5. Complex eigenvalues give rise to (a) unstable foci ($\alpha > 0$), (b) centres ($\alpha = 0$) and (c) stable foci ($\alpha < 0$).

Example 5 Sketch the phase portrait of the system

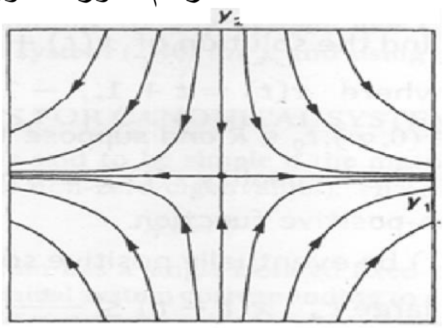
$$y_1' = 2y_1, \ \ y_2' = -2y_2; \text{and} \ \ y_1' = -2y_2, \ \ y_2' = 2y_1 \qquad (2.12)$$

and the corresponding phase portraits in the x_1-x_2 plane where

$$M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, M_5 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, M_6 = \begin{bmatrix} 1 & 1 \\ 4 & -4 \end{bmatrix} \qquad (2.13)$$

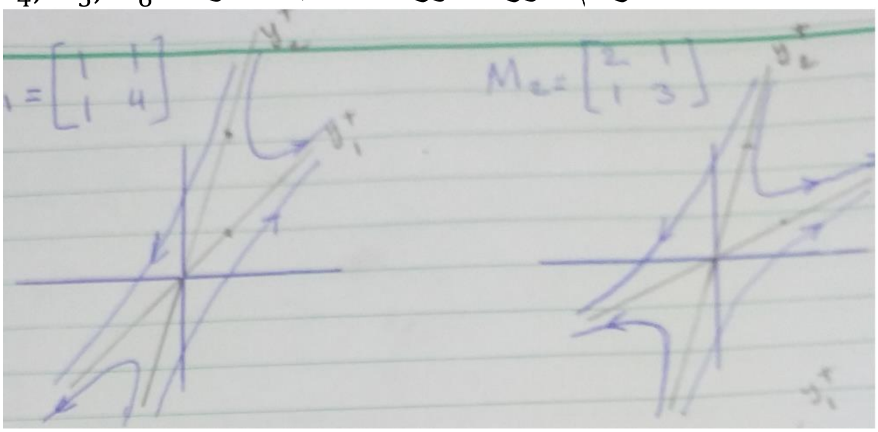
$$J = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \lambda_1 = 2, \lambda_2 = -2$$

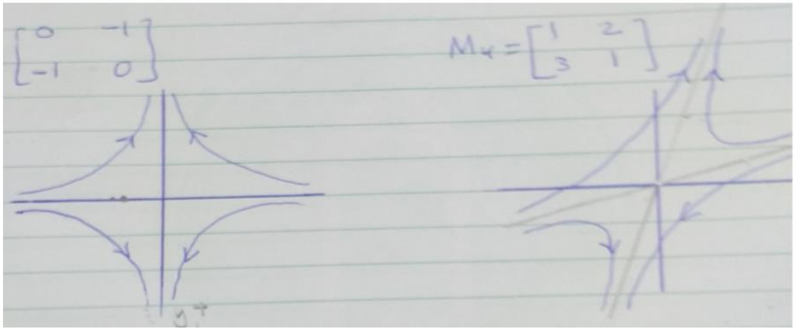
Jordan canonical form رسم صورة الطورالى

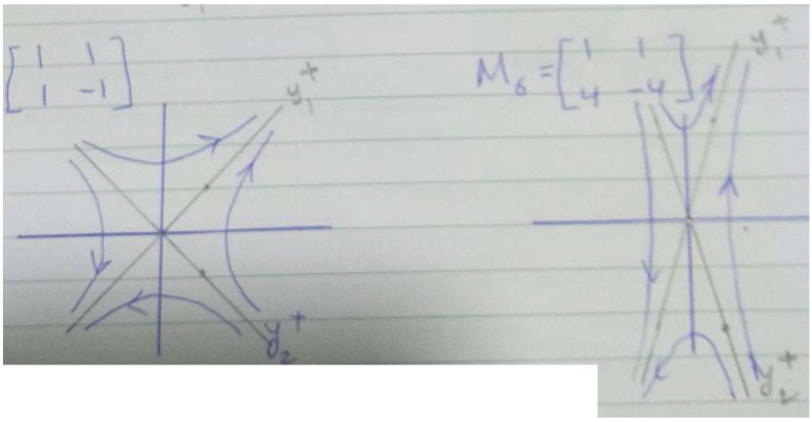


λ=2, λ=-2: spiral unstable

$M_1, M_2, M_3, M_4, M_5, M_6$ رسم صورة الطور الخاصة بالمصفوفات





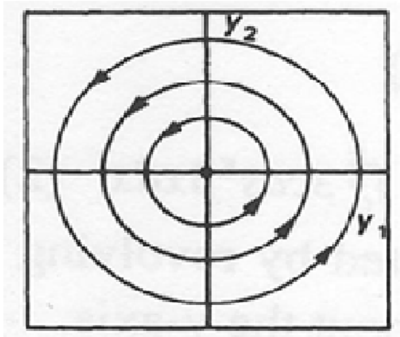


Example 6 Sketch the phase portrait of the system

$$x_1' = 2x_1 + 2x_2, \quad x_2' = 4x_1 - 2x_2 \qquad (2.14)$$

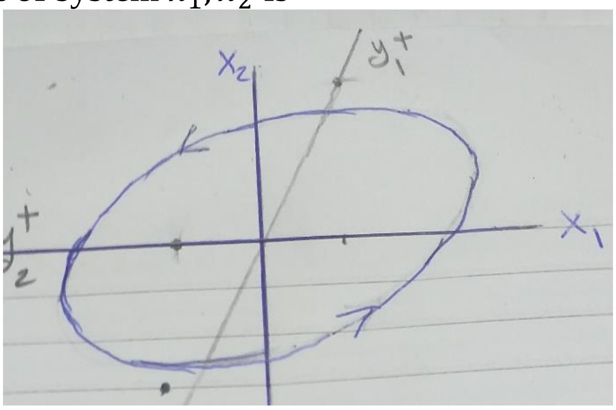
The eigenvalue are $\lambda_1 = 2i, \lambda_2 = -2i, \ \alpha = 0, \beta = 2$

Then $J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, and $M = \begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix}$ the phase portrait of Jordan form is



centers λ1=2i, λ2=-2i

And the phase portrait of system x_1, x_2 is



2.4 Phase Portraits for Canonical Systems in Plane:

Theorem 2.1 (Linearization theorem)

Let the non-linear system $X' = F(X)$ (2.15), $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, F = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$

have a simple fixed point at $(c_1, c_2) = (0,0)$. Then, in a neighborhood of the origin the phase portraits of the system and its linearization are qualitatively equivalent provided the linearized system is not a center.

كيف استخرج النظام الخطي من النظام غير الخطي؟

Definition 4: Jacobian Matrix: Let a system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$ then the jacobian matrix at critical point (c_1, c_2) is defined by $J_{(c_1,c_2)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(c_1,c_2)}$.

Example7 Find critical points and Jacobian matrix at each of them of the system

$$x_1' = 2x_1 - x_1 x_2, \quad x_2' = 2x_1 + x_2 \qquad (2.15)$$

$2x_1 - x_1 x_2 = 0 \Longrightarrow x_1(2 - x_2) = 0$

Either $x_1 = 0$ or $x_2 = 2$

If $x_1 = 0$ then $2x_1 + x_2 = 0 \Longrightarrow x_2 = -2x_1 \Longrightarrow x_2 = 0$ the first critical point (0,0).

If $x_2 = 2$ then $x_2 = -2x_1 \Longrightarrow x_1 = -1$, the second critical point $(-1,2)$

$$J_{(0,0)} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 2 & 1 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$J_{(-1,2)} = \begin{bmatrix} 2 - x_2 & -x_1 \\ 2 & 1 \end{bmatrix}_{(-1,2)} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

Example 8 Sketch the phase portrait of the system (2.15)

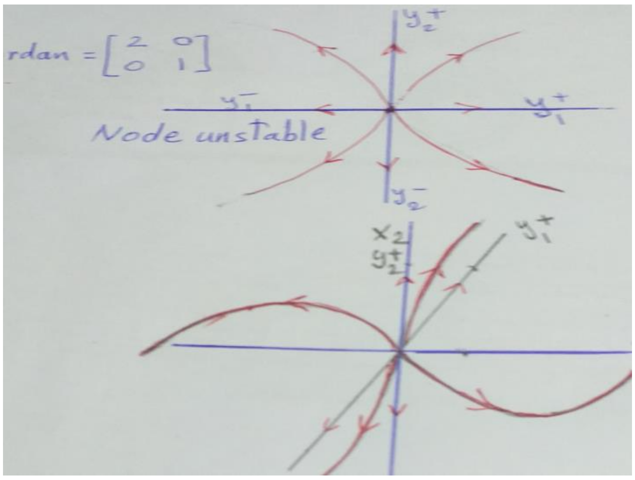
From example 7 we get the first critical point (0,0) and $J_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$

so we have the first system $x_1' = 2x_1, \ x_2' = 2x_1 + x_2$ that is $A_1 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$

the eigenvalue are $\lambda^2 - 3\lambda + 2 = 0 \Longrightarrow (\lambda - 2)(\lambda - 1) = 0 \Longrightarrow$

$$\lambda_1 = 2, \lambda_2 = 1$$

$$J = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$



the second critical point $(-1,2)$ and $J_{(-1,2)} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

so we have the second system $x_1' = -x_2, \;\; x_2' = 2x_1 + x_2$ that is $A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

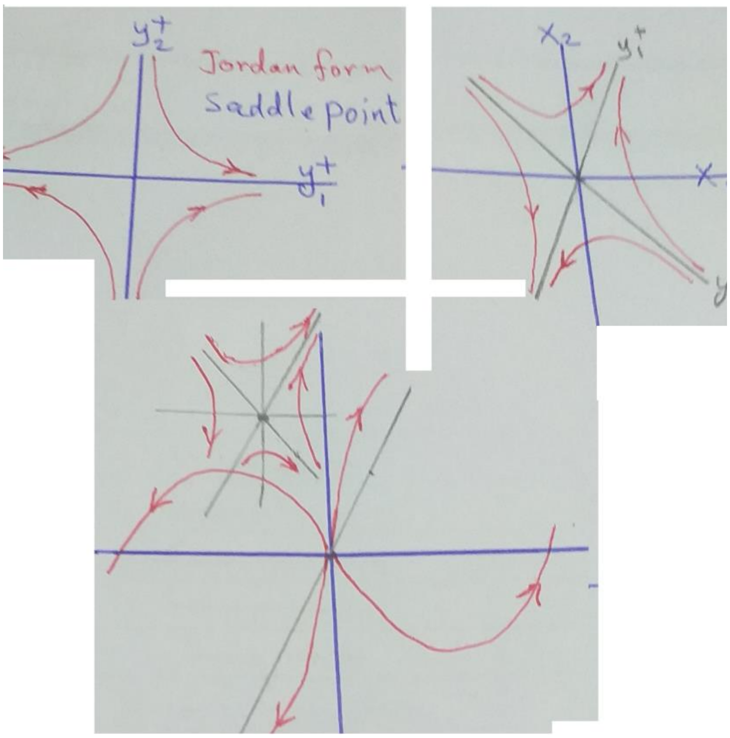
the eigenvalue are $\lambda^2 - \lambda - 2 = 0 \Longrightarrow (\lambda - 2)(\lambda + 1) = 0 \Longrightarrow$

$$\lambda_1 = 2, \lambda_2 = -1$$

$$J = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{2} & \frac{1}{2} \end{bmatrix}, \;\; M = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{7}}{2} \\ 2 & 0 \end{bmatrix}$$

The final phase portrait is



Example 9 Sketch the phase portrait of the system $x_1' = x_2^2 - 3x_1 + 2, x_2' = x_1^2 - x_2^2$

To find the critical points: $x_1 = \pm x_2$ if $x_1 = x_2$ then from second equation $x_2^2 - 3x_2 + 2 = 0$ we get the critical points $(2,2), (1,1)$, and if $x_1 = -x_2$ then from second equation $x_2^2 + 3x_2 + 2 = 0$ we get the critical points $(2,-2), (1,-1)$

$$J_{(2,2)} = \begin{bmatrix} -3 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}_{(2,2)} = \begin{bmatrix} -3 & 4 \\ 4 & -4 \end{bmatrix}$$

$$J_{(1,1)} = \begin{bmatrix} -3 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}_{(1,1)} = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}$$

$$J_{(2,-2)} = \begin{bmatrix} -3 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}_{(2,-2)} = \begin{bmatrix} -3 & -4 \\ 4 & 4 \end{bmatrix}$$

$$J_{(1,-1)} = \begin{bmatrix} -3 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}_{(1,-1)} = \begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix}$$

For $J_{(2,2)} = \begin{bmatrix} -3 & 4 \\ 4 & -4 \end{bmatrix}$ the eigenvalue are $\lambda^2 + 7\lambda - 4 = 0 \Rightarrow \lambda_{1,2} = \frac{-7 \pm \sqrt{65}}{2} = \begin{cases} 0.53 \\ -7.53 \end{cases}$ saddle point.

For $J_{(1,1)} = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}$ the eigenvalue are $\lambda^2 + 5\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = \frac{-5 \pm \sqrt{17}}{2} = \begin{cases} -0.438 \\ -4.56 \end{cases}$ node stable.

$J_{(2,-2)} = \begin{bmatrix} -3 & -4 \\ 4 & 4 \end{bmatrix}$ the eigenvalue are $\lambda^2 - \lambda + 4 = 0 \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{15}i}{2}$ spiral unstable.

$J_{(1,-1)} = \begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix}$ the eigenvalue are $\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -2$ saddle point.

Example 10 Sketch the phase portrait of the system $x_1' = -x_2 - x_1^3, x_2' = x_1 - x_2^3$

To get the critical points from the first equation: $x_2 = -x_1^3$ then from second equation $x_1 + x_1^6 = 0 \Rightarrow x_1(1 + x_1^5) = 0$ so either $x_1 = 0$ then $x_2 = 0$ we get the critical point $(0,0)$, or $x_1^5 = -1 \Rightarrow x_1 = -1$ then $x_2 = 1$ the critical point $(-1,1)$